

Roots and Coefficients of Polynomials

1 (3 marks)

$$\alpha + \beta + \gamma = -\frac{-5}{2} = \frac{5}{2} \quad \text{B1}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{4}{2} = 2 \quad \text{B1}$$

$$\alpha\beta\gamma = -\frac{-7}{2} = \frac{7}{2} \quad \text{B1}$$

$$\text{Answer: } \alpha + \beta + \gamma = \frac{5}{2}, \alpha\beta + \beta\gamma + \gamma\alpha = 2, \alpha\beta\gamma = \frac{7}{2} \quad \text{3 marks}$$

2 (3 marks)

$$\alpha + \beta = 6 \text{ and } \alpha\beta = 4 \quad \text{B1}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 36 - 8 \quad \text{M1}$$

$$= 28 \quad \text{A1}$$

$$\text{Answer: } 28 \quad \text{3 marks}$$

3 (3 marks)

$$\Sigma\alpha = 2, \Sigma\alpha\beta = 5, \alpha\beta\gamma = 4 \quad \text{B1}$$

$$(\alpha + 2)(\beta + 2)(\gamma + 2) = \alpha\beta\gamma + 2\Sigma\alpha\beta + 4\Sigma\alpha + 8 \quad \text{M1}$$

$$= 4 + 10 + 8 + 8 = 30 \quad \text{A1}$$

$$\text{Answer: } 30 \quad \text{3 marks}$$

4 (3 marks)

$$p = -(\alpha + \beta + \gamma) = -5 \quad \text{B1}$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha = 2 \quad \text{B1}$$

$$r = -\alpha\beta\gamma = 6 \quad \text{B1}$$

$$\text{Answer: } p = -5, q = 2, r = 6 \quad \text{3 marks}$$

5 (3 marks)

$$\alpha + \beta = -\frac{k}{3} = 4, \text{ so } k = -12 \quad \text{M1 A1}$$

$$\alpha\beta = \frac{12}{3} = 4 \quad \text{B1}$$

$$\text{Answer: } k = -12, \alpha\beta = 4 \quad \text{3 marks}$$

6 (2 marks)

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} \quad \text{M1}$$

$$= \frac{5}{3} \quad \text{A1}$$

$$\text{Answer: } \frac{5}{3} \quad \text{2 marks}$$

7 (5 marks)

$$\text{(a) } \Sigma\alpha = \frac{1}{2}, \Sigma\alpha\beta = 3, \alpha\beta\gamma = -2 \quad \text{B1}$$

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = \frac{1}{4} - 6 \quad \text{M1}$$

$$= -\frac{23}{4} \quad \text{A1}$$

$$\text{Answer: } -\frac{23}{4} \quad \text{3 marks}$$

$$\text{(b) } \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\Sigma\alpha\beta}{\alpha\beta\gamma} = \frac{3}{-2} \quad \text{M1}$$

$$= -\frac{3}{2} \quad \text{A1}$$

$$\text{Answer: } -\frac{3}{2} \quad \text{2 marks}$$

8 (4 marks)

$(x - 2)$ is a factor: $x^3 - 6x^2 + 13x - 10 = (x - 2)(x^2 - 4x + 5)$

M1 A1

$$x = \frac{4 \pm \sqrt{16 - 20}}{2}$$

M1

$$x = 2 \pm i$$

A1

Answer: $2 + i$ and $2 - i$

4 marks

9 (5 marks)

Let the roots be α and $\alpha + 3$. Sum of roots = $\frac{15}{3} = 5$, so $2\alpha + 3 = 5$ and $\alpha = 1$

M1 A1

The roots are 1 and 4

A1

Product of roots = $\frac{c}{3} = 4$, so $c = 12$

M1 A1

Answer: $c = 12$; roots 1 and 4

5 marks

10 (4 marks)

$$\text{Sum of roots} = \frac{1}{2} - 3 + 4 = \frac{3}{2}$$

M1

$$\text{Sum of pairs} = -\frac{3}{2} + 2 - 12 = -\frac{23}{2}$$

A1

$$\text{Product} = -6, \text{ so } x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = 0$$

A1

$$\text{Multiply by 2: } 2x^3 - 3x^2 - 23x + 12 = 0$$

B1

Answer: $2x^3 - 3x^2 - 23x + 12 = 0$

4 marks

11 (5 marks)

The third root is $1 - i$

B1

Sum of roots = 4, so $a = -4$

M1 A1

Sum of pairs = $2(1 + i) + 2(1 - i) + 2 = 6$, so $b = 6$

A1

Product = $2 \times 2 = 4$, so $c = -4$

A1

Answer: $x^3 - 4x^2 + 6x - 4 = 0$

5 marks

12 (5 marks)

The other roots are $1 - i$ and $3 + 2i$

B1

Quadratic factors $x^2 - 2x + 2$ and $x^2 - 6x + 13$

M1 A1

Multiply: $x^4 - 8x^3 + (13 + 12 + 2)x^2 - (26 + 12)x + 26$

M1

$$x^4 - 8x^3 + 27x^2 - 38x + 26 = 0$$

A1

Answer: $x^4 - 8x^3 + 27x^2 - 38x + 26 = 0$

5 marks

13 (4 marks)

$$\alpha + \beta = \frac{5}{3}, \alpha\beta = -\frac{2}{3}$$

B1

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

M1

$$= \frac{25}{9} + \frac{8}{3} = \frac{49}{9}$$

M1 A1

Answer: $\frac{49}{9}$

4 marks

14 (6 marks)

$$\Sigma\alpha = 4, \Sigma\alpha\beta = 1, \alpha\beta\gamma = -5$$

B1

$$\Sigma\alpha^2 = 16 - 2 = 14$$

M1 A1

$$\alpha^3 = 4\alpha^2 - \alpha - 5, \text{ and similarly for } \beta \text{ and } \gamma$$

M1

$$\Sigma\alpha^3 = 4(14) - 4 - 15$$

M1

$$= 37$$

A1

Answer: 37

6 marks

15 (6 marks)

$\alpha + \beta = (\alpha + \beta + \gamma) - \gamma$ and so on, so the product is $(e_1 - \alpha)(e_1 - \beta)(e_1 - \gamma)$

M1

$= e_1^3 - e_1 \cdot e_1^2 + e_2 e_1 - e_3$, with $e_1 = \Sigma\alpha$, $e_2 = \Sigma\alpha\beta$, $e_3 = \alpha\beta\gamma$

M1 A1

$= e_1 e_2 - e_3$

A1

$= (-2)(-5) - 1 = 9$

M1 A1

Answer: 9

6 marks

16 (6 marks)

Roots $a - d$, a , $a + d$; sum $= 3a = 6$, so $a = 2$

M1 A1

Product: $a(a^2 - d^2) = -10$, so $2(4 - d^2) = -10$

M1

$d^2 = 9$, so $d = \pm 3$ and the roots are $-1, 2, 5$

A1

$k = \Sigma\alpha\beta = -2 + 10 - 5$

M1

$k = 3$

A1

Answer: Roots $-1, 2, 5$; $k = 3$

6 marks

17 (6 marks)

Roots $\frac{a}{r}$, a , ar ; product $a^3 = 8$, so $a = 2$

M1 A1

Sum: $2(\frac{1}{r} + 1 + r) = 7$, so $2r^2 - 5r + 2 = 0$

M1

$(2r - 1)(r - 2) = 0$, so the roots are $1, 2, 4$

M1 A1

$k = \Sigma\alpha\beta = 2 + 8 + 4 = 14$

A1

Answer: Roots $1, 2, 4$; $k = 14$

6 marks

18 (6 marks)

Sum of roots: $\alpha - \alpha + \beta = 3$, so $\beta = 3$

M1 A1

3 is a root: $27 - 27 + 3p + q = 0$, so $q = -3p$

M1 A1

$p = -4$, $q = 12$: $x^3 - 3x^2 - 4x + 12 = (x - 3)(x^2 - 4)$

M1

Roots $3, 2, -2$

A1

Answer: $3, 2, -2$

6 marks

19 (5 marks)

$\Sigma\alpha = -\frac{3}{2}$, $\Sigma\alpha\beta = -2$, $\alpha\beta\gamma = -\frac{1}{2}$

B1

$\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha)$

M1 A1

$= 4 - 2(-\frac{1}{2})(-\frac{3}{2})$

M1

$= \frac{5}{2}$

A1

Answer: $\frac{5}{2}$

5 marks

20 (7 marks)

Let the roots be α , $\frac{1}{\alpha}$ and β . Product of roots $= \frac{18}{6} = 3$, so $\beta = 3$

M1 A1

$x = 3$: $162 - 297 + 3k - 18 = 0$, so $k = 51$

M1 A1

$6x^3 - 33x^2 + 51x - 18 = 3(x - 3)(2x^2 - 5x + 2)$

M1

$= 3(x - 3)(2x - 1)(x - 2)$

A1

$x = 3, 2$ or $\frac{1}{2}$

A1

Answer: $k = 51$; $x = 3, 2, \frac{1}{2}$

7 marks

21 (10 marks)

(a) $\Sigma\alpha = 2$, $\Sigma\alpha\beta = 3$

B1

$\Sigma\alpha^2 = 4 - 6 = -2$

M1 A1

Answer: -2

3 marks

(b) If all three roots were real, $\Sigma\alpha^2$ would be positive

E1

But $\Sigma\alpha^2 < 0$, so there are non-real roots, which occur in conjugate pairs; hence exactly one real root

E1

Answer: One real root

2 marks

(c) $\alpha^3 = 2\alpha^2 - 3\alpha - 5$, so $\Sigma\alpha^3 = 2(-2) - 3(2) - 15$
 $= -25$

M1 A1

A1

$\alpha^4 = 2\alpha^3 - 3\alpha^2 - 5\alpha$, so $\Sigma\alpha^4 = -50 + 6 - 10$
 $= -54$

M1

A1

Answer: -25 and -54

5 marks

22 (8 marks)

$\Sigma\alpha = 4$, $\Sigma\alpha\beta = 2$, $\alpha\beta\gamma = -3$

B1

$$\Sigma \frac{1}{\alpha^2} = \frac{(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma \cdot \Sigma\alpha}{(\alpha\beta\gamma)^2} = \frac{4 + 24}{9} = \frac{28}{9}$$

M1 A1

$$\Sigma \frac{1}{(\alpha\beta)^2} = \frac{(\Sigma\alpha)^2 - 2\Sigma\alpha\beta}{(\alpha\beta\gamma)^2} = \frac{12}{9} = \frac{4}{3}$$

M1 A1

$$\text{Product of new roots} = \frac{1}{(\alpha\beta\gamma)^2} = \frac{1}{9}$$

B1

$$y^3 - \frac{28}{9}y^2 + \frac{4}{3}y - \frac{1}{9} = 0, \text{ so } 9y^3 - 28y^2 + 12y - 1 = 0$$

M1 A1

Answer: $9y^3 - 28y^2 + 12y - 1 = 0$

8 marks

23 (8 marks)

Let the roots be $a - 3d$, $a - d$, $a + d$, $a + 3d$

B1

Sum of roots: $4a = 8$, so $a = 2$

M1 A1

$$\Sigma\alpha^2 = 4a^2 + 20d^2, \text{ so } \Sigma\alpha\beta = \frac{(\Sigma\alpha)^2 - \Sigma\alpha^2}{2} = 6a^2 - 10d^2$$

M1 A1

$$24 - 10d^2 = 14, \text{ so } d^2 = 1$$

M1 A1

Roots -1 , 1 , 3 , 5

A1

Answer: -1, 1, 3, 5

8 marks

24 (10 marks)

(a) $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

M1

$$4 = 14 + 2(xy + yz + zx), \text{ so } xy + yz + zx = -5$$

A1

Answer: $xy + yz + zx = -5$

2 marks

(b) $20 - 3xyz = 2(14 + 5)$

M1 A1

$$xyz = -6$$

A1

Answer: $xyz = -6$

3 marks

(c) x , y and z are the roots of $t^3 - 2t^2 - 5t + 6 = 0$

M1 A1

$$(t - 1)(t - 3)(t + 2) = 0$$

M1 A1

x , y , $z = 1, 3, -2$ in any order

A1

Answer: 1, 3 and -2 in any order

5 marks

25 (11 marks)

(a) $\alpha\beta\gamma\delta = -3$, so $\gamma\delta = \frac{-3}{-1} = 3$

M1 A1

Answer: 3

2 marks

(b) $s + t = 3$

B1

$$\Sigma\alpha\beta = \alpha\beta + \gamma\delta + st = 4, \text{ so } -1 + 3 + st = 4 \text{ and } st = 2$$

M1 A1

$$\Sigma\alpha\beta\gamma = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 5, \text{ so } -t + 3s = 5$$

M1

$$\text{Then } -t + 3(3 - t) = 5, \text{ so } t = 1 \text{ and } s = 2$$

M1 A1

Answer: $s = 2$, $t = 1$

6 marks

(c) $(x^2 - 2x - 1)(x^2 - x + 3) = 0$

M1

$$x = 1 \pm \sqrt{2} \text{ or } x = \frac{1 \pm i\sqrt{11}}{2}$$

A1 A1

Answer: $1 \pm \sqrt{2}$ and $\frac{1 \pm i\sqrt{11}}{2}$

3 marks

26 (5 marks)

(a) The identity is $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$; the student has added instead of subtracting

E1 E1

Answer: Wrong sign in the identity

2 marks

(b) $\Sigma\alpha^2 = \frac{9}{4} - 4 = -\frac{7}{4}$

M1 A1

This is negative, so the roots cannot all be real

E1

Answer: $-\frac{7}{4}$; not all roots are real

3 marks

27 (11 marks)

(a) The x^2 coefficient is 0, so $\Sigma\alpha = 0$

B1

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = 0 - 2p$$

M1 A1

Answer: $\alpha + \beta + \gamma = 0$ and $\alpha^2 + \beta^2 + \gamma^2 = -2p$

3 marks

(b) $\alpha^3 = -p\alpha - q$, and similarly for β, γ

M1

$$\Sigma\alpha^3 = -p\Sigma\alpha - 3q = -3q$$

M1 A1

$$\alpha\beta\gamma = -q, \text{ so } \Sigma\alpha^3 = 3\alpha\beta\gamma$$

A1

Answer: $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$

4 marks

(c) $\alpha^5 = -p\alpha^3 - q\alpha^2$

M1

$$\Sigma\alpha^5 = -p\Sigma\alpha^3 - q\Sigma\alpha^2$$

M1

$$= -p(-3q) - q(-2p) = 5pq$$

M1 A1

Answer: $\alpha^5 + \beta^5 + \gamma^5 = 5pq$

4 marks

28 (12 marks)

(a) $x^3 + px + q = (x - \alpha)(x - \beta)(x - \gamma)$; differentiate: $3x^2 + p = (x - \beta)(x - \gamma) + (x - \alpha)(x - \gamma) + (x - \alpha)(x - \beta)$

M1 A1

Put $x = \alpha$: $3\alpha^2 + p = (\alpha - \beta)(\alpha - \gamma)$

A1

Answer: $3\alpha^2 + p = (\alpha - \beta)(\alpha - \gamma)$

3 marks

(b) Multiply the three results: $\prod(3\alpha^2 + p) = -[(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)]^2$

M1 A1

$$\prod(3\alpha^2 + p) = 27(\alpha\beta\gamma)^2 + 9p\Sigma\alpha^2\beta^2 + 3p^2\Sigma\alpha^2 + p^3$$

M1

$$\Sigma\alpha^2\beta^2 = p^2 \text{ (as } \Sigma\alpha = 0), \Sigma\alpha^2 = -2p, (\alpha\beta\gamma)^2 = q^2$$

M1

$$= 27q^2 + 9p^3 - 6p^3 + p^3 = 4p^3 + 27q^2$$

A1

$$\text{So the required product is } -4p^3 - 27q^2$$

Answer: $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = -4p^3 - 27q^2$

5 marks

(c) Distinct real roots need $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 > 0$

M1

$$p = -3, q = k: -4(-27) - 27k^2 > 0$$

M1

$$108 - 27k^2 > 0, \text{ so } k^2 < 4$$

A1

$$-2 < k < 2$$

A1

Answer: $-2 < k < 2$

4 marks

29 (12 marks)

(a) $\alpha^3 = 2\alpha^2 + \alpha - 3$, so $\alpha^{n+3} = 2\alpha^{n+2} + \alpha^{n+1} - 3\alpha^n$

M1 A1

Add the corresponding equations for β and γ

A1

Answer: $S_{n+3} = 2S_{n+2} + S_{n+1} - 3S_n$

3 marks

(b) $S_1 = 2, S_2 = 4 + 2 = 6$

B1

$$S_3 = 2(6) + 2 - 9 = 5$$

M1 A1

$$S_4 = 2(5) + 6 - 3(2) = 10, S_5 = 2(10) + 5 - 18 = 7$$

M1 A1

$$S_6 = 2(7) + 10 - 3(5) = 9$$

M1 A1

Answer: $S_1 = 2, S_2 = 6, S_3 = 5, S_6 = 9$

7 marks

(c) S_1, S_2, S_3 are integers and the recurrence has integer coefficients

E1

so by induction every S_n is an integer

E1

Answer: Integer recurrence

2 marks

30 (10 marks)

(a) Sum of three squares ≥ 0 , so $2\Sigma\alpha^2 - 2\Sigma\alpha\beta \geq 0$

M1 A1

$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$, so $(\Sigma\alpha)^2 - 3\Sigma\alpha\beta \geq 0$

M1 A1

$\Sigma\alpha = 6$, $\Sigma\alpha\beta = k$: $36 \geq 3k$

M1

So $k \leq 12$

A1

Answer: $(\alpha + \beta + \gamma)^2 \geq 3(\alpha\beta + \beta\gamma + \gamma\alpha)$ for real α, β, γ

6 marks

(b) Equality holds, so each square is zero and $\alpha = \beta = \gamma$

M1 A1

Sum 6 gives each root 2, and $2^3 = 8$ agrees with $\alpha\beta\gamma$

M1 A1

Answer: 2, 2, 2

4 marks

Transformations of Polynomial Roots**1 (3 marks)**

$\alpha + \beta = 5$ and $\alpha\beta = 2$

B1

Sum of new roots $= 2(\alpha + \beta) = 10$, product $= 4\alpha\beta = 8$

M1 A1

Answer: $y^2 - 10y + 8 = 0$

3 marks

2 (3 marks)

Let $y = x + 1$, so $x = y - 1$

B1

$(y - 1)^3 - 2(y - 1)^2 + 3(y - 1) - 4 = 0$

M1

$y^3 - 5y^2 + 10y - 10 = 0$

A1

Answer: $y^3 - 5y^2 + 10y - 10 = 0$

3 marks

3 (3 marks)

Let $y = \frac{1}{x}$, so $x = \frac{1}{y}$

B1

$\frac{2}{y^2} - \frac{3}{y} + 7 = 0$, so $7y^2 - 3y + 2 = 0$

M1 A1

Answer: $7y^2 - 3y + 2 = 0$

3 marks

4 (3 marks)

Let $y = 3x$, so $x = \frac{y}{3}$

B1

$\frac{y^3}{27} + 2\frac{y^2}{9} - \frac{y}{3} + 6 = 0$

M1

$y^3 + 6y^2 - 9y + 162 = 0$

A1

Answer: $y^3 + 6y^2 - 9y + 162 = 0$

3 marks

5 (4 marks)

$\alpha + \beta = 4$, $\alpha\beta = 1$

B1

$\alpha^2 + \beta^2 = 16 - 2 = 14$, $\alpha^2\beta^2 = 1$

M1 A1

$y^2 - 14y + 1 = 0$

A1

Answer: $y^2 - 14y + 1 = 0$

4 marks

6 (3 marks)

Let $y = x - 2$, so $x = y + 2$

B1

$(y + 2)^4 - 3(y + 2)^2 + (y + 2) + 2 = 0$

M1

$y^4 + 8y^3 + 21y^2 + 21y + 8 = 0$

A1

Answer: $y^4 + 8y^3 + 21y^2 + 21y + 8 = 0$

3 marks

7 (6 marks)

$$\alpha + \beta = 6, \alpha\beta = 7$$

B1

$$\text{Sum of new roots} = 4(\alpha + \beta) = 24$$

M1 A1

$$\text{Product} = 3\alpha^2 + 10\alpha\beta + 3\beta^2 = 3(\alpha^2 + \beta^2) + 10\alpha\beta$$

M1

$$= 3(36 - 14) + 70 = 136$$

A1

$$y^2 - 24y + 136 = 0$$

A1

$$\text{Answer: } y^2 - 24y + 136 = 0$$

6 marks

8 (3 marks)

$$\text{Let } y = \frac{1}{x}, \text{ so } x = \frac{1}{y}$$

B1

$$\frac{2}{y^4} - \frac{3}{y^3} + \frac{5}{y} + 1 = 0$$

M1

$$\text{Multiply by } y^4: y^4 + 5y^3 - 3y + 2 = 0$$

A1

$$\text{Answer: } y^4 + 5y^3 - 3y + 2 = 0$$

3 marks

9 (5 marks)

$$\alpha + \beta = \frac{5}{2}, \alpha\beta = \frac{1}{2}$$

B1

$$\text{Sum of new roots} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{\frac{25}{4} - 1}{\frac{1}{2}} = \frac{21}{2}$$

M1 A1

$$\text{Product} = 1$$

B1

$$2y^2 - 21y + 2 = 0$$

A1

$$\text{Answer: } 2y^2 - 21y + 2 = 0$$

5 marks

10 (4 marks)

$$\alpha + \beta = 4 - \gamma, \text{ and so on, so each new root is } y = 4 - x$$

M1 A1

$$\text{Substitute } x = 4 - y: (4 - y)^3 - 4(4 - y)^2 + (4 - y) + 3 = 0$$

M1

$$y^3 - 8y^2 + 17y - 7 = 0$$

A1

$$\text{Answer: } y^3 - 8y^2 + 17y - 7 = 0$$

4 marks

11 (4 marks)

$$\text{Let } y = 2x + 1, \text{ so } x = \frac{y - 1}{2}$$

B1

$$\frac{(y - 1)^3}{8} - 2(y - 1) + 5 = 0$$

M1

$$(y - 1)^3 - 16(y - 1) + 40 = 0$$

M1

$$y^3 - 3y^2 - 13y + 55 = 0$$

A1

$$\text{Answer: } y^3 - 3y^2 - 13y + 55 = 0$$

4 marks

12 (6 marks)

$$\alpha + \beta = 4, \alpha\beta = 2$$

B1

$$\text{Sum} = (\alpha + \beta) + \frac{\alpha + \beta}{\alpha\beta} = 4 + 2 = 6$$

M1 A1

$$\text{Product} = \alpha\beta + 1 + 1 + \frac{1}{\alpha\beta} = \frac{9}{2}$$

M1 A1

$$2y^2 - 12y + 9 = 0$$

A1

$$\text{Answer: } 2y^2 - 12y + 9 = 0$$

6 marks

13 (6 marks)

$$\alpha + \beta = -3k, \alpha\beta = 2k$$

B1

$$\text{Sum of new roots} = (\alpha + \beta)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{9k^2}{2k} = \frac{9k}{2}$$

M1 A1

$$\text{Product} = \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{9k}{2}$$

M1 A1

$$y^2 - \frac{9k}{2}y + \frac{9k}{2} = 0, \text{ so } 2y^2 - 9ky + 9k = 0$$

A1

$$\text{Answer: } 2y^2 - 9ky + 9k = 0$$

6 marks

14 (6 marks)

$$(a) \text{ Let } y = \frac{1}{x}: \frac{1}{y^4} - \frac{2}{y^3} + \frac{3}{y} - 1 = 0$$

M1 A1

$$1 - 2y + 3y^3 - y^4 = 0, \text{ so } y^4 - 3y^3 + 2y - 1 = 0$$

A1

$$\text{Answer: } y^4 - 3y^3 + 2y - 1 = 0$$

3 marks

$$(b) \text{ For the new equation } \Sigma y = 3 \text{ and } \Sigma yy' = 0$$

B1

$$\Sigma y^2 = 3^2 - 2(0) = 9$$

M1 A1

$$\text{Answer: } 9$$

3 marks

15 (5 marks)

$$\alpha\beta\gamma = -3, \text{ so } \beta\gamma = \frac{-3}{\alpha} \text{ etc.; let } y = -\frac{3}{x}, \text{ so } x = -\frac{3}{y}$$

M1 A1

$$-\frac{27}{y^3} - \frac{36}{y^2} - \frac{3}{y} + 3 = 0$$

M1

$$\text{Multiply by } -y^3/3: y^3 - y^2 - 12y - 9 = 0$$

A1 A1

$$\text{Answer: } y^3 - y^2 - 12y - 9 = 0$$

5 marks

16 (6 marks)

$$\alpha\beta\gamma = -3, \text{ so } \frac{\alpha}{\beta\gamma} = \frac{\alpha^2}{\alpha\beta\gamma} = -\frac{\alpha^2}{3}$$

M1 A1

$$\text{Let } y = -\frac{x^2}{3}, \text{ so } x^2 = -3y$$

M1

$$\text{Write the equation as } x(x^2 - 1) = 2x^2 - 3 \text{ and square: } x^2(x^2 - 1)^2 = (2x^2 - 3)^2$$

M1

$$-3y(-3y - 1)^2 = (-6y - 3)^2$$

M1

$$9y^3 + 18y^2 + 13y + 3 = 0$$

A1

$$\text{Answer: } 9y^3 + 18y^2 + 13y + 3 = 0$$

6 marks

17 (6 marks)

$$\Sigma\alpha = 1, \Sigma\alpha\beta = -1, \alpha\beta\gamma = 1$$

B1

$$\Sigma\alpha^2 = 1 + 2 = 3, \Sigma\alpha^3 = (1)(3) - (-1)(1) + 3(1) = 7$$

M1 A1

$$\Sigma\alpha^3\beta^3 = (\Sigma\alpha\beta)^3 - 3(\Sigma\alpha)(\Sigma\alpha\beta)(\alpha\beta\gamma) + 3(\alpha\beta\gamma)^2 = -1 + 3 + 3 = 5$$

M1 A1

$$\alpha^3\beta^3\gamma^3 = 1, \text{ so } y^3 - 7y^2 + 5y - 1 = 0$$

A1

$$\text{Answer: } y^3 - 7y^2 + 5y - 1 = 0$$

6 marks

18 (6 marks)

$$y = 2x - 1, \text{ so substitute } y = 2x - 1$$

M1

$$(2x - 1)^3 - 3(2x - 1)^2 + 7(2x - 1) + 9 = 0$$

M1 A1

$$8x^3 - 24x^2 + 32x - 2 = 0, \text{ so } 4x^3 - 12x^2 + 16x - 1 = 0$$

A1

$$\Sigma\alpha = 3, \Sigma\alpha\beta = 4, \text{ so } \Sigma\alpha^2 = 9 - 8 = 1$$

M1 A1

$$\text{Answer: } 4x^3 - 12x^2 + 16x - 1 = 0; \Sigma\alpha^2 = 1$$

6 marks

19 (7 marks)

(a) Let $y = \frac{1}{(x-1)}$, so $x = 1 + \frac{1}{y}$

M1

$$(1 + \frac{1}{y})^3 - (1 + \frac{1}{y})^2 - 1 = 0$$

M1

Multiply by y^3 : $(y+1)^3 - y(y+1)^2 - y^3 = 0$, giving $y^3 - y^2 - 2y - 1 = 0$

A1 A1

Answer: $y^3 - y^2 - 2y - 1 = 0$

4 marks

(b) $\Sigma y = 1$, $\Sigma yy' = -2$

B1

$$\Sigma y^2 = 1 + 4 = 5$$

M1 A1

Answer: 5

3 marks

20 (6 marks)

Let $y = x^2 - x$. From the equation $x^3 - 2x^2 + 3x - 1 = 0$: $x(x^2 - x) - (x^2 - x) + 2x - 1 = 0$,
i.e. $(x-1)y + 2x - 1 = 0$

M1 A1

So $x = \frac{y+1}{y+2}$

M1

Substitute into $y = x^2 - x$: $y(y+2)^2 = (y+1)^2 - (y+1)(y+2)$

M1

$$y(y+2)^2 = -(y+1)$$

M1

$$y^3 + 4y^2 + 5y + 1 = 0$$

A1

Answer: $y^3 + 4y^2 + 5y + 1 = 0$

6 marks

21 (8 marks)

$\Sigma \alpha = 3$, so $\beta + \gamma = 3 - \alpha$, and $\frac{\beta + \gamma}{\alpha} = \frac{3 - \alpha}{\alpha} = \frac{3}{\alpha} - 1$

M1 A1

Let $y = \frac{3}{x} - 1$, so $x = \frac{3}{y+1}$

M1

$$\frac{27}{(y+1)^3} - \frac{27}{(y+1)^2} + \frac{6}{y+1} + 4 = 0$$

M1 A1

Multiply by $(y+1)^3$: $27 - 27(y+1) + 6(y+1)^2 + 4(y+1)^3 = 0$

M1

$$4y^3 + 18y^2 - 3y + 10 = 0$$

A1 A1

Answer: $4y^3 + 18y^2 - 3y + 10 = 0$

8 marks

22 (10 marks)

(a) Let $y = \frac{x+1}{x-1}$, so $x = \frac{y+1}{y-1}$

M1 A1

$$(\frac{y+1}{y-1})^3 - 3(\frac{y+1}{y-1})^2 + 3(\frac{y+1}{y-1}) + 3 = 0$$

M1

Multiply by $(y-1)^3$: $(y+1)^3 - 3(y+1)^2(y-1) + 3(y+1)(y-1)^2 + 3(y-1)^3 = 0$

M1

Expanding: $4y^3 - 12y^2 + 12y + 4 = 0$, so $y^3 - 3y^2 + 3y + 1 = 0$

M1 A1

Answer: the equation with roots $\frac{\alpha+1}{\alpha-1}$, $\frac{\beta+1}{\beta-1}$ and $\frac{\gamma+1}{\gamma-1}$ is $y^3 - 3y^2 + 3y + 1 = 0$

6 marks

(b) $y^3 - 3y^2 + 3y - 1 = (y-1)^3$, so the equation is $(y-1)^3 + 2 = 0$

M1 A1

Each root satisfies $(y-1)^3 = -2$, so the sum over three roots is -6

M1 A1

Answer: -6

4 marks

23 (9 marks)

(a) $\Sigma \alpha = 4$, $\Sigma \alpha \beta = 5$, $\alpha \beta \gamma = 1$

B1

$$\Sigma \alpha^2 = 16 - 10 = 6$$
, $\Sigma \alpha^2 \beta^2 = 25 - 8 = 17$

M1 A1

$$\Sigma(\alpha^2 + 1) = 9$$
; $\Sigma(\alpha^2 + 1)(\beta^2 + 1) = 17 + 2(6) + 3 = 32$

M1 A1

$$\Pi(\alpha^2 + 1) = (\alpha \beta \gamma)^2 + \Sigma \alpha^2 \beta^2 + \Sigma \alpha^2 + 1 = 25$$

M1

$$y^3 - 9y^2 + 32y - 25 = 0$$

A1

Answer: $y^3 - 9y^2 + 32y - 25 = 0$

7 marks

(b) The product of the roots of the new equation is 25

M1 A1

Answer: $(\alpha^2 + 1)(\beta^2 + 1)(\gamma^2 + 1) = 25$

2 marks

24 (5 marks)

(a) Replacing x by $\frac{1}{y}$ gives $\frac{2}{y^3} - \frac{5}{y^2} + \frac{4}{y} - 7 = 0$; the student replaced x by y instead

E1 E1

Answer: Substituted y for x instead of $\frac{1}{y}$

2 marks

(b) Multiply by y^3 : $2 - 5y + 4y^2 - 7y^3 = 0$, so $7y^3 - 4y^2 + 5y - 2 = 0$

M1 A1

Sum of roots = $\frac{4}{7}$

A1

Answer: $7y^3 - 4y^2 + 5y - 2 = 0$; $\frac{4}{7}$

3 marks

25 (9 marks)

$\Sigma\alpha = 2$, $\Sigma\alpha\beta = -1$, $\alpha\beta\gamma = -3$

B1

$\Sigma\alpha^2 = 4 + 2 = 6$, $\Sigma\alpha^2\beta^2 = (-1)^2 - 2(-3)(2) = 13$

M1 A1

$\beta^2 + \gamma^2 = 6 - \alpha^2$, so each new root is $y = 6 - x^2$

M1 A1

The roots α^2 of the cubic in w have $\Sigma w = 6$, $\Sigma ww' = 13$, $\Pi w = 9$

M1

Equation with roots $6 - w$: sum 12, pairs $3(36) - 12(6) + 13 = 49$, product $216 - 216 + 78 - 9 = 69$

M1 A1

$y^3 - 12y^2 + 49y - 69 = 0$

A1

Answer: $y^3 - 12y^2 + 49y - 69 = 0$

9 marks

26 (9 marks)

$\Sigma\alpha = 3$, $\Sigma\alpha\beta = 2$, $\alpha\beta\gamma = -4$, and $\Sigma\frac{1}{\alpha} = \frac{\Sigma\alpha\beta}{\alpha\beta\gamma} = -\frac{1}{2}$

B1 M1

Sum of new roots = $3 - \frac{1}{2} = \frac{5}{2}$

A1

$\Sigma(\alpha + \frac{1}{\alpha})(\beta + \frac{1}{\beta}) = \Sigma\alpha\beta + \Sigma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}) + \Sigma\frac{1}{\alpha\beta}$, where $\Sigma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}) = (\Sigma\alpha)(\Sigma\frac{1}{\alpha}) - 3 = -\frac{9}{2}$ and

M1 A1

$\Sigma\frac{1}{\alpha\beta} = \frac{\Sigma\alpha}{\alpha\beta\gamma} = -\frac{3}{4}$

Sum of pairs = $2 - \frac{9}{2} - \frac{3}{4} = -\frac{13}{4}$

A1

$\Pi(\alpha + \frac{1}{\alpha}) = \frac{\Pi(\alpha^2 + 1)}{\alpha\beta\gamma} = \frac{(\alpha\beta\gamma)^2 + \Sigma\alpha^2\beta^2 + \Sigma\alpha^2 + 1}{\alpha\beta\gamma} = \frac{16 + 28 + 5 + 1}{-4} = -\frac{25}{2}$

M1 A1

$y^3 - \frac{5}{2}y^2 - \frac{13}{4}y + \frac{25}{2} = 0$, so $4y^3 - 10y^2 - 13y + 50 = 0$

A1

Answer: $4y^3 - 10y^2 - 13y + 50 = 0$

9 marks

27 (11 marks)

(a) $\Sigma\alpha = -p$, $\Sigma\alpha\beta = q$, $\alpha\beta\gamma = -r$

B1

$\Sigma\alpha^2 = p^2 - 2q$

M1 A1

$\Sigma\alpha^2\beta^2 = q^2 - 2(\alpha\beta\gamma)(\Sigma\alpha) = q^2 - 2(-r)(-p) = q^2 - 2pr$

M1 A1

$\alpha^2\beta^2\gamma^2 = r^2$, and the equation follows

A1

Answer: the equation with roots α^2 , β^2 and γ^2 is $y^3 - (p^2 - 2q)y^2 + (q^2 - 2pr)y - r^2 = 0$

6 marks

(b) $p^2 - 2q = 36 - 22 = 14$, $q^2 - 2pr = 121 - 72 = 49$, $r^2 = 36$

M1 A1

$y^3 - 14y^2 + 49y - 36 = 0$

A1

Answer: $y^3 - 14y^2 + 49y - 36 = 0$

3 marks

(c) α^2 , β^2 , γ^2 are non-negative, so their sum and the sum of their pairwise products are non-negative

M1

$p^2 - 2q \geq 0$ and $q^2 - 2pr \geq 0$

A1

Answer: $p^2 \geq 2q$ and $q^2 \geq 2pr$

2 marks

28 (11 marks)

(a) Let the roots be $\frac{m}{k}$, m , mk ; the product is $m^3 = -r$

M1 A1

$$\Sigma\alpha\beta = m^2\left(\frac{1}{k} + 1 + k\right) = m\left(m\left(\frac{1}{k} + 1 + k\right)\right) = m(-p)$$

M1 A1

So $q = -pm$, hence $q^3 = -p^3m^3 = p^3r$

M1 A1

Answer: $q^3 = p^3r$

6 marks

(b) $p = -14$ and $r = -64$: $q^3 = (-14)^3(-64) = 175\,616 = 56^3$

M1 A1

$q = 56$

A1

$m^3 = 64$, so $m = 4$; sum of roots $4\left(\frac{1}{k} + 1 + k\right) = 14$, so $2k^2 - 5k + 2 = 0$

M1

Roots 2, 4, 8

A1

Answer: $q = 56$; roots 2, 4, 8

5 marks

29 (12 marks)

(a) $(y + 2)^3 - 6(y + 2)^2 + 9(y + 2) + k = 0$

M1

$$y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 9y + 18 + k = 0$$

M1 A1

$$y^3 - 3y + (k + 2) = 0$$

A1

Answer: the equation can be written as $y^3 - 3y + (k + 2) = 0$

4 marks

(b) A repeated root of $f(y) = y^3 - 3y + k + 2$ satisfies $f'(y) = 3y^2 - 3 = 0$, so $y = \pm 1$

M1 A1

$y = 1$: $-2 + k + 2 = 0$, so $k = 0$ and $x = y + 2 = 3$

M1 A1

$y = -1$: $2 + k + 2 = 0$, so $k = -4$ and $x = 1$

A1

Answer: $k = 0$ ($x = 3$ repeated), $k = -4$ ($x = 1$ repeated)

5 marks

(c) The graph of f has a local maximum at $y = -1$ with value $k + 4$ and a local minimum at $y = 1$ with value k

M1

Three distinct real roots need $k + 4 > 0$ and $k < 0$, so $-4 < k < 0$

A1 A1

Answer: $-4 < k < 0$

3 marks

30 (11 marks)

(a) $\Sigma\alpha = 0$, $\Sigma\alpha\beta = -1$, $\alpha\beta\gamma = 1$

B1

$$\Sigma\alpha^2 = 0 + 2 = 2, \Sigma\alpha^2\beta^2 = 1 - 0 = 1, \alpha^2\beta^2\gamma^2 = 1$$

M1 A1

$$y^3 - 2y^2 + y - 1 = 0$$

A1

Answer: $y^3 - 2y^2 + y - 1 = 0$

4 marks

(b) $\Sigma\alpha^4 = \Sigma(\alpha^2)^2 = (\Sigma\alpha^2)^2 - 2\Sigma\alpha^2\beta^2$

M1

$$= 4 - 2 = 2$$

A1

Answer: 2

2 marks

(c) $\alpha^3 = \alpha + 1$, so $\alpha^{n+3} = \alpha^{n+1} + \alpha^n$; add the three similar results

M1 A1

$$S_3 = S_1 + S_0 = 0 + 3 = 3, S_4 = S_2 + S_1 = 2$$

M1

$$S_5 = S_3 + S_2 = 3 + 2 = 5$$

M1 A1

Answer: $S_5 = 5$

5 marks

Polynomial Equations**1 (3 marks)**

$1 - 2 - 5 + 6 = 0$, so $(x - 1)$ is a factor

B1

$$x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6) = (x - 1)(x - 3)(x + 2)$$

M1 A1

Answer: $x = 1, 3, -2$

3 marks

2 (2 marks)

$$x = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm 6i}{2}$$

M1

$$x = 2 \pm 3i$$

A1

Answer: $2 + 3i$ and $2 - 3i$

2 marks

3 (3 marks)

$f(1) = 0$, so $(x - 1)$ is a factor

B1

$$x^3 - 7x + 6 = (x - 1)(x^2 + x - 6) = (x - 1)(x - 2)(x + 3)$$

M1

$$x = 1, 2, -3$$

A1

Answer: $x = 1, 2, -3$

3 marks

4 (3 marks)

$(x + 3)$ is a factor: $2x^3 + x^2 - 13x + 6 = (x + 3)(2x^2 - 5x + 2)$

M1 A1

$$(2x - 1)(x - 2) = 0, \text{ so } x = \frac{1}{2} \text{ and } x = 2$$

A1

Answer: $\frac{1}{2}$ and 2

3 marks

5 (3 marks)

The other root is $2 - 3i$

B1

Sum = 4, product = 13, so $x^2 - 4x + 13 = 0$

M1 A1

Answer: $x^2 - 4x + 13 = 0$

3 marks

6 (3 marks)

$$x^4 - 16 = (x^2 - 4)(x^2 + 4)$$

M1

$$x^2 = 4 \text{ or } x^2 = -4$$

A1

$$x = \pm 2 \text{ or } x = \pm 2i$$

A1

Answer: $x = \pm 2, \pm 2i$

3 marks

7 (5 marks)

$$f(1) = 1 - 5 + 17 - 13 = 0$$

B1

$$f(x) = (x - 1)(x^2 - 4x + 13)$$

M1 A1

$$x = \frac{4 \pm \sqrt{16 - 52}}{2}$$

M1

$$x = 1, 2 \pm 3i$$

A1

Answer: $x = 1, 2 + 3i, 2 - 3i$

5 marks

8 (5 marks)

The coefficients are real, so $1 - 2i$ is also a root

B1

Quadratic factor: $x^2 - 2x + 5$

M1 A1

Third root: sum of roots is 5, so the root is $5 - 2 = 3$

M1 A1

Answer: $1 - 2i$ and 3

5 marks

9 (5 marks)

$-2i$ is also a root, so $x^2 + 4$ is a factor

M1 A1

$$x^4 - 2x^3 + 6x^2 - 8x + 8 = (x^2 + 4)(x^2 - 2x + 2)$$

M1 A1

$$x^2 - 2x + 2 = 0 \text{ gives } x = 1 \pm i$$

A1

Answer: $x = \pm 2i, 1 \pm i$

5 marks

10 (5 marks)

$1 - i$ is also a root, giving the factor $x^2 - 2x + 2$

B1

$$x^3 + ax^2 + bx + 10 = (x^2 - 2x + 2)(x + 5), \text{ since } 2 \times 5 = 10$$

M1 A1

Expanding gives $x^3 + 3x^2 - 8x + 10$, so $a = 3$ and $b = -8$

A1

Other roots: $1 - i$ and -5

A1

Answer: $a = 3, b = -8$; roots $1 - i$ and -5

5 marks

11 (5 marks)

$$f'(x) = 3x^2 - 10x + 8 = (3x - 4)(x - 2)$$

M1

$$f(2) = 8 - 20 + 16 - 4 = 0 \text{ and } f'(2) = 0, \text{ so } 2 \text{ is a repeated root}$$

M1 A1

$$f(x) = (x - 2)^2(x - 1)$$

M1

$$x = 2 \text{ (twice) and } x = 1$$

A1

Answer: $x = 1, 2, 2$

5 marks

12 (5 marks)

$$p(1) = 0: 1 + a + b - 6 = 0, \text{ so } a + b = 5$$

M1 A1

$$p(-2) = 0: -8 + 4a - 2b - 6 = 0, \text{ so } 2a - b = 7$$

M1

$$a = 4 \text{ and } b = 1$$

A1

Product of roots is 6, so the third root is -3 and the factor is $(x + 3)$

A1

Answer: $a = 4$, $b = 1$; third factor $(x + 3)$

5 marks

13 (4 marks)

$$\text{Let } u = x^2: u^2 + 3u - 4 = 0$$

M1

$$(u + 4)(u - 1) = 0, \text{ so } x^2 = 1 \text{ or } x^2 = -4$$

M1 A1

$$x = \pm 1 \text{ or } x = \pm 2i$$

A1

Answer: $x = \pm 1, \pm 2i$

4 marks

14 (7 marks)

$$2 + i \text{ is also a root; } (x - 2 + i)(x - 2 - i) = x^2 - 4x + 5$$

M1 A1

$$\text{Product of roots} = -15: 5r = -15, \text{ so the third root is } -3$$

M1 A1

$$\text{Sum of roots} = 2 + i + 2 - i - 3 = 1, \text{ so } a = -1$$

M1

$$\Sigma \alpha\beta = 5 + (-3)(4) = -7, \text{ so } b = -7$$

M1 A1

Answer: $a = -1$, $b = -7$; other roots $2 + i$, -3

7 marks

15 (5 marks)

$$(x^2 + 2)^2 - (2x)^2 = x^4 + 4x^2 + 4 - 4x^2$$

M1 A1

$$= x^4 + 4, \text{ a difference of two squares: } (x^2 + 2 + 2x)(x^2 + 2 - 2x)$$

A1

$$x^2 + 2x + 2 = 0 \text{ gives } x = -1 \pm i; x^2 - 2x + 2 = 0 \text{ gives } x = 1 \pm i$$

M1 A1

Answer: $x = \pm 1 \pm i$

5 marks

16 (6 marks)

$$\text{Divide by } x^2: 2(x^2 + \frac{1}{x^2}) - 9(x + \frac{1}{x}) + 14 = 0$$

M1

$$x^2 + \frac{1}{x^2} = z^2 - 2, \text{ so } 2z^2 - 9z + 10 = 0$$

M1 A1

$$(2z - 5)(z - 2) = 0, \text{ so } z = \frac{5}{2} \text{ or } z = 2$$

A1

$$x + \frac{1}{x} = \frac{5}{2} \text{ gives } x = 2 \text{ or } \frac{1}{2}; x + \frac{1}{x} = 2 \text{ gives } x = 1 \text{ (twice)}$$

M1 A1

Answer: $x = 2, \frac{1}{2}, 1, 1$

6 marks

17 (7 marks)

$$1 - 2i \text{ is also a root, so the product of these two roots is } 5$$

B1

$$\text{Product of all roots} = 30: 5 \times 3 \times r = 30, \text{ so the fourth root is } 2$$

M1 A1

$$\Sigma \alpha = -a: 2 + 3 + 2 = 7, \text{ so } a = -7$$

M1 A1

$$(x^2 - 2x + 5)(x - 3)(x - 2) \text{ expanded gives } c = -37$$

M1 A1

Answer: $a = -7$, $c = -37$; remaining roots $1 - 2i$ and 2

7 marks

18 (5 marks)

$$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$$

M1 A1

$$x = -2$$

A1

$$x = \frac{2 \pm \sqrt{4 - 16}}{2}$$

M1

$$x = 1 \pm i\sqrt{3}$$

A1

Answer: $x = -2, 1 + i\sqrt{3}, 1 - i\sqrt{3}$

5 marks

19 (6 marks)

$$f'(x) = 3x^2 - 6x = 0, \text{ so the repeated root is } x = 0 \text{ or } x = 2$$

M1 A1

$$x = 0: k = 0, \text{ and } x^3 - 3x^2 = x^2(x - 3), \text{ roots } 0, 0, 3$$

M1 A1

$$x = 2: 8 - 12 + k = 0, \text{ so } k = 4$$

M1

$$x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2, \text{ roots } -1, 2, 2$$

A1

Answer: k = 0: roots 0, 0, 3; k = 4: roots -1, 2, 2

6 marks

20 (6 marks)

$$x^4 - 4x^3 + 4x^2 = (x^2 - 2x)^2, \text{ so the equation is } u^2 + 3x^2 - 6x + 2 = 0$$

M1 A1

$$u^2 + 3u + 2 = 0$$

A1

$$(u + 1)(u + 2) = 0$$

M1

$$x^2 - 2x + 1 = 0 \text{ gives } x = 1 \text{ (twice); } x^2 - 2x + 2 = 0 \text{ gives } x = 1 \pm i$$

A1 A1

Answer: x = 1, 1, 1 ± i

6 marks

21 (9 marks)

$$z = 3: 27 - 54 + 3(12 - i) + c = 0, \text{ so } c = -9 + 3i$$

M1 A1

$$z^3 - 6z^2 + (12 - i)z - 9 + 3i = (z - 3)(z^2 - 3z + 3 - i)$$

M1 A1

$$z = \frac{3 \pm \sqrt{9 - 4(3 - i)}}{2} = \frac{3 \pm \sqrt{-3 + 4i}}{2}$$

M1

$$(1 + 2i)^2 = -3 + 4i, \text{ so } \sqrt{-3 + 4i} = \pm(1 + 2i)$$

M1 A1

$$z = 2 + i \text{ or } z = 1 - i$$

A1

The coefficients are not all real, so the conjugate root theorem does not apply

E1

Answer: c = -9 + 3i; z = 2 + i and z = 1 - i; the coefficients are not all real

9 marks

22 (7 marks)

$$\text{Divide by } x^2: 3(x^2 + \frac{1}{x^2}) - 10(x + \frac{1}{x}) + 14 = 0$$

M1

$$\text{With } z = x + \frac{1}{x}: 3(z^2 - 2) - 10z + 14 = 0, \text{ so } 3z^2 - 10z + 8 = 0$$

M1 A1

$$(3z - 4)(z - 2) = 0, \text{ so } z = \frac{4}{3} \text{ or } z = 2$$

A1

$$z = 2: x^2 - 2x + 1 = 0, \text{ so } x = 1 \text{ (twice)}$$

A1

$$z = \frac{4}{3}: 3x^2 - 4x + 3 = 0, \text{ so } x = \frac{4 \pm \sqrt{16 - 36}}{6} = \frac{2 \pm i\sqrt{5}}{3}$$

M1 A1

Answer: x = 1, 1, $\frac{2 \pm i\sqrt{5}}{3}$

7 marks

23 (8 marks)

The other factor is monic quadratic $x^2 + px + q$ with $13q = 39$, so $q = 3$

M1 A1

x^3 coefficient: $4 + p = 4$, so $p = 0$

M1 A1

$$(x^2 + 4x + 13)(x^2 + 3) = x^4 + 4x^3 + 16x^2 + 12x + 39, \text{ so } a = 16, b = 12$$

M1 A1

$$x = -2 \pm 3i \text{ (from the first factor) and } x = \pm i\sqrt{3}$$

A1 A1

Answer: a = 16, b = 12; x = -2 ± 3i, ±i√3

8 marks

24 (8 marks)

$$(a) f\left(\frac{1}{\alpha}\right) = \frac{2}{\alpha^3} - \frac{7}{\alpha^2} + \frac{7}{\alpha} - 2 = \frac{2 - 7\alpha + 7\alpha^2 - 2\alpha^3}{\alpha^3}$$

M1 A1

$$= -\frac{f(\alpha)}{\alpha^3} = 0$$

M1 A1

Answer: if α is a root of $f(x) = 0$, then so is $\frac{1}{\alpha}$

4 marks

(b) $f(1) = 0$, so $(x - 1)$ is a factor

B1

$$f(x) = (x - 1)(2x^2 - 5x + 2) = (x - 1)(2x - 1)(x - 2)$$

M1 A1

$$x = 1, \frac{1}{2}, 2$$

A1

Answer: $x = 1, \frac{1}{2}, 2$

4 marks

25 (5 marks)

(a) The equation has degree 2 and is meant to be a cubic

E1

A real cubic with root $1 + i$ must also have root $1 - i$, and the coefficients should be real

E1

Answer: Not a cubic; coefficients not real; missing $1 - i$

2 marks

(b) $(x - 2)(x - 1 - i)(x - 1 + i)$

M1

$$(x - 2)(x^2 - 2x + 2) = x^3 - 4x^2 + 6x - 4$$

M1 A1

Answer: $x^3 - 4x^2 + 6x - 4 = 0$

3 marks

26 (7 marks)

$$x^6 - 1 = (x^3 - 1)(x^3 + 1)$$

M1

$$= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$$

M1 A1

$$x^2 + x + 1 = 0: x = \frac{-1 \pm i\sqrt{3}}{2}$$

M1 A1

$$x^2 - x + 1 = 0: x = \frac{1 \pm i\sqrt{3}}{2}; \text{ with } x = \pm 1 \text{ these are all six roots}$$

A1 A1

Answer: $x = \pm 1, \frac{-1 \pm i\sqrt{3}}{2}, \frac{1 \pm i\sqrt{3}}{2}$

7 marks

27 (11 marks)

(a) Substitute $x = ki$: $-k^3i - pk^2 + qki + r = 0$

M1 A1

Real part: $r = pk^2$; imaginary part: $k(q - k^2) = 0$, so $q = k^2$

M1 A1

Hence $r = pq$

A1

Answer: $r = pq$

5 marks

(b) $r = 18$, and $k^2 = 9$ so the imaginary roots are $\pm 3i$

M1 A1

Sum of roots = -2 , so the third root is -2

M1 A1

Answer: $x = \pm 3i, -2$

4 marks

(c) $(x^2 + 9)(x + 2) = x^3 + 2x^2 + 9x + 18$

M1 A1

Answer: $(x + 2)(x^2 + 9)$

2 marks

28 (11 marks)

(a) $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$

M1

$$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta \cos\theta$$

M1 A1

$$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta = 4\cos^3\theta - 3\cos\theta$$

A1

Answer: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

4 marks

(b) $x^3 - 3x = 8\cos^3\theta - 6\cos\theta = 2(4\cos^3\theta - 3\cos\theta) = 2\cos 3\theta$

M1 A1

Answer: $x^3 - 3x = 2\cos 3\theta$

2 marks

(c) With $x = 2\cos\theta$: $2\cos 3\theta = 1$, so $\cos 3\theta = \frac{1}{2}$

M1 A1

$$3\theta = \pm \frac{\pi}{3} + 2k\pi, \text{ so } \theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9} \text{ (distinct values of } 2\cos\theta)$$

M1 A1

$$x = 2\cos\left(\frac{\pi}{9}\right), 2\cos\left(\frac{5\pi}{9}\right), 2\cos\left(\frac{7\pi}{9}\right)$$

A1

Answer: $x = 2\cos\left(\frac{\pi}{9}\right), 2\cos\left(\frac{5\pi}{9}\right), 2\cos\left(\frac{7\pi}{9}\right)$

5 marks

29 (11 marks)

(a) $(x - 1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$

M1 A1

So the equation is $(x - 1)^4 + 16 = 0$

A1

Answer: the equation can be written as $(x - 1)^4 = -16$

3 marks

(b) $(y^2 + 8)^2 - 16y^2 = y^4 + 16$, a difference of two squares

M1 A1

$y^2 + 4y + 8 = 0$ gives $y = -2 \pm 2i$; $y^2 - 4y + 8 = 0$ gives $y = 2 \pm 2i$

M1 A1

$x = y + 1$: $x = -1 \pm 2i$ or $x = 3 \pm 2i$

A1

Answer: $x = -1 \pm 2i, 3 \pm 2i$

5 marks

(c) The points are $(-1, \pm 2)$ and $(3, \pm 2)$

M1

Horizontal sides of length 4 and vertical sides of length 4, with right angles

A1

Area = 16

A1

Answer: Square of area 16

3 marks

30 (10 marks)

(a) The remainder has the form $ax^2 + bx + c$

B1

$p(1) = 7$, so $a + b + c = 7$

M1 A1

$ax^2 + bx + c = a(x^2 + 1) + bx + (c - a)$, so $b = 2$ and $c - a = 3$

M1 A1

Then $2a + 5 = 7$, so $a = 1$ and $c = 4$

M1

Remainder $x^2 + 2x + 4$

A1

Answer: $x^2 + 2x + 4$

7 marks

(b) $x^{100} = (x^2)^{50}$ and $x^2 \equiv -1$ modulo $x^2 + 1$

M1 A1

$(-1)^{50} = 1$, so the remainder is 1

A1

Answer: 1

3 marks

Rational Functions

1 (2 marks)

Vertical asymptote where the denominator is zero: $x = 3$

B1

As $x \rightarrow \infty$, $y \rightarrow 2$, so the horizontal asymptote is $y = 2$

B1

Answer: $x = 3$ and $y = 2$

2 marks

2 (2 marks)

$y = 0$: $x = -4$, so $(-4, 0)$

B1

$x = 0$: $y = \frac{4}{-2} = -2$, so $(0, -2)$

B1

Answer: $(-4, 0)$ and $(0, -2)$

2 marks

3 (3 marks)

$3x - 5 = 3(x + 1) - 8$, so $\frac{3x - 5}{x + 1} = 3 - \frac{8}{x + 1}$

M1 A1

Asymptotes $x = -1$ and $y = 3$

B1

Answer: $3 - \frac{8}{x + 1}$; $x = -1$, $y = 3$

3 marks

4 (2 marks)

$x^2 - 4 = 0$ gives vertical asymptotes $x = 2$ and $x = -2$

B1

As $x \rightarrow \infty$, $y \rightarrow 1$, so $y = 1$ is the horizontal asymptote

B1

Answer: $x = 2$, $x = -2$, $y = 1$

2 marks

5 (2 marks)

$y = x + \frac{1}{x}$

M1

As $x \rightarrow \infty$, $\frac{1}{x} \rightarrow 0$, so the oblique asymptote is $y = x$

A1

Answer: $y = x$

2 marks

6 (2 marks)

Multiply by $(x - 1)(x + 3)$: $x + 3 = 2(x - 1)$

M1

$x = 5$

A1

Answer: $x = 5$

2 marks

7 (5 marks)

Vertical asymptote $x = 2$; horizontal asymptote $y = 1$ (since $y = 1 + \frac{3}{x-2}$)

B1 B1

Crosses the x -axis at $(-1, 0)$ and the y -axis at $(0, -\frac{1}{2})$

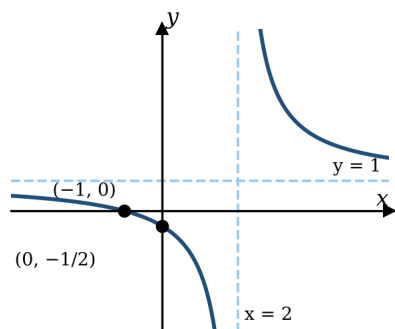
B1 B1

Two branches: one above $y = 1$ for $x > 2$ and one below $y = 1$ for $x < 2$, each approaching its asymptotes

B1

Answer: Asymptotes $x = 2$ and $y = 1$; crosses the axes at $(-1, 0)$ and $(0, -\frac{1}{2})$

5 marks

**8 (6 marks)**

$$x^2 + x - 6 = (x-1)(x+2) - 4, \text{ so } y = x + 2 - \frac{4}{x-1}$$

M1 A1

Asymptotes $x = 1$ and $y = x + 2$

A1 A1

$y = 0$: $(x+3)(x-2) = 0$, so $(-3, 0)$ and $(2, 0)$; $x = 0$: $(0, 6)$

B1 B1

Answer: $x = 1$, $y = x + 2$; $(-3, 0)$, $(2, 0)$, $(0, 6)$

6 marks

9 (6 marks)

$$\frac{x^2 + bx + c}{x-2} = x + (b+2) + \frac{2b+c+4}{x-2}$$

M1 A1

The oblique asymptote is $y = x + b + 2 = x + 5$, so $b = 3$

M1 A1

$$(0, -2): \frac{c}{-2} = -2, \text{ so } c = 4$$

M1 A1

Answer: $b = 3$, $c = 4$

6 marks

10 (5 marks)

Vertical asymptotes $x = 1$ and $x = 4$

B1

As $x \rightarrow \infty$, $y \rightarrow 1$, so $y = 1$ is the horizontal asymptote

B1

$$y = 1: x^2 = x^2 - 5x + 4$$

M1

$$x = \frac{4}{5}, \text{ so the point is } (\frac{4}{5}, 1)$$

A1 A1

Answer: $x = 1$, $x = 4$, $y = 1$; $(\frac{4}{5}, 1)$

5 marks

11 (5 marks)

$$\frac{3x-1}{x+2} = 3 - \frac{7}{x+2}$$

M1 A1

Stretch (and reflect) $y = \frac{1}{x}$ by scale factor -7 in the y -direction, then translate by $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

B1 B1

(or translate by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, stretch in y with factor -7 , translate by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$)

B1

Answer: $3 - \frac{7}{x+2}$; translation by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, then y -stretch of factor -7 , then translation by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

5 marks

12 (3 marks)

Vertical asymptote $x = -c$, so $c = -2$

B1

The curve passes through $(0, 1)$: $1 = \frac{b}{-2}$, so $b = -2$

M1 A1

Answer: $b = -2$, $c = -2$

3 marks

13 (5 marks)

Vertical asymptotes $x = \pm 3$; horizontal asymptote $y = 0$

B1 B1

Crosses the y-axis at $(0, -\frac{1}{9})$, a maximum

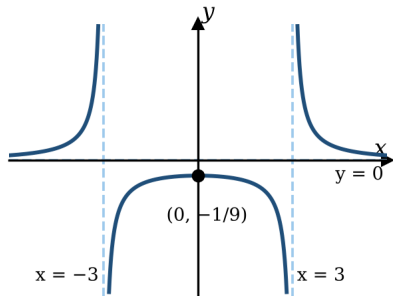
B1 B1

$y > 0$ for $|x| > 3$, $y < 0$ for $|x| < 3$, curve symmetric about the y-axis

B1

Answer: $x = \pm 3$, $y = 0$; $(0, -\frac{1}{9})$ is a maximum

5 marks



14 (7 marks)

Vertical asymptotes $x = \pm 1$; $y = 1 + \frac{1}{x^2 - 1}$ so horizontal asymptote $y = 1$

B1 B1

Passes through the origin only, $y = 0$ there

B1

Even function, so symmetric in the y-axis

B1

$\frac{dy}{dx} = -\frac{2x}{(x^2 - 1)^2} = 0$ only at $x = 0$: a maximum at $(0, 0)$

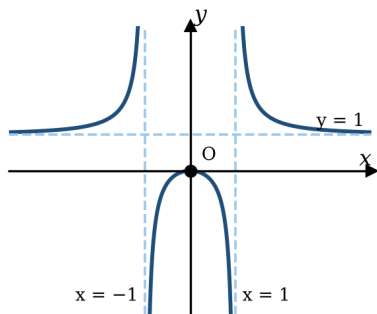
M1 A1

Sketch: branches $y > 1$ outside the asymptotes, and $y \leq 0$ between them

B1

Answer: $x = \pm 1$, $y = 1$; maximum at $(0, 0)$

7 marks



15 (6 marks)

$$y = x - 4 + \frac{4}{x+1}$$

M1 A1

$$\frac{dy}{dx} = 1 - \frac{4}{(x+1)^2} = 0, \text{ so } (x+1)^2 = 4$$

M1 A1

$$x = 1: y = -1; x = -3: y = -9$$

M1 A1

Answer: $(1, -1)$ and $(-3, -9)$

6 marks

16 (5 marks)

$$yx^2 - x + y = 0$$

M1

For real x , the discriminant satisfies $1 - 4y^2 \geq 0$

M1 A1

$$-\frac{1}{2} \leq y \leq \frac{1}{2}$$

A1

($y = 0$ gives $x = 0$, so the case $y = 0$ is also attained)

B1

Answer: $-\frac{1}{2} \leq y \leq \frac{1}{2}$

5 marks

17 (10 marks)

C_1 : asymptotes $x = 0$ and $y = 0$, with no axis crossings

B1

C_2 : $\frac{x+1}{x+3} = 1 - \frac{2}{x+3}$, asymptotes $x = -3$ and $y = 1$, crossing the axes at $(-1, 0)$ and $(0, \frac{1}{3})$

B1 B1

Both curves drawn correctly on one set of axes

B1 B1

$\frac{2}{x} = \frac{x+1}{x+3}$: $2(x+3) = x(x+1)$, so $x^2 - x - 6 = 0$

M1 A1

$x = -2$ or $x = 3$: points $(-2, -1)$ and $(3, \frac{2}{3})$

A1

From the sketch, C_1 is above C_2 for $-3 < x < -2$ and $0 < x < 3$

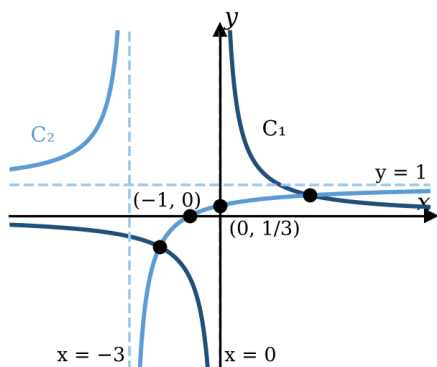
M1

$-3 < x \leq -2$ or $0 < x \leq 3$

A1

Answer: $(-2, -1)$ and $(3, \frac{2}{3})$; $-3 < x \leq -2$ or $0 < x \leq 3$

10 marks

**18 (5 marks)**

$$\frac{(x+1)(x+3) - x(x-2)}{(x-2)(x+3)} \leq 0$$

M1

$$\frac{6x+3}{(x-2)(x+3)} \leq 0$$

A1

Critical values $-3, -\frac{1}{2}, 2$

B1

Sign table / sketch: negative for $x < -3$ and for $-\frac{1}{2} < x < 2$

M1

$x < -3$ or $-\frac{1}{2} \leq x < 2$

A1

Answer: $x < -3$ or $-\frac{1}{2} \leq x < 2$

5 marks

19 (6 marks)

$x^2 - 4x + 3 = (x-1)(x-3)$, so vertical asymptotes $x = 1$ and $x = 3$

B1 B1

Horizontal asymptote $y = 1$

B1

$$x^2 - x - 2 = x^2 - 4x + 3$$

M1

$3x = 5$, so the point is $(\frac{5}{3}, 1)$

A1 A1

Answer: $x = 1, x = 3, y = 1; (\frac{5}{3}, 1)$

6 marks

20 (6 marks)

$3x + 3 = (x+1)(x-1)$, so $x^2 - 3x - 4 = 0$

M1 A1

$x = -1$ and $x = 4$; points $(-1, 0)$ and $(4, 5)$

A1 A1

Sketch: C has asymptote $x = 1$; C is below l for $-1 < x < 1$ and for $x > 4$

M1

$-1 < x < 1$ or $x > 4$

A1

Answer: $(-1, 0), (4, 5); -1 < x < 1$ or $x > 4$

6 marks

21 (9 marks)

Vertical asymptotes $x = -1$ and $x = 2$; horizontal asymptote $y = 1$

B1 B1

C meets the axes at $(-2, 0)$, $(3, 0)$ and $(0, 3)$

B1 B1

$y = 1: x^2 - x - 6 = x^2 - x - 2$, which gives $-6 = -2$, impossible

M1 A1

$y - 1 = -\frac{4}{(x+1)(x-2)}$, so C approaches $y = 1$ from below as $x \rightarrow \pm\infty$

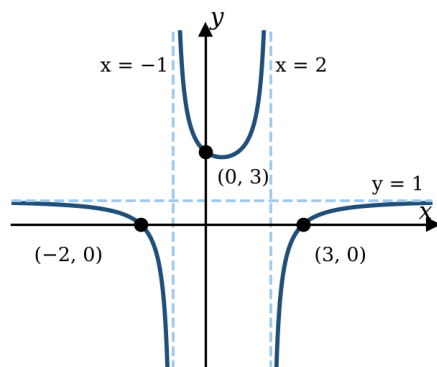
M1

Sketch showing three branches and the features above

B1 B1

Answer: $x = -1$, $x = 2$, $y = 1$; $(-2, 0)$, $(3, 0)$, $(0, 3)$

9 marks



22 (10 marks)

$y = x + 4 + \frac{5}{x-2}$; asymptotes $x = 2$ and $y = x + 4$

M1 A1 A1

$\frac{dy}{dx} = 1 - \frac{5}{(x-2)^2} = 0$, so $x = 2 \pm \sqrt{5}$

M1 A1

$x = 2 + \sqrt{5}$: $y = 6 + 2\sqrt{5}$, a minimum; $x = 2 - \sqrt{5}$: $y = 6 - 2\sqrt{5}$, a maximum (second derivative $\frac{10}{(x-2)^3}$)

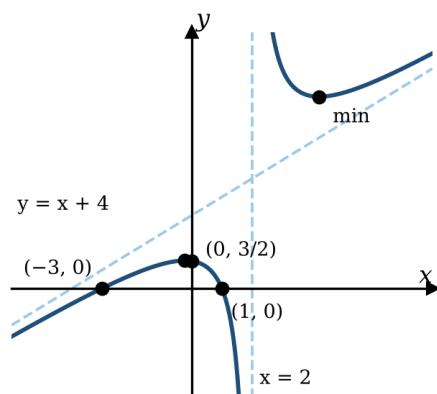
M1 A1 A1

Crosses the axes at $(1, 0)$, $(-3, 0)$, $(0, \frac{3}{2})$; sketch with two branches

B1 B1

Answer: $x = 2$, $y = x + 4$; min $(2 + \sqrt{5}, 6 + 2\sqrt{5})$, max $(2 - \sqrt{5}, 6 - 2\sqrt{5})$

10 marks



23 (8 marks)

(a) $y(x-1) = x^2$, so $x^2 - yx + y = 0$

M1 A1

Real x requires $y^2 - 4y \geq 0$

M1

$y(y-4) \geq 0$, so $y \leq 0$ or $y \geq 4$

M1 A1

Answer: Every point on C has $y \leq 0$ or $y \geq 4$

5 marks

(b) The boundary values $y = 0$ and $y = 4$ give repeated roots, hence turning points

M1

$y = 0$: $x^2 = 0$, so $(0, 0)$; $y = 4$: $x^2 - 4x + 4 = 0$, so $(2, 4)$

A1 A1

Answer: $(0, 0)$ and $(2, 4)$

3 marks

24 (11 marks)

(a) Divide each term by n

M1 A1

Answer: $A = n + 60 + \frac{3600}{n}$

2 marks

$$(b) \frac{dA}{dn} = 1 - \frac{3600}{n^2} = 0$$

M1 A1

$$n = 60 \ (n > 0)$$

A1

$A = 60 + 60 + 60 = 180$, so the least average cost is £180

M1 A1

$$\frac{d^2A}{dn^2} = \frac{7200}{n^3} > 0, \text{ so this is a minimum}$$

B1

Answer: $n = 60$; £180

6 marks

$$(c) A - 180 = \frac{n^2 - 120n + 3600}{n} = \frac{(n - 60)^2}{n}$$

M1 A1

Since $n > 0$ and $(n - 60)^2 \geq 0$, $A \geq 180$ with equality at $n = 60$

E1

$$\text{Answer: } A - 180 = \frac{(n - 60)^2}{n}$$

3 marks

25 (7 marks)

(a) Multiplying by $(x - 1)$ is only valid when $x - 1 > 0$; the direction of the inequality reverses when $x < 1$

E1 E1

$x = 0$ satisfies $x \leq 2$ but $\frac{0}{-1} = 0$, which is not ≥ 2

B1

Answer: Sign of $x - 1$ ignored; $x = 0$ is a counter-example

3 marks

$$(b) \frac{x}{x-1} - 2 \geq 0, \text{ so } \frac{2-x}{x-1} \geq 0$$

M1 A1

Critical values 1 and 2; the expression is positive between them

M1

$$1 < x \leq 2$$

A1

Answer: $1 < x \leq 2$

4 marks

26 (7 marks)

$$\text{Equate: } (k - x)(x - 2) = 2x + 1$$

M1

$$x^2 - kx + 2k + 1 = 0$$

A1

A tangent gives a repeated root: $k^2 - 4(2k + 1) = 0$

M1 A1

$$k^2 - 8k - 4 = 0$$

A1

$$k = \frac{8 \pm \sqrt{64 + 16}}{2} = 4 \pm 2\sqrt{5}$$

M1 A1

Answer: $k = 4 \pm 2\sqrt{5}$

7 marks

27 (12 marks)

$$(a) \text{ At } x = 0: \frac{q}{-2} = -\frac{3}{2}, \text{ so } q = 3$$

M1 A1

$$x^2 + px + q = (x - 2)(x + p + 2) + (2p + 4 + q), \text{ so } y = x + p + 2 + \frac{2p + q + 4}{x - 2}$$

M1

$$\frac{dy}{dx} = 1 - \frac{2p + q + 4}{(x - 2)^2} = 0 \text{ at } x = 5, \text{ so } 2p + q + 4 = 9$$

M1 A1

$$p = 1$$

A1

Answer: $p = 1$, $q = 3$

6 marks

$$(b) y = x + 3 + \frac{9}{x - 2}, \text{ with } (x - 2)^2 = 9, \text{ so } x = 5 \text{ or } x = -1$$

M1 A1

(5, 11) a minimum and (-1, -1) a maximum

A1

Answer: (5, 11) minimum; (-1, -1) maximum

3 marks

$$(c) x = 2 \text{ and } y = x + 3$$

B1 B1

y cannot lie strictly between the two turning values: $-1 < y < 11$

B1

Answer: $x = 2$, $y = x + 3$; $-1 < y < 11$ not attained

3 marks

28 (12 marks)

(a) Vertical asymptotes $x = 2$ and $x = 3$; horizontal asymptote $y = 1$

B1 B1

$$y = 1 - \frac{2}{x^2 - 5x + 6}, \text{ and } \frac{2}{x^2 - 5x + 6} \text{ is never } 0$$

M1 A1

Answer: $x = 2$, $x = 3$, $y = 1$

4 marks

$$(b) \frac{dy}{dx} = \frac{2(2x - 5)}{(x^2 - 5x + 6)^2}$$

M1

Zero only when $x = \frac{5}{2}$

A1

$y = 1 - \frac{2}{-\frac{1}{4}} = 9$, so the turning point is $(\frac{5}{2}, 9)$

M1 A1

Answer: $(\frac{5}{2}, 9)$

4 marks

(c) Between the asymptotes ($2 < x < 3$) the curve goes to $+\infty$ at both ends, so $(\frac{5}{2}, 9)$ is a minimum there

M1 A1

Outside this interval $y < 1$ and takes all such values

M1

$y < 1$ or $y \geq 9$

A1

Answer: $y < 1$ or $y \geq 9$

4 marks

29 (9 marks)

(a) $x^2 + 3 = (x + 1)(x - 1) + 4$

M1 A1

$y = x - 1$

B1

Answer: $y = x - 1$

3 marks

(b) Area = $\int_1^3 (x - 1 + \frac{4}{x+1}) dx$

M1

$= [\frac{x^2}{2} - x + 4 \ln(x+1)]_1^3$

M1 A1 A1

$= (\frac{9}{2} - 3 + 4 \ln 4) - (\frac{1}{2} - 1 + 4 \ln 2)$

M1

$= 2 + 4 \ln 2$

A1

Answer: $2 + 4 \ln 2$

6 marks

30 (10 marks)

(a) $k(x - 2)(x - 1) = 1$

M1

$kx^2 - 3kx + 2k - 1 = 0$

A1

Answer: the x-coordinates of any points of intersection satisfy $kx^2 - 3kx + 2k - 1 = 0$

2 marks

(b) Discriminant $9k^2 - 4k(2k - 1) = k^2 + 4k = k(k + 4)$

M1 A1

$k(k + 4) > 0$, so $k < -4$ or $k > 0$

M1 A1

Answer: $k < -4$ or $k > 0$

4 marks

(c) $k = -4$: $-4x^2 + 12x - 9 = 0$, so $(2x - 3)^2 = 0$

M1 A1

$x = \frac{3}{2}$, $y = \frac{1}{\frac{1}{2}} = 2$, so the point is $(\frac{3}{2}, 2)$

M1 A1

Answer: $(\frac{3}{2}, 2)$

4 marks

Partial Fractions

1 (3 marks)

$\frac{5}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$, so $5 = A(x+4) + B(x-1)$

M1

$x = 1$: $A = 1$; $x = -4$: $B = -1$

A1

$\frac{1}{x-1} - \frac{1}{x+4}$

A1

Answer: $\frac{1}{x-1} - \frac{1}{x+4}$

3 marks

2 (3 marks)

$$3x + 7 = A(x + 3) + B(x + 1)$$

M1

$$x = -1: 4 = 2A, \text{ so } A = 2; x = -3: -2 = -2B, \text{ so } B = 1$$

A1

$$\frac{2}{x+1} + \frac{1}{x+3}$$

A1

$$\text{Answer: } \frac{2}{x+1} + \frac{1}{x+3}$$

3 marks

3 (3 marks)

$$x^2 - 9 = (x - 3)(x + 3), \text{ so } 6 = A(x + 3) + B(x - 3)$$

M1

$$x = 3: A = 1; x = -3: B = -1$$

A1

$$\frac{1}{x-3} - \frac{1}{x+3}$$

A1

$$\text{Answer: } \frac{1}{x-3} - \frac{1}{x+3}$$

3 marks

4 (3 marks)

$$7x - 1 = A(2x + 1) + B(x - 1)$$

M1

$$x = 1: 6 = 3A, \text{ so } A = 2; x = -\frac{1}{2}: -\frac{9}{2} = -\frac{3}{2}B, \text{ so } B = 3$$

A1 A1

$$\text{Answer: } A = 2, B = 3$$

3 marks

5 (5 marks)

$$1 = A(x + 5) + B(x + 2), \text{ so } A = \frac{1}{3} \text{ and } B = -\frac{1}{3}$$

M1 A1

$$\text{Integral} = \frac{1}{3}[\ln(x + 2) - \ln(x + 5)]_0^1$$

M1 A1

$$= \frac{1}{3}(\ln \frac{3}{6} - \ln \frac{2}{5}) = \frac{1}{3} \ln \frac{5}{4}$$

A1

$$\text{Answer: } \frac{1}{3} \ln \frac{5}{4}$$

5 marks

6 (3 marks)

$$2 = A(x + 2) + Bx: A = 1, B = -1, \text{ so } \frac{2}{x(x+2)} = \frac{1}{x} - \frac{1}{x+2}$$

M1 A1

$$x = 2: \frac{2}{8} = \frac{1}{4}; \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, \text{ which agrees}$$

B1

$$\text{Answer: } \frac{1}{x} - \frac{1}{x+2}; \text{ both give } \frac{1}{4}$$

3 marks

7 (5 marks)

$$\frac{x^2 + 10x + 7}{(x - 1)(x + 2)^2} = \frac{A}{x - 1} + \frac{B}{x + 2} + \frac{C}{(x + 2)^2}$$

M1

$$x^2 + 10x + 7 = A(x + 2)^2 + B(x - 1)(x + 2) + C(x - 1)$$

M1

$$x = 1: 18 = 9A, \text{ so } A = 2; x = -2: -9 = -3C, \text{ so } C = 3$$

A1 A1

$$x^2 \text{ coefficient: } 1 = A + B, \text{ so } B = -1$$

A1

$$\text{Answer: } \frac{2}{x-1} - \frac{1}{x+2} + \frac{3}{(x+2)^2}$$

5 marks

8 (4 marks)

$$4x^2 + 3x - 7 = A(x - 2)(x + 3) + B(x + 1)(x + 3) + C(x + 1)(x - 2)$$

M1

$$x = -1: -6 = -6A, \text{ so } A = 1$$

A1

$$x = 2: 15 = 15B, \text{ so } B = 1$$

A1

$$x = -3: 20 = 10C, \text{ so } C = 2$$

A1

$$\text{Answer: } \frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{x+3}$$

4 marks

9 (5 marks)

$$\frac{3x^2 + x + 8}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

M1

$$3x^2 + x + 8 = A(x^2 + 4) + (Bx + C)(x + 1)$$

M1

$$x = -1: 10 = 5A, \text{ so } A = 2$$

A1

$$x^2: 3 = A + B, \text{ so } B = 1; \text{ constant: } 8 = 4A + C, \text{ so } C = 0$$

A1 A1

$$\text{Answer: } \frac{2}{x+1} + \frac{x}{x^2+4}$$

5 marks

10 (4 marks)

The numerator and denominator have the same degree: $x^2 + 3x + 1 = (x^2 + x) + 2x + 1$, so the

M1 A1

expression is $1 + \frac{2x+1}{x(x+1)}$

$$\frac{2x+1}{x(x+1)} = \frac{B}{x} + \frac{C}{x+1}; 2x+1 = B(x+1) + Cx, \text{ giving } B = 1 \text{ and } C = 1$$

M1 A1

$$\text{Answer: } 1 + \frac{1}{x} + \frac{1}{x+1}$$

4 marks

11 (4 marks)

$$4x^2 - 3x - 4 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

M1

$$x = 0: -4 = -2A, \text{ so } A = 2$$

A1

$$x = 1: -3 = 3B, \text{ so } B = -1$$

A1

$$x = -2: 18 = 6C, \text{ so } C = 3$$

A1

$$\text{Answer: } A = 2, B = -1, C = 3$$

4 marks

12 (6 marks)

$$x + 8 = A(x+2) + B(x-1): A = 3, B = -2$$

M1 A1

$$\text{Integral} = [3 \ln(x-1) - 2 \ln(x+2)]_2^4$$

M1 A1

$$= (3 \ln 3 - 2 \ln 6) - (0 - 2 \ln 4)$$

M1

$$= \ln \frac{27 \times 16}{36} = \ln 12$$

A1

$$\text{Answer: } \ln 12$$

6 marks

13 (6 marks)

$$5x + 1 = A(x+2) + B(x-1): A = 2, B = 3, \text{ so } f(x) = \frac{2}{x-1} + \frac{3}{x+2}$$

M1 A1 A1

$$f'(x) = -\frac{2}{(x-1)^2} - \frac{3}{(x+2)^2}$$

M1 A1

Both terms are negative for $x > 1$, so $f'(x) < 0$ and f is decreasing

E1

$$\text{Answer: } f(x) = \frac{2}{x-1} + \frac{3}{x+2}; f'(x) < 0$$

6 marks

14 (5 marks)

$$x^2 - 3x + 14 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

M1

$$x = -1: 18 = 9A, \text{ so } A = 2$$

A1

$$x = 2: 12 = 3C, \text{ so } C = 4$$

A1

$$x^2 \text{ coefficient: } 1 = A + B, \text{ so } B = -1$$

M1 A1

$$\text{Answer: } A = 2, B = -1, C = 4$$

5 marks

15 (5 marks)

The numerator has higher degree, so divide: $(x-1)(x+2) = x^2 + x - 2$

M1

$$x^3 + 2x^2 - x + 4 = (x+1)(x^2 + x - 2) + 6$$

A1

$$\frac{6}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}; 6 = A(x+2) + B(x-1)$$

M1

$$A = 2 \text{ and } B = -2$$

A1 A1

$$\text{Answer: } x + 1 + \frac{2}{x-1} - \frac{2}{x+2}$$

5 marks

16 (7 marks)

$$5 - x = A(1 + x) + B(1 - 2x); x = -1 \text{ gives } B = 2; x = \frac{1}{2} \text{ gives } A = 3$$

M1 A1 A1

$$3(1 - 2x)^{-1} = 3(1 + 2x + 4x^2 + 8x^3 + \dots)$$

M1

$$2(1 + x)^{-1} = 2(1 - x + x^2 - x^3 + \dots)$$

M1

$$\text{Sum: } 5 + 4x + 14x^2 + 22x^3$$

A1

$$\text{Valid for } |x| < \frac{1}{2}$$

B1

$$\text{Answer: } 5 + 4x + 14x^2 + 22x^3 + \dots; |x| < \frac{1}{2}$$

7 marks

17 (7 marks)

$$3x + 2 = A(x^2 + 1) + (Bx + C)(x + 1); x = -1 \text{ gives } A = -\frac{1}{2}$$

M1 A1

$$x^2: 0 = A + B \text{ so } B = \frac{1}{2}; \text{ constant: } 2 = A + C \text{ so } C = \frac{5}{2}$$

A1

$$\text{Integral} = \int_0^1 \left[-\frac{1}{2(x+1)} + \frac{x}{2(x^2+1)} + \frac{5}{2(x^2+1)} \right] dx$$

M1

$$= \left[-\frac{1}{2} \ln(x+1) + \frac{1}{4} \ln(x^2+1) + \frac{5}{2} \arctan x \right]_0^1$$

M1 A1

$$= -\frac{1}{2} \ln 2 + \frac{1}{4} \ln 2 + \frac{5\pi}{8} = \frac{5\pi}{8} - \frac{1}{4} \ln 2$$

A1

$$\text{Answer: } \frac{5\pi}{8} - \frac{1}{4} \ln 2$$

7 marks

18 (5 marks)

$$3x^3 - 2x^2 + 9x + 1 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 1)$$

M1

$$x^3: A + C = 3; x: 4A + C = 9, \text{ so } A = 2 \text{ and } C = 1$$

M1 A1

$$x^2: B + D = -2; \text{ constant: } 4B + D = 1, \text{ so } B = 1 \text{ and } D = -3$$

M1 A1

$$\text{Answer: } \frac{2x+1}{x^2+1} + \frac{x-3}{x^2+4}$$

5 marks

19 (4 marks)

$$x^2 + 3x - 4 = (x - 1)(x + 4)$$

M1

$$\text{So the expression is } \frac{1}{x}, \text{ giving } A = 1, B = 0, C = 0$$

A1 A1

$$B = C = 0 \text{ because the numerator has factors } (x - 1) \text{ and } (x + 4), \text{ which cancel with the denominator}$$

E1

$$\text{Answer: } A = 1, B = 0, C = 0$$

4 marks

20 (6 marks)

$$2 = A(y^2 - 1) + By(y + 1) + Cy(y - 1); y = 0: A = -2; y = 1: B = 1; y = -1: C = 1$$

M1 A1 A1

$$\text{Integral} = [-2 \ln y + \ln(y - 1) + \ln(y + 1)]_2^3$$

M1 A1

$$= \ln \frac{y^2 - 1}{y^2} \text{ evaluated: } \ln \frac{8}{9} - \ln \frac{3}{4} = \ln \frac{32}{27}$$

A1

$$\text{Answer: } \ln \frac{32}{27}$$

6 marks

21 (7 marks)

$$\text{The degrees are equal, so write it as } 1 + \text{remainder: } (x - 1)^2(x + 2) = x^3 - 3x + 2$$

M1

$$x^3 + x^2 + 4x + 3 - (x^3 - 3x + 2) = x^2 + 7x + 1$$

A1

$$\frac{x^2 + 7x + 1}{(x - 1)^2(x + 2)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 2}$$

M1

$$x = 1: 9 = 3B, \text{ so } B = 3; x = -2: -9 = 9C, \text{ so } C = -1$$

A1 A1

$$x^2 \text{ coefficient: } 1 = A + C, \text{ so } A = 2$$

M1 A1

$$\text{Answer: } 1 + \frac{2}{x - 1} + \frac{3}{(x - 1)^2} - \frac{1}{x + 2}$$

7 marks

22 (8 marks)

$$\frac{3x^2 + x - 2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}, \text{ so } 3x^2 + x - 2 = Ax(x+2) + B(x+2) + Cx^2$$

M1

$$x = 0: -2 = 2B, \text{ so } B = -1; x = -2: 8 = 4C, \text{ so } C = 2$$

A1 A1

$$x^2 \text{ coefficient: } 3 = A + C, \text{ so } A = 1$$

A1

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x+2} \right) dx = \left[\ln x + \frac{1}{x} + 2 \ln(x+2) \right]_1^2$$

M1 A1

$$= \left(\ln 2 + \frac{1}{2} + 2 \ln 4 \right) - \left(0 + 1 + 2 \ln 3 \right)$$

M1

$$= \ln \frac{32}{9} - \frac{1}{2}$$

A1

$$\text{Answer: } \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x+2}; \ln \frac{32}{9} - \frac{1}{2}$$

8 marks

23 (7 marks)

$$(a) 1 = A(1 - 2x) + B(1 - x): x = 1 \text{ gives } A = -1; x = \frac{1}{2} \text{ gives } B = 2$$

M1 A1 A1

$$\text{Answer: } \frac{2}{1-2x} - \frac{1}{1-x}$$

3 marks

$$(b) \frac{2}{1-2x} = 2 \sum (2x)^n, \text{ so the coefficient of } x^n \text{ is } 2 \times 2^n = 2^{n+1}$$

M1 A1

$$\frac{1}{1-x} = \sum x^n, \text{ with coefficient } 1$$

B1

$$\text{Coefficient} = 2^{n+1} - 1$$

A1

$$\text{Answer: the coefficient of } x^n \text{ in the expansion of } \frac{1}{(1-x)(1-2x)} \text{ in ascending powers of } x \text{ is } 2^{n+1} - 1$$

4 marks

24 (9 marks)

$$(a) x^4 - 1 = (x-1)(x+1)(x^2+1)$$

B1

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$$

M1

$$x = 1: A = \frac{1}{4}; x = -1: B = -\frac{1}{4}$$

A1 A1

$$x^3: 0 = A + B + C \text{ so } C = 0; \text{ constant: } 1 = A - B - D \text{ so } D = -\frac{1}{2}$$

M1 A1

$$\text{Answer: } \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$$

6 marks

$$(b) \int = \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| - \frac{1}{2} \arctan x + c$$

M1 A1 A1

$$\text{Answer: } \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \arctan x + c$$

3 marks

25 (7 marks)

$$(a) \text{ The repeated factor } (x+2)^2 \text{ also needs a term } \frac{C}{x+2}$$

E1

With only A and B, the expression would not be able to match all of the numerator coefficients

E1

$$\text{Answer: Missing } \frac{C}{x+2} \text{ term}$$

2 marks

$$(b) 3x = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

M1

$$x = 1: A = \frac{1}{3}; x = -2: C = 2$$

A1 A1

$$x^2: 0 = A + B, \text{ so } B = -\frac{1}{3}$$

M1 A1

$$\text{Answer: } \frac{1}{3(x-1)} - \frac{1}{3(x+2)} + \frac{2}{(x+2)^2}$$

5 marks

26 (10 marks)

$$(x+2)(x^2+1) = x^3 + 2x^2 + x + 2, \text{ so } A = 1 \text{ and the remainder is } 3x^2 + x \quad \text{M1 A1}$$

$$3x^2 + x = B(x^2 + 1) + (Cx + D)(x + 2); x = -2 \text{ gives } 10 = 5B, \text{ so } B = 2 \quad \text{M1 A1}$$

$$x^2: 3 = B + C, \text{ so } C = 1; \text{ constant: } 0 = B + 2D, \text{ so } D = -1 \quad \text{M1 A1}$$

$$\text{Integral} = [x + 2 \ln(x+2) + \frac{1}{2} \ln(x^2+1) - \arctan x]_0^1 \quad \text{M1 A1}$$

$$= 1 + 2 \ln 3 - 2 \ln 2 + \frac{1}{2} \ln 2 - \frac{\pi}{4} \quad \text{M1}$$

$$= 1 + 2 \ln 3 - \frac{3}{2} \ln 2 - \frac{\pi}{4} \quad \text{A1}$$

$$\text{Answer: } 1 + 2 \ln 3 - \frac{3}{2} \ln 2 - \frac{\pi}{4} \quad \text{10 marks}$$

27 (10 marks)

$$(a) 1 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b) \quad \text{M1}$$

$$x = a: 1 = A(a-b)(a-c), \text{ so } A = \frac{1}{(a-b)(a-c)} \quad \text{A1}$$

$$\text{Similarly } B = \frac{1}{(b-a)(b-c)} \text{ and } C = \frac{1}{(c-a)(c-b)} \quad \text{A1}$$

$$\text{Answer: } f(x) = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}, \text{ where } A = \frac{1}{(a-b)(a-c)} \quad \text{3 marks}$$

$$(b) \text{ Multiply } f(x) \text{ by } x \text{ and let } x \rightarrow \infty: x f(x) \rightarrow 0 \quad \text{M1}$$

$$x f(x) = \frac{Ax}{x-a} + \frac{Bx}{x-b} + \frac{Cx}{x-c} \rightarrow A + B + C \quad \text{M1 A1}$$

$$\text{Answer: } A + B + C = 0 \quad \text{3 marks}$$

$$(c) A = \frac{1}{(-1)(-3)} = \frac{1}{3}, B = \frac{1}{(1)(-2)} = -\frac{1}{2}, C = \frac{1}{(3)(2)} = \frac{1}{6} \quad \text{M1 A1}$$

$$\frac{1}{3(x-1)} - \frac{1}{2(x-2)} + \frac{1}{6(x-4)}; \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0 \quad \text{A1 B1}$$

$$\text{Answer: } \frac{1}{3(x-1)} - \frac{1}{2(x-2)} + \frac{1}{6(x-4)} \quad \text{4 marks}$$

28 (12 marks)

$$(a) 2x^3 + x^2 + 5x + 3 = (Ax + B)(x^2 + 1) + Cx + D \quad \text{M1}$$

$$x^3: A = 2; x^2: B = 1 \quad \text{A1 A1}$$

$$x: 5 = A + C, \text{ so } C = 3; \text{ constant: } 3 = B + D, \text{ so } D = 2 \quad \text{A1 A1}$$

$$\text{Answer: } A = 2, B = 1, C = 3, D = 2 \quad \text{5 marks}$$

$$(b) \int_0^1 \frac{2x}{x^2+1} dx = \ln 2; \int_0^1 \frac{1}{x^2+1} dx = \frac{\pi}{4} \quad \text{M1 A1}$$

$$\int_0^1 \frac{3x}{(x^2+1)^2} dx = \left[-\frac{3}{2(x^2+1)} \right]_0^1 = \frac{3}{4} \quad \text{M1 A1}$$

$$\int_0^1 \frac{2}{(x^2+1)^2} dx = \left[\frac{x}{x^2+1} + \arctan x \right]_0^1 = \frac{1}{2} + \frac{\pi}{4} \quad \text{M1 A1}$$

$$\text{Total: } \ln 2 + \frac{\pi}{4} + \frac{3}{4} + \frac{1}{2} + \frac{\pi}{4} = \frac{5}{4} + \ln 2 + \frac{\pi}{2} \quad \text{A1}$$

$$\text{Answer: } \frac{5}{4} + \ln 2 + \frac{\pi}{2} \quad \text{7 marks}$$

29 (12 marks)

$$(a) 1 = A(1-x)^2 + B(1-x)(1-2x) + C(1-2x) \quad \text{M1}$$

$$x = 1: C = -1; x = \frac{1}{2}: A = 4 \quad \text{A1 A1}$$

$$x = 0: 1 = A + B + C, \text{ so } B = -2 \quad \text{M1 A1}$$

$$\text{Answer: } \frac{4}{1-2x} - \frac{2}{1-x} - \frac{1}{(1-x)^2} \quad \text{5 marks}$$

$$(b) \frac{4}{1-2x} = 4 \sum (2x)^n \text{ has coefficient } 2^{n+2}, \text{ and } \frac{2}{1-x} \text{ has coefficient } 2 \quad \text{M1 A1}$$

$$(1-x)^{-2} = \sum (n+1)x^n, \text{ coefficient } n+1 \quad \text{M1 A1}$$

$$\text{Coefficient} = 2^{n+2} - 2 - (n+1) = 2^{n+2} - n - 3 \quad \text{A1}$$

Answer: For $|x| < \frac{1}{2}$, the coefficient of x^n is $2^{n+2} - n - 3$

5 marks

(c) $2^{12} - 10 - 3 = 4096 - 13$
 $= 4083$

M1

A1

Answer: 4083

2 marks

30 (12 marks)

(a) $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$

B1

$1 = (Ax + B)(x^2 - 2x + 2) + (Cx + D)(x^2 + 2x + 2)$

M1

$x^3: A + C = 0; x: 2A - 2B + 2C + 2D = 0$, so $D = B$

M1 A1

Constant: $2B + 2D = 1$, so $B = D = \frac{1}{4}$

A1

$x^2: -2A + B + 2C + D = 0$, so $A = \frac{1}{8}$ and $C = -\frac{1}{8}$

M1 A1

Answer: $A = \frac{1}{8}, B = \frac{1}{4}, C = -\frac{1}{8}, D = \frac{1}{4}$

7 marks

(b) $\frac{x+2}{8(x^2+2x+2)} = \frac{1}{16} \cdot \frac{2x+2}{x^2+2x+2} + \frac{1}{8} \cdot \frac{1}{(x+1)^2+1}$

M1 A1

$\frac{-(x-2)}{8(x^2-2x+2)} = -\frac{1}{16} \cdot \frac{2x-2}{x^2-2x+2} + \frac{1}{8} \cdot \frac{1}{(x-1)^2+1}$

A1

Integrate each and combine the logarithms: $\frac{1}{16} \ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) + \frac{1}{8}(\arctan(x+1) + \arctan(x-1))$

M1 A1

+ c

Answer: $\int \frac{1}{x^4+4} dx = \frac{1}{16} \ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) + \frac{1}{8}(\arctan(x+1) + \arctan(x-1)) + c$

5 marks

Inequalities and Modulus Functions

1 (2 marks)

$-5 < 2x - 3 < 5$

M1

$-2 < 2x < 8$, so $-1 < x < 4$

A1

Answer: $-1 < x < 4$

2 marks

2 (2 marks)

$x + 4 \geq 3$ or $x + 4 \leq -3$

M1

$x \geq -1$ or $x \leq -7$

A1

Answer: $x \leq -7$ or $x \geq -1$

2 marks

3 (3 marks)

Critical values $-3, 1, 5$

B1

Cubic with positive leading coefficient: negative for $x < -3$ and for $1 < x < 5$

M1 A1

Answer: $x < -3$ or $1 < x < 5$

3 marks

4 (2 marks)

$3x - 1 = 8$ or $3x - 1 = -8$

M1

$x = 3$ or $x = -\frac{7}{3}$

A1

Answer: $x = 3$ or $x = -\frac{7}{3}$

2 marks

5 (3 marks)

V-shaped graph with vertex on the x-axis

B1

Meets the x-axis at $(2, 0)$

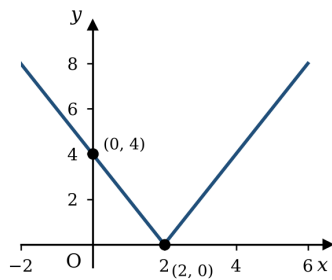
B1

Meets the y-axis at $(0, 4)$

B1

Answer: Vertex $(2, 0)$, y-intercept $(0, 4)$

3 marks

**6 (2 marks)**

$$(x - 4)(x + 2) \geq 0$$

$$x \leq -2 \text{ or } x \geq 4$$

Answer: $x \leq -2$ or $x \geq 4$

M1

A1

2 marks

7 (4 marks)

$$x - 3 = 2x + 1 \text{ gives } x = -4; \text{ or } 3 - x = 2x + 1 \text{ gives } x = \frac{2}{3}$$

M1 A1

Check: $x = -4$ gives $|-7| = 7$ but $2x + 1 = -7$, so reject

M1

$$x = \frac{2}{3} \text{ gives } \frac{7}{3} = \frac{7}{3}, \text{ valid}$$

A1

Answer: $x = \frac{2}{3}$

4 marks

8 (3 marks)

$$\text{Square both sides: } (x + 1)^2 < (x - 3)^2$$

M1

$$x^2 + 2x + 1 < x^2 - 6x + 9$$

A1

$$8x < 8, \text{ so } x < 1$$

A1

Answer: $x < 1$

3 marks

9 (4 marks)

$$-5 < x^2 - 4 < 5$$

M1

$$x^2 - 4 > -5 \text{ is true for all } x; x^2 - 4 < 5 \text{ gives } x^2 < 9$$

M1 A1

$$-3 < x < 3$$

A1

Answer: $-3 < x < 3$

4 marks

10 (4 marks)

$$x^2 - 3x = 4: (x - 4)(x + 1) = 0, \text{ so } x = 4 \text{ or } x = -1$$

M1 A1

$$x^2 - 3x = -4: x^2 - 3x + 4 = 0 \text{ has discriminant } 9 - 16 < 0, \text{ no real roots}$$

M1 A1

Answer: $x = 4$ or $x = -1$

4 marks

11 (6 marks)

$$x^2 - x - 6 = (x - 3)(x + 2), \text{ so it meets the } x\text{-axis at } (-2, 0) \text{ and } (3, 0)$$

B1 B1

Meets the y-axis at (0, 6)

B1

The part of the parabola between -2 and 3 is reflected in the x-axis; the minimum of the parabola

M1 A1

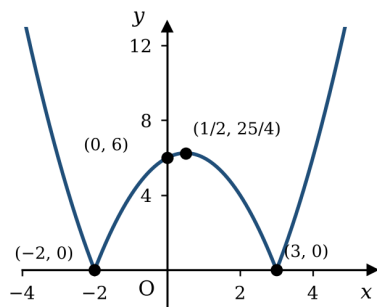
is at $(\frac{1}{2}, -\frac{25}{4})$, so the local maximum is $(\frac{1}{2}, \frac{25}{4})$

Correct overall shape (W-shape with cusps at the x-intercepts)

B1

Answer: Axes: $(-2, 0)$, $(3, 0)$, $(0, 6)$; local maximum $(\frac{1}{2}, \frac{25}{4})$

6 marks



12 (4 marks)

$$2x + 5 > 1, \text{ so } x > -2$$

B1

$$x^2 - x - 12 = (x - 4)(x + 3) < 0, \text{ so } -3 < x < 4$$

M1 A1

Both hold when $-2 < x < 4$

A1

Answer: $-2 < x < 4$

4 marks

13 (4 marks)

$$\frac{3}{x+2} - 2 < 0, \text{ so } \frac{3 - 2(x+2)}{x+2} < 0, \text{ i.e. } \frac{-2x-1}{x+2} < 0$$

M1 A1

$$\frac{2x+1}{x+2} > 0, \text{ critical values } -2 \text{ and } -\frac{1}{2}$$

M1

$$x < -2 \text{ or } x > -\frac{1}{2}$$

A1

Answer: $x < -2 \text{ or } x > -\frac{1}{2}$

4 marks

14 (6 marks)

Consider the intervals $x < -2$, $-2 \leq x < 1$ and $x \geq 1$

M1

$x \geq 1$: $(x - 1) + (x + 2) = 5$, so $2x = 4$ and $x = 2$

M1 A1

$-2 \leq x < 1$: $(1 - x) + (x + 2) = 3 \neq 5$, no solution

A1

$x < -2$: $(1 - x) + (-x - 2) = -2x - 1 = 5$, so $x = -3$

M1 A1

Answer: $x = 2 \text{ or } x = -3$

6 marks

15 (5 marks)

$$(2x + 1)^2 > (x - 4)^2$$

M1

$$4x^2 + 4x + 1 > x^2 - 8x + 16, \text{ so } 3x^2 + 12x - 15 > 0$$

M1 A1

$$x^2 + 4x - 5 > 0, \text{ so } (x + 5)(x - 1) > 0$$

M1

$$x < -5 \text{ or } x > 1$$

A1

Answer: $x < -5 \text{ or } x > 1$

5 marks

16 (7 marks)

$$\text{Area: } (x + 2)(x + 6) \geq 60, \text{ so } x^2 + 8x - 48 \geq 0$$

M1 A1

$$(x + 12)(x - 4) \geq 0, \text{ so } x \leq -12 \text{ or } x \geq 4$$

M1 A1

$$\text{Perimeter: } 2(2x + 8) \leq 44, \text{ so } x \leq 7$$

M1 A1

A length must be positive, so $4 \leq x \leq 7$

A1

Answer: $4 \leq x \leq 7$

7 marks

17 (6 marks)

$$3x - 2 \geq x + 1 \text{ gives } x \geq \frac{3}{2}$$

M1 A1

$$2 - 3x \geq x + 1 \text{ gives } x \leq \frac{1}{4}$$

M1 A1

These cover the values with $x + 1 < 0$ as well, since the left side is non-negative

E1

$$x \leq \frac{1}{4} \text{ or } x \geq \frac{3}{2}$$

A1

Answer: $x \leq \frac{1}{4} \text{ or } x \geq \frac{3}{2}$

6 marks

18 (6 marks)

$$\frac{(x-1)(x+1)}{x-3} \geq 0; \text{ critical values } -1, 1, 3$$

M1 A1

Sign table: negative for $x < -1$, positive for $-1 < x < 1$, negative for $1 < x < 3$, positive for $x > 3$

M1 A1

The value $x = 3$ is excluded

B1

$$-1 \leq x \leq 1 \text{ or } x > 3$$

A1

Answer: $-1 \leq x \leq 1$ or $x > 3$

6 marks

19 (5 marks)Let $u = |x|$, with $x^2 = u^2$: $u^2 - 3u - 4 < 0$

M1

$$(u-4)(u+1) < 0, \text{ so } -1 < u < 4$$

M1 A1

Since $u = |x| \geq 0$, $|x| < 4$

A1

$$-4 < x < 4$$

A1

Answer: $-4 < x < 4$

5 marks

20 (6 marks)

$y = |2x - 3|$ is V-shaped with vertex $(\frac{3}{2}, 0)$ and y-intercept $(0, 3)$; $y = x + 1$ is a line through $(0, 1)$ and $(-1, 0)$

B1 B1 B1

Intersections: $3 - 2x = x + 1$ gives $x = \frac{2}{3}$; $2x - 3 = x + 1$ gives $x = 4$

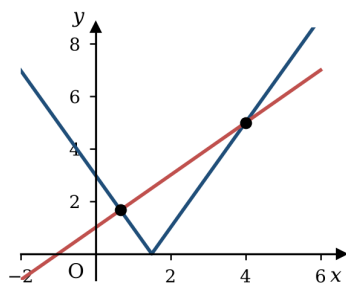
M1 A1

The V lies below the line between the intersections: $\frac{2}{3} < x < 4$

A1

Answer: $\frac{2}{3} < x < 4$

6 marks

**21 (7 marks)**Need $2x \geq 0$, so $x \geq 0$

B1

$$-2x \leq x^2 - 3x \leq 2x$$

M1

$$x^2 - 3x \leq 2x: x(x-5) \leq 0, \text{ so } 0 \leq x \leq 5$$

M1 A1

$$x^2 - 3x \geq -2x: x(x-1) \geq 0, \text{ so } x \leq 0 \text{ or } x \geq 1$$

M1 A1

Combine: $x = 0$ or $1 \leq x \leq 5$

A1

Answer: $x = 0$ or $1 \leq x \leq 5$

7 marks

22 (7 marks) $x \neq -1$. Multiply by $2(x+1)^2$ (positive): $2|x - 2|(x+1) < (x+1)^2$

M1

Case $x > -1$: $2|x - 2| < x + 1$

M1

 $x \geq 2$: $2x - 4 < x + 1$, so $x < 5$; $x < 2$: $4 - 2x < x + 1$, so $x > 1$

A1 A1

Case $x < -1$: dividing by the negative $x + 1$ gives $2|x - 2| > x + 1$, which is always true

M1 A1

$$x < -1 \text{ or } 1 < x < 5$$

A1

Answer: $x < -1$ or $1 < x < 5$

7 marks

23 (8 marks)

$$(a)(|a| + |b|)^2 - |a + b|^2 = a^2 + 2|ab| + b^2 - (a^2 + 2ab + b^2)$$

M1 A1

$$= 2(|ab| - ab) \geq 0$$

M1

Both sides are non-negative, so $|a + b| \leq |a| + |b|$

A1

Answer: $|a + b| \leq |a| + |b|$ for all real a and b

4 marks

$$(b) |x - 1| + |4 - x| \geq |(x - 1) + (4 - x)| = 3$$

M1 A1

Equality holds when $(x - 1)$ and $(4 - x)$ have the same sign (or one is zero), i.e. $1 \leq x \leq 4$

M1 A1

Answer: $|x - 1| + |x - 4| \geq 3$ for all real x ; equality for $1 \leq x \leq 4$

4 marks

24 (6 marks)

(a) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

M1 A1

Each square is non-negative, so the expression is ≥ 0

E1

Answer: $a^2 + b^2 + c^2 \geq ab + bc + ca$ for all real numbers a, b and c

3 marks

(b) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \geq 3(ab + bc + ca)$

M1 A1

$36 \geq 3(ab + bc + ca)$, so $ab + bc + ca \leq 12$

A1

Answer: $ab + bc + ca \leq 12$

3 marks

25 (5 marks)

(a) Dividing by x is invalid when x is negative (and the inequality sign would reverse), and $x = 0$ is lost

E1 E1

$x = -1$: $1 > -4$ is true, but -1 is not in $x > 4$

B1

Answer: Sign of x not considered; $x = -1$ satisfies the inequality

3 marks

(b) $x^2 - 4x > 0$, so $x(x - 4) > 0$

M1

$x < 0$ or $x > 4$

A1

Answer: $x < 0$ or $x > 4$

2 marks

26 (6 marks)

Square both sides: $(x + 3)^2 < 4(x - 1)^2$

M1

$x^2 + 6x + 9 < 4x^2 - 8x + 4$

A1

$3x^2 - 14x - 5 > 0$

A1

$(3x + 1)(x - 5) > 0$

M1

$x < -\frac{1}{3}$ or $x > 5$

A1 A1

Answer: $x < -\frac{1}{3}$ or $x > 5$

6 marks

27 (8 marks)

(a) $x^2 - 4x + 3 = (x - 1)(x - 3)$, so it meets the x -axis at $(1, 0)$ and $(3, 0)$

B1

y -intercept $(0, 3)$

B1

The part between $x = 1$ and $x = 3$ is reflected: the parabola's minimum $(2, -1)$ becomes the local maximum $(2, 1)$

M1 A1

W-shaped graph with cusps at $(1, 0)$ and $(3, 0)$

B1

Answer: Axes $(1, 0)$, $(3, 0)$, $(0, 3)$; local maximum $(2, 1)$

5 marks

(b) Intersections of $y = k$ with the graph

M1

(i) $0 < k < 1$

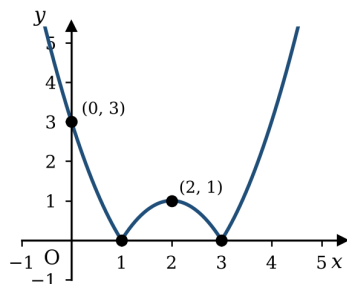
A1

(ii) $k = 1$ (the line passes through the local maximum)

A1

Answer: (i) $0 < k < 1$; (ii) $k = 1$

3 marks



28 (8 marks)

$$\text{Expand: } (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3 + \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{a}{c} + \frac{c}{a}\right)$$

M1 A1

$$\frac{a}{b} + \frac{b}{a} - 2 = \frac{a^2 + b^2 - 2ab}{ab} = \frac{(a - b)^2}{ab} \geq 0$$

M1 A1

So each bracket is at least 2 (similarly for the others)

A1

The expression is at least $3 + 6 = 9$

A1

Equality holds when $(a - b)^2 = (b - c)^2 = (c - a)^2 = 0$, i.e. $a = b = c$

M1 A1

Answer: $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$, with equality when $a = b = c$

8 marks

29 (9 marks)

$$\text{Square both sides: } (x^2 - 4)^2 - 4(x - 1)^2 \geq 0$$

M1 A1

$$\text{Difference of two squares: } (x^2 - 4 - 2x + 2)(x^2 - 4 + 2x - 2) \geq 0$$

M1 A1

$$(x^2 - 2x - 2)(x^2 + 2x - 6) \geq 0$$

A1

$$\text{Critical values: } 1 \pm \sqrt{3} \text{ and } -1 \pm \sqrt{7}$$

M1 A1

In increasing order $-1 - \sqrt{7}$, $1 - \sqrt{3}$, $-1 + \sqrt{7}$, $1 + \sqrt{3}$; sign table (positive, negative, positive, negative, positive)

M1

$$x \leq -1 - \sqrt{7} \text{ or } 1 - \sqrt{3} \leq x \leq -1 + \sqrt{7} \text{ or } x \geq 1 + \sqrt{3}$$

A1

Answer: $x \leq -1 - \sqrt{7}$ or $1 - \sqrt{3} \leq x \leq -1 + \sqrt{7}$ or $x \geq 1 + \sqrt{3}$

9 marks

30 (8 marks)

$x \neq -2$. Case $x < -2$: the numerator is non-negative and the denominator negative, so the left side is ≤ 0 and the inequality always holds

M1 A1

Case $x > -2$: multiply by $x + 2 > 0$: $|2x - 1| \leq x + 2$

M1

$$-(x + 2) \leq 2x - 1 \leq x + 2$$

A1

$$2x - 1 \leq x + 2 \text{ gives } x \leq 3; 2x - 1 \geq -x - 2 \text{ gives } x \geq -\frac{1}{3}$$

M1 A1

$$-\frac{1}{3} \leq x \leq 3 \text{ (within } x > -2)$$

A1

$$x < -2 \text{ or } -\frac{1}{3} \leq x \leq 3$$

A1

Answer: $x < -2$ or $-\frac{1}{3} \leq x \leq 3$

8 marks

Summation of Series**1 (2 marks)**

$$\Sigma r = \frac{n(n+1)}{2} \text{ with } n = 20: \frac{20 \times 21}{2}$$

M1

$$= 210$$

A1

Answer: 210

2 marks

2 (2 marks)

$$\Sigma r^2 = \frac{n(n+1)(2n+1)}{6} \text{ with } n = 10: \frac{10 \times 11 \times 21}{6}$$

M1

$$= 385$$

A1

Answer: 385

2 marks

3 (2 marks)

$$\Sigma r^3 = \frac{n^2(n+1)^2}{4} \text{ with } n = 8: \frac{64 \times 81}{4}$$

M1

$$= 1296$$

A1

Answer: 1296

2 marks

4 (2 marks)

$$\Sigma(2r + 3) = 2 \cdot \frac{n(n+1)}{2} + 3n$$

M1

$$= n^2 + n + 3n = n(n+4)$$

A1

$$\text{Answer: } \sum_{r=1}^n (2r + 3) = n(n+4)$$

2 marks

5 (2 marks)

$$3\Sigma r + \Sigma 1 = 3 \cdot \frac{12 \times 13}{2} + 12$$

M1

$$= 234 + 12 = 246$$

A1

$$\text{Answer: } 246$$

2 marks

6 (2 marks)

$$\Sigma_{r=11}^{20} r = \Sigma_{r=1}^{20} r - \Sigma_{r=1}^{10} r = 210 - 55$$

M1

$$= 155$$

A1

$$\text{Answer: } 155$$

2 marks

7 (4 marks)

$$\Sigma(r^2 + 3r) = \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}$$

M1 A1

$$= \frac{n(n+1)}{6} [(2n+1) + 9]$$

M1

$$= \frac{n(n+1)(2n+10)}{6} = \frac{n(n+1)(n+5)}{3}$$

A1

$$\text{Answer: } \sum_{r=1}^n (r^2 + 3r) = \frac{n(n+1)(n+5)}{3}$$

4 marks

8 (4 marks)

$$6\Sigma r^2 - 2\Sigma r + \Sigma 1 = n(n+1)(2n+1) - n(n+1) + n$$

M1 A1

$$= n(n+1)(2n) + n$$

M1

$$= n(2n^2 + 2n + 1)$$

A1

$$\text{Answer: } n(2n^2 + 2n + 1)$$

4 marks

9 (5 marks)

$$\Sigma r(r+2) = \Sigma r^2 + 2\Sigma r = \frac{n(n+1)(2n+1)}{6} + n(n+1)$$

M1 A1

$$= \frac{n(n+1)}{6} (2n+1+6) = \frac{n(n+1)(2n+7)}{6}$$

A1

$$n = 15: \frac{15 \times 16 \times 37}{6} = 1480$$

M1 A1

$$\text{Answer: } \sum_{r=1}^n r(r+2) = \frac{n(n+1)(2n+7)}{6}; 1480$$

5 marks

10 (4 marks)

$$\Sigma(r^3 + r) = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)}{2}$$

M1 A1

$$= \frac{n(n+1)}{4} [n(n+1) + 2]$$

M1

$$= \frac{n(n+1)(n^2 + n + 2)}{4}$$

A1

$$\text{Answer: } \frac{n(n+1)(n^2 + n + 2)}{4}$$

4 marks

11 (3 marks)

$$\begin{aligned}\sum_{r=n+1}^{2n} r &= \sum_{r=1}^{2n} r - \sum_{r=1}^n r && \text{M1} \\ &= \frac{2n(2n+1)}{2} - \frac{n(n+1)}{2} && \text{A1} \\ &= \frac{n(4n+2-n-1)}{2} = \frac{n(3n+1)}{2} && \text{A1} \\ \text{Answer: } \sum_{r=n+1}^{2n} r &= \frac{n(3n+1)}{2} && \text{3 marks}\end{aligned}$$

12 (4 marks)

$$\begin{aligned}\text{The } r\text{th term is } r(r+2), \text{ so the sum is } \sum_{r=1}^{20} r(r+2) &&& \text{M1} \\ &= \frac{20 \times 21 \times 47}{6} && \text{M1 A1} \\ &= 3290 && \text{A1} \\ \text{Answer: } 3290 &&& \text{4 marks}\end{aligned}$$

13 (4 marks)

$$\begin{aligned}\frac{n(n+1)}{2} > 1000, \text{ so } n(n+1) > 2000 &&& \text{M1} \\ n = 44: 44 \times 45/2 = 990 < 1000; n = 45: 45 \times 46/2 = 1035 > 1000 &&& \text{M1 A1} \\ n = 45 &&& \text{A1} \\ \text{Answer: } 45 &&& \text{4 marks}\end{aligned}$$

14 (5 marks)

$$\begin{aligned}r(r+1)(r+2) &= r^3 + 3r^2 + 2r && \text{B1} \\ \Sigma &= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{2} + n(n+1) && \text{M1 A1} \\ &= \frac{n(n+1)}{4} [n(n+1) + 2(2n+1) + 4] && \text{M1} \\ &= \frac{n(n+1)}{4} (n^2 + 5n + 6) = \frac{1}{4} n(n+1)(n+2)(n+3) && \text{A1} \\ \text{Answer: } \sum_{r=1}^n r(r+1)(r+2) &= \frac{1}{4} n(n+1)(n+2)(n+3) && \text{5 marks}\end{aligned}$$

15 (5 marks)

$$\begin{aligned}\Sigma(2r-1)^2 &= 4\Sigma r^2 - 4\Sigma r + n && \text{M1 A1} \\ &= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n && \text{M1} \\ &= \frac{n}{3} [2(n+1)(2n+1) - 6(n+1) + 3] = \frac{n}{3} (4n^2 - 1) && \text{M1} \\ &= \frac{1}{3} n(2n-1)(2n+1) && \text{A1} \\ \text{Answer: } 1^2 + 3^2 + \dots + (2n-1)^2 &= \frac{1}{3} n(2n-1)(2n+1) && \text{5 marks}\end{aligned}$$

16 (6 marks)

$$\begin{aligned}(\text{a}) \Sigma(3r^2 - r) &= \frac{n(n+1)(2n+1)}{2} - \frac{n(n+1)}{2} && \text{M1 A1} \\ &= \frac{n(n+1)}{2} (2n) = n^2(n+1) && \text{A1} \\ \text{Answer: } \sum_{r=1}^n (3r^2 - r) &= n^2(n+1) && \text{3 marks} \\ (\text{b}) n^2(n+1) &= 1100 && \text{M1} \\ n = 10 \text{ gives } 100 \times 11 = 1100, \text{ and } n^2(n+1) \text{ is increasing, so } n &= 10 && \text{A1 A1} \\ \text{Answer: } n &= 10 && \text{3 marks}\end{aligned}$$

17 (8 marks)

$$\sum_{r=1}^n r(r+p)(r+q) = \sum r^3 + (p+q)\sum r^2 + pq\sum r \quad \text{M1}$$

$$= \frac{n^2(n+1)^2}{4} + \frac{(p+q)n(n+1)(2n+1)}{6} + \frac{pq \cdot n(n+1)}{2} \quad \text{M1 A1}$$

$$= \frac{n(n+1)}{12} [3n(n+1) + 2(p+q)(2n+1) + 6pq] \quad \text{M1}$$

$$(n+2)(3n+13) = 3n^2 + 19n + 26; \text{ coefficient of } n: 3 + 4(p+q) = 19, \text{ so } p+q = 4 \quad \text{M1 A1}$$

$$\text{Constant: } 2(p+q) + 6pq = 26, \text{ so } pq = 3 \quad \text{A1}$$

$$p = 1 \text{ and } q = 3 \quad \text{A1}$$

$$\text{Answer: } p = 1, q = 3 \quad \text{8 marks}$$

18 (5 marks)

$$\sum (r^3 + r^2) = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} \quad \text{M1 A1}$$

$$= \frac{n(n+1)}{12} [3n(n+1) + 2(2n+1)] \quad \text{M1 A1}$$

$$= \frac{n(n+1)}{12} (3n^2 + 7n + 2) = \frac{1}{12} n(n+1)(n+2)(3n+1) \quad \text{A1}$$

$$\text{Answer: } \sum_{r=1}^n r^2(r+1) = \frac{1}{12} n(n+1)(n+2)(3n+1) \quad \text{5 marks}$$

19 (7 marks)

$$\sum (2r)^2 = 4\sum r^2 = \frac{4n(n+1)(2n+1)}{6} \quad \text{M1 A1}$$

$$= \frac{2}{3} n(n+1)(2n+1) \quad \text{A1}$$

$$n = 20: \text{ even sum} = \frac{2}{3} \times 20 \times 21 \times 41 = 11\,480 \quad \text{M1 A1}$$

$$n = 20: \text{ odd sum} = \frac{1}{3} \times 20 \times 39 \times 41 = 10\,660 \quad \text{M1}$$

$$\text{Difference} = 820 \quad \text{A1}$$

$$\text{Answer: } 2^2 + 4^2 + 6^2 + \dots + (2n)^2 = \frac{2}{3} n(n+1)(2n+1); 820 \quad \text{7 marks}$$

20 (6 marks)

$$\sum_{r=1}^{12} r^2 = \frac{12 \times 13 \times 25}{6} \quad \text{M1 A1}$$

$$= 650 \quad \text{A1}$$

$$\sum_{r=1}^{13} r^2 = 819 < 1000 \text{ and } \sum_{r=1}^{14} r^2 = 1015 \quad \text{M1 A1}$$

$$\text{So 14 layers are needed} \quad \text{A1}$$

$$\text{Answer: 650 tins; 14 layers} \quad \text{6 marks}$$

21 (7 marks)

$$\text{The first factors are } 5, 7, 9, \dots \text{ and the second factors are } 2, 5, 8, \dots, \text{ so the } r\text{th term is } (2r+3)(3r-1) \quad \text{B1 B1}$$

$$(2r+3)(3r-1) = 6r^2 + 7r - 3 \quad \text{M1}$$

$$\text{Sum} = n(n+1)(2n+1) + \frac{7n(n+1)}{2} - 3n \quad \text{M1 A1}$$

$$= \frac{n}{2} [2(2n^2 + 3n + 1) + 7(n+1) - 6] \quad \text{M1}$$

$$= \frac{n}{2} (4n^2 + 13n + 3) = \frac{1}{2} n(n+3)(4n+1) \quad \text{A1}$$

$$\text{Answer: } \frac{1}{2} n(n+3)(4n+1) \quad \text{7 marks}$$

22 (5 marks)

$$\sum r^2 = \frac{n(n+1)(2n+1)}{6}$$

B1

 $\sum 3^r$ is geometric with first term 3, common ratio 3, n terms

M1

$$\text{Sum} = \frac{3(3^n - 1)}{3 - 1} = \frac{3^{n+1} - 3}{2}$$

M1 A1

$$\text{Total} = \frac{n(n+1)(2n+1)}{6} + \frac{3^{n+1} - 3}{2}$$

A1

$$\text{Answer: } \frac{n(n+1)(2n+1)}{6} + \frac{3^{n+1} - 3}{2}$$

5 marks

23 (7 marks)

The rth term is $r(r+1)^2 = r^3 + 2r^2 + r$

B1 M1

$$\Sigma = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}$$

M1 A1

$$= \frac{n(n+1)}{12} [3n(n+1) + 4(2n+1) + 6]$$

M1

$$= \frac{n(n+1)}{12} (3n^2 + 11n + 10) = \frac{1}{12} n(n+1)(n+2)(3n+5)$$

A1 A1

$$\text{Answer: } \frac{1}{12} n(n+1)(n+2)(3n+5)$$

7 marks

24 (6 marks)

$$\sum_{r=1}^m u_r = \frac{m(m+1)(2m+1)}{2} - \frac{5m(m+1)}{2} + 2m$$

M1 A1

$$\text{Simplify: } = \frac{m}{2} [(m+1)(2m+1) - 5(m+1) + 4] = \frac{m}{2} (2m^2 - 2m) = m^2(m-1)$$

M1 A1

$$\sum_{r=n+1}^{2n} (2n)^2(2n-1) - n^2(n-1)$$

M1

$$= n^2 [4(2n-1) - (n-1)] = n^2(7n-3)$$

A1

$$\text{Answer: } \sum_{r=n+1}^{2n} u_r = n^2(7n-3)$$

6 marks

25 (6 marks)

(a) $(r+1)^2 = r^2 + 2r + 1$ and the sum of the 1s is n, not 1; the middle term 2r has also been left out

E1 E1

Answer: Terms 2r omitted and the constant summed incorrectly

2 marks

$$(b) \sum (r^2 + 2r + 1) = \frac{n(n+1)(2n+1)}{6} + n(n+1) + n$$

M1 A1

$$= \frac{n}{6} [(n+1)(2n+1) + 6(n+1) + 6] = \frac{n}{6} (2n^2 + 9n + 13)$$

M1 A1

$$\text{Answer: } \frac{n}{6} (2n^2 + 9n + 13)$$

4 marks

26 (7 marks)

$$(2r+1)(r+2) = 2r^2 + 5r + 2$$

B1

$$\text{With upper limit } n-1: \sum r = \frac{(n-1)n}{2} \text{ and } \sum r^2 = \frac{(n-1)n(2n-1)}{6}$$

M1 A1

$$\text{Sum} = \frac{(n-1)n(2n-1)}{3} + \frac{5(n-1)n}{2} + 2(n-1)$$

M1

$$= \frac{n-1}{6} [2n(2n-1) + 15n + 12]$$

M1 A1

$$= \frac{1}{6} (n-1)(4n^2 + 13n + 12)$$

A1

$$\text{Answer: } \sum_{r=1}^{n-1} (2r+1)(r+2) = \frac{1}{6} (n-1)(4n^2 + 13n + 12)$$

7 marks

27 (6 marks)

$$(a) (2r - 1)^2 - (2r)^2 = (2r - 1 - 2r)(2r - 1 + 2r) = -(4r - 1)$$

M1 A1

$$\text{Answer: } (2r - 1)^2 - (2r)^2 = 1 - 4r$$

3 marks

$$(b) \text{ Group the terms in pairs: the sum is } \sum_{r=1}^n (1 - 4r)$$

M1 A1

$$= n - 4 \cdot \frac{n(n+1)}{2}$$

M1

$$= n - 2n(n+1) = -n(2n+1)$$

A1

$$\text{Answer: } 1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2 = -n(2n+1)$$

4 marks

28 (8 marks)

$$\text{The } r\text{th term is } r(n+1-r), \text{ so the sum is } \sum_{r=1}^n r(n+1-r)$$

M1 A1

$$= (n+1)\sum r - \sum r^2 = \frac{n(n+1)^2}{2} - \frac{n(n+1)(2n+1)}{6}$$

M1 A1

$$= \frac{n(n+1)}{6} [3(n+1) - (2n+1)]$$

M1

$$= \frac{n(n+1)(n+2)}{6}$$

A1

$$n = 20: \frac{20 \times 21 \times 22}{6} = 1540$$

M1 A1

$$\text{Answer: } 1 \times n + 2 \times (n-1) + 3 \times (n-2) + \dots + n \times 1 = \frac{1}{6} n(n+1)(n+2); 1540$$

8 marks

29 (10 marks)

$$\sum (2r-1)^3 = \sum (8r^3 - 12r^2 + 6r - 1)$$

M1 A1

$$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$$

M1 A1

$$= n[2n(n+1)^2 - 2(n+1)(2n+1) + 3(n+1) - 1] = n(2n^3 - n)$$

M1 A1

$$= n^2(2n^2 - 1)$$

A1

$$21 \text{ is the 11th odd integer and } 39 \text{ is the 20th: sum} = 20^2(799) - 10^2(199)$$

M1 A1

$$= 319\,600 - 19\,900 = 299\,700$$

A1

$$\text{Answer: } n^2(2n^2 - 1); 299\,700$$

10 marks

30 (11 marks)

$$(a) \sum_{r=1}^n \frac{r(r+1)}{2} = \frac{1}{2}(\sum r^2 + \sum r)$$

M1 A1

$$= \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right]$$

M1

$$= \frac{n(n+1)}{12} (2n+1+3) = \frac{n(n+1)(n+2)}{6}$$

M1 A1

$$\text{Answer: a pyramid with } n \text{ layers contains } \frac{1}{6} n(n+1)(n+2) \text{ cannonballs}$$

5 marks

$$(b) n(n+1)(n+2) = 1320$$

M1

$$n = 10: 10 \times 11 \times 12 = 1320, \text{ and the expression is increasing, so } n = 10$$

A1 A1

$$\text{Answer: } 10 \text{ layers}$$

3 marks

$$(c) \text{ Layers 1 to 5: } \frac{5 \times 6 \times 7}{6} = 35$$

M1 A1

$$220 - 35 = 185$$

A1

$$\text{Answer: } 185$$

3 marks

Method of Differences**1 (2 marks)**

$$\frac{1}{r} - \frac{1}{r+1} = \frac{(r+1) - r}{r(r+1)}$$

M1

$$= \frac{1}{r(r+1)}$$

A1

$$\text{Answer: } \frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r+1}$$

2 marks

2 (2 marks)

$$(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{6}) \quad \text{M1}$$

The middle terms cancel, leaving $1 - \frac{1}{6} = \frac{5}{6}$ A1

Answer: $\frac{5}{6}$ 2 marks

3 (2 marks)

Terms: $(2^2 - 1^2) + (3^2 - 2^2) + \dots + ((n+1)^2 - n^2)$ M1

All the middle terms cancel, leaving $(n+1)^2 - 1 = n^2 + 2n$ A1

Answer: $\sum_{r=1}^n [f(r+1) - f(r)] = n^2 + 2n$ 2 marks

4 (2 marks)

$$\ln\left(\frac{r+1}{r}\right) = \ln(r+1) - \ln r \quad \text{M1}$$

The sum telescopes to $\ln(n+1) - \ln 1 = \ln(n+1)$ A1

Answer: $\sum_{r=1}^n \ln\left(\frac{r+1}{r}\right) = \ln(n+1)$ 2 marks

5 (2 marks)

$$1 = A(r+2) + Br: r=0 \text{ gives } A = \frac{1}{2}; r=-2 \text{ gives } B = -\frac{1}{2} \quad \text{M1}$$

$$\frac{1}{r(r+2)} = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+2}\right) \quad \text{A1}$$

Answer: $\frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+2}\right)$ 2 marks

6 (2 marks)

$$\text{As } n \rightarrow \infty, \frac{1}{n+1} \rightarrow 0 \quad \text{M1}$$

The sum to infinity is 1 A1

Answer: 1 2 marks

7 (4 marks)

$$\text{Write out: } (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{n} - \frac{1}{n+1}) \quad \text{M1 A1}$$

Cancelling consecutive terms leaves $1 - \frac{1}{n+1}$ M1

$$= \frac{n}{n+1} \quad \text{A1}$$

Answer: $\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$ 4 marks

8 (6 marks)

$$\text{Write out the terms: } \frac{1}{2}[(1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \dots + (\frac{1}{n} - \frac{1}{n+2})] \quad \text{M1 A1}$$

The terms $\frac{1}{3}, \dots, \frac{1}{n}$ cancel, leaving $1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$ M1 A1

$$\text{Sum} = \frac{3}{4} - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right) = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)} \quad \text{A1}$$

$$\text{Sum to infinity} = \frac{3}{4} \quad \text{B1}$$

Answer: $\frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$; sum to infinity $\frac{3}{4}$ 6 marks

9 (5 marks)

$$1 = A(2r + 1) + B(2r - 1): A = \frac{1}{2}, B = -\frac{1}{2}$$

M1 A1

$$\Sigma = \frac{1}{2}[(1 - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{5}) + \dots + (\frac{1}{2n-1} - \frac{1}{2n+1})]$$

M1

$$= \frac{1}{2}(1 - \frac{1}{2n+1}) = \frac{n}{2n+1}$$

A1 A1

$$\text{Answer: } \frac{n}{2n+1}$$

5 marks

10 (6 marks)

$$\frac{1}{r^2 - 1} = \frac{1}{(r-1)(r+1)} = \frac{1}{2}(\frac{1}{r-1} - \frac{1}{r+1})$$

M1 A1

$$\Sigma = \frac{1}{2}[(1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + \dots + (\frac{1}{n-1} - \frac{1}{n+1})]$$

M1

$$= \frac{1}{2}(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1})$$

M1 A1

$$= \frac{3}{4} - \frac{2n+1}{2n(n+1)}$$

A1

$$\text{Answer: } \sum_{r=2}^n \frac{1}{r^2 - 1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$$

6 marks

11 (5 marks)

$$\text{Multiply by the conjugate: } \frac{\sqrt{r+1} - \sqrt{r}}{(\sqrt{r+1} + \sqrt{r})(\sqrt{r+1} - \sqrt{r})} = \frac{\sqrt{r+1} - \sqrt{r}}{(r+1) - r}$$

M1 A1

$$= \sqrt{r+1} - \sqrt{r}$$

A1

$$\Sigma = (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + \dots + (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1} - 1$$

M1 A1

$$\text{Answer: } \sqrt{n+1} - 1$$

5 marks

12 (4 marks)

$$(a) (r+1)! = (r+1) \times r!$$

B1

$$(r+1)! - r! = r! (r+1 - 1) = r \times r!$$

A1

$$\text{Answer: } (r+1)! - r! = r \times r!$$

2 marks

$$(b) \Sigma[(r+1)! - r!] = (2! - 1!) + (3! - 2!) + \dots + ((n+1)! - n!)$$

M1

$$= (n+1)! - 1$$

A1

$$\text{Answer: } (n+1)! - 1$$

2 marks

13 (4 marks)

$$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

M1

$$\Sigma = \frac{1}{11} - \frac{1}{21}$$

M1 A1

$$= \frac{10}{231}$$

A1

$$\text{Answer: } \frac{10}{231}$$

4 marks

14 (8 marks)

$$(a) \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2) - r}{r(r+1)(r+2)}$$

M1 A1

$$= \frac{2}{r(r+1)(r+2)}, \text{ so the identity holds}$$

A1

$$\text{Answer: } \frac{1}{r(r+1)(r+2)} \equiv \frac{1}{2}[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}]$$

3 marks

$$(b) \text{ Let } f(r) = \frac{1}{r(r+1)}, \text{ so the sum is } \frac{1}{2}\Sigma[f(r) - f(r+1)]$$

M1

$$= \frac{1}{2}[f(1) - f(n+1)] = \frac{1}{2}[\frac{1}{2} - \frac{1}{(n+1)(n+2)}]$$

M1 A1

$$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

A1

$$\text{Sum to infinity} = \frac{1}{4}$$

B1

$$\text{Answer: } \frac{1}{4} - \frac{1}{2(n+1)(n+2)}; \frac{1}{4}$$

5 marks

15 (6 marks)

$$f(r+1) - f(r) = (r-1)2^{r+1} - (r-2)2^r$$

M1

$$= 2^r[2(r-1) - (r-2)] = r \times 2^r$$

M1 A1

$$\Sigma[f(r+1) - f(r)] = f(n+1) - f(1)$$

M1

$$= (n-1)2^{n+1} + 2$$

A1 A1

$$\text{Answer: } (n-1)2^{n+1} + 2$$

6 marks

16 (5 marks)

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2}$$

M1 A1

$$= \frac{2r+1}{r^2(r+1)^2}$$

A1

$$\Sigma = (1 - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{9}) + \dots + (\frac{1}{n^2} - \frac{1}{(n+1)^2}) = 1 - \frac{1}{(n+1)^2}$$

M1 A1

$$\text{Answer: } 1 - \frac{1}{(n+1)^2}$$

5 marks

17 (6 marks)

$$\frac{1}{r(r+3)} = \frac{1}{3}(\frac{1}{r} - \frac{1}{r+3})$$

M1 A1

$$\Sigma = \frac{1}{3}[(1 - \frac{1}{4}) + (\frac{1}{2} - \frac{1}{5}) + (\frac{1}{3} - \frac{1}{6}) + \dots + (\frac{1}{n} - \frac{1}{n+3})]$$

M1

$$\text{Remaining terms: } 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

M1 A1

$$= \frac{11}{18} - \frac{1}{3}(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3})$$

A1

$$\text{Answer: } \frac{11}{18} - \frac{1}{3}(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3})$$

6 marks

18 (5 marks)

$$\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{(r+1) - 1}{(r+1)!}$$

M1 A1

$$= \frac{r}{(r+1)!}$$

A1

$$\Sigma = (\frac{1}{1!} - \frac{1}{2!}) + (\frac{1}{2!} - \frac{1}{3!}) + \dots + (\frac{1}{n!} - \frac{1}{(n+1)!}) = 1 - \frac{1}{(n+1)!}$$

M1 A1

$$\text{Answer: } 1 - \frac{1}{(n+1)!}$$

5 marks

19 (6 marks)

$$(r+1)^3 - r^3 = 3r^2 + 3r + 1$$

B1

$$\Sigma[(r+1)^3 - r^3] = (n+1)^3 - 1$$

M1 A1

$$3\Sigma r^2 = (n+1)^3 - 1 - 3 \cdot \frac{n(n+1)}{2} - n$$

M1

$$= \frac{n(n+1)(2n+1)}{2}, \text{ so } \Sigma r^2 = \frac{1}{6} n(n+1)(2n+1)$$

M1 A1

$$\text{Answer: } \sum_{r=1}^n r^2 = \frac{1}{6} n(n+1)(2n+1)$$

6 marks

20 (5 marks)

$$\Sigma = \frac{n}{n+1} \text{ (from the partial fractions } \frac{1}{r} - \frac{1}{r+1} \text{)}$$

M1 A1

$$\frac{n}{n+1} > 0.99, \text{ so } n > 0.99n + 0.99$$

M1

$$0.01n > 0.99, \text{ so } n > 99$$

A1

Least n is 100

A1

Answer: n = 100

5 marks

21 (7 marks)

$$\frac{1}{r2^{r-1}} - \frac{1}{(r+1)2^r} = \frac{2(r+1) - r}{r(r+1)2^r}$$

M1 A1

$$= \frac{r+2}{r(r+1)2^r}$$

A1

$$\text{Let } f(r) = \frac{1}{r2^{r-1}}; \text{ the sum is } \Sigma[f(r) - f(r+1)] = f(1) - f(n+1)$$

M1

$$= 1 - \frac{1}{(n+1)2^n}$$

A1

As $n \rightarrow \infty$ the second term tends to 0, so the sum to infinity is 1

B1 B1

Answer: $1 - \frac{1}{(n+1)2^n}$; sum to infinity 1

7 marks

22 (7 marks)(a) Common denominator $r(r+1)(r+2)(r+3)$: numerator $(r+3) - r = 3$

M1 A1

A1

$$\text{Answer: } \frac{1}{r(r+1)(r+2)} - \frac{1}{(r+1)(r+2)(r+3)} = \frac{3}{r(r+1)(r+2)(r+3)}$$

3 marks

(b) With $f(r) = \frac{1}{r(r+1)(r+2)}$, the sum is $\frac{1}{3} \Sigma[f(r) - f(r+1)]$

M1

$$= \frac{1}{3}[f(1) - f(n+1)]$$

M1

$$= \frac{1}{3}\left[\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right]$$

A1

$$= \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$$

A1

$$\text{Answer: } \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$$

4 marks

23 (6 marks)(a) The terms cancel in steps of two, not one: $\frac{1}{r+2}$ cancels with $\frac{1}{r}$ two terms later

E1

So the terms $\frac{1}{2}$ (from $r=2$) and the last-but-one $\frac{1}{n+1}$ do not cancel

E1

Answer: Two terms at each end do not cancel

2 marks

$$(b) \Sigma = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

M1 A1

$$n=2: \frac{1}{3} + \frac{1}{8} = \frac{11}{24}; \text{ formula } \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2} \times \frac{11}{12} = \frac{11}{24}$$

M1 A1

$$\text{Answer: } \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

4 marks

24 (9 marks)(a) $(r+1)^4 = r^4 + 4r^3 + 6r^2 + 4r + 1$ by the binomial theorem

M1 A1

$$\text{Answer: } (r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$$

2 marks

$$(b) \Sigma[(r+1)^4 - r^4] = (n+1)^4 - 1$$

M1 A1

$$4\Sigma r^3 = (n+1)^4 - 1 - n(n+1)(2n+1) - 2n(n+1) - n$$

M1 A1

$$\text{Expand: } (n+1)^4 - 1 = n^4 + 4n^3 + 6n^2 + 4n, n(n+1)(2n+1) = 2n^3 + 3n^2 + n, 2n(n+1) = 2n^2 + 2n$$

M1

$$4\Sigma r^3 = n^4 + 2n^3 + n^2 = n^2(n+1)^2$$

M1 A1

Answer: $\sum_{r=1}^n r^3 = \frac{1}{4} n^2(n+1)^2$

7 marks

25 (7 marks)

$$1 - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - 1}{(r+1)^2} = \frac{r(r+2)}{(r+1)^2}$$

M1 A1

$$\ln\left(1 - \frac{1}{(r+1)^2}\right) = \ln r + \ln(r+2) - 2 \ln(r+1) = [\ln(r+2) - \ln(r+1)] - [\ln(r+1) - \ln r]$$

A1

Let $g(r) = \ln(r+1) - \ln r$, so the sum is $\sum[g(r+1) - g(r)] = g(n+1) - g(1)$

M1

$$= \ln\left(\frac{n+2}{n+1}\right) - \ln 2 = \ln\left(\frac{n+2}{2(n+1)}\right), \text{ so } a = 2$$

A1

As $n \rightarrow \infty$, $\frac{n+2}{n+1} \rightarrow 1$, so the sum to infinity is $\ln \frac{1}{2} = -\ln 2$

M1 A1

Answer: $\ln\left(\frac{n+2}{2(n+1)}\right)$; $a = 2$; sum to infinity $-\ln 2$

7 marks

26 (5 marks)

$$\sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^n r^2$$

M1

$$= \frac{2n(2n+1)}{3} - \frac{n(n+1)}{3}$$

A1

$$= \frac{2n(2n+1) - n(2n+1)}{3}$$

M1 A1

$$= \frac{n}{3}$$

A1

Answer: $\sum_{r=n+1}^{2n} \frac{1}{r(r+1)} = \frac{n}{(n+1)(2n+1)}$

5 marks

27 (9 marks)

(a) Let $A = \arctan(r+1)$, $B = \arctan r$: $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

M1

$$= \frac{(r+1) - r}{1 + r(r+1)} = \frac{1}{r^2 + r + 1}$$

M1 A1

$A - B$ lies between 0 and $\frac{\pi}{2}$, so $A - B = \arctan\left(\frac{1}{r^2 + r + 1}\right)$

A1

Answer: $\arctan(r+1) - \arctan r = \arctan\left(\frac{1}{r^2 + r + 1}\right)$

4 marks

(b) $\sum[\arctan(r+1) - \arctan r] = \arctan(n+1) - \arctan 1$

M1 A1

$$= \arctan(n+1) - \frac{\pi}{4}$$

A1

As $n \rightarrow \infty$, $\arctan(n+1) \rightarrow \frac{\pi}{2}$, so the sum to infinity is $\frac{\pi}{4}$

M1 A1

Answer: $\arctan(n+1) - \frac{\pi}{4}$; $\frac{\pi}{4}$

5 marks

28 (11 marks)

(a) $\cos(A - B) - \cos(A + B) = 2 \sin A \sin B$

M1

With $A = r\theta$ and $B = \frac{\theta}{2}$: $\cos\left((r - \frac{1}{2})\theta\right) - \cos\left((r + \frac{1}{2})\theta\right) = 2 \sin r\theta \sin\left(\frac{\theta}{2}\right)$

A1

Answer: $2 \sin\left(\frac{\theta}{2}\right) \sin r\theta = \cos\left((r - \frac{1}{2})\theta\right) - \cos\left((r + \frac{1}{2})\theta\right)$

2 marks

(b) $\sum \sin r\theta = \frac{1}{2 \sin(\frac{\theta}{2})} \sum[\cos\left((r - \frac{1}{2})\theta\right) - \cos\left((r + \frac{1}{2})\theta\right)]$

M1

Telescoping: $= \frac{\cos(\frac{\theta}{2}) - \cos((n + \frac{1}{2})\theta)}{2 \sin(\frac{\theta}{2})}$

M1 A1

$\cos A - \cos B = 2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{B-A}{2}\right)$ with $A = \frac{\theta}{2}$, $B = (n + \frac{1}{2})\theta$

M1

$$= \frac{2 \sin(\frac{(n+1)\theta}{2}) \sin(\frac{n\theta}{2})}{2 \sin(\frac{\theta}{2})}, \text{ which is the required result}$$

A1

$$\text{Answer: } \sum_{r=1}^n \sin r\theta = \frac{\sin(\frac{n\theta}{2}) \sin(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})}$$

5 marks

$$(c) \text{ Use } \theta = 30^\circ, n = 5: \frac{\sin 75^\circ \sin 90^\circ}{\sin 15^\circ}$$

M1 A1

$$\sin 75^\circ = \cos 15^\circ, \text{ so the value is } \cot 15^\circ = 2 + \sqrt{3}$$

M1 A1

$$\text{Answer: } 2 + \sqrt{3}$$

4 marks

29 (11 marks)

$$(a) r = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2)$$

M1

$$r = -1: -1 = 2A, \text{ so } A = -\frac{1}{2}; r = -2: -2 = -B, \text{ so } B = 2; r = -3: -3 = 2C, \text{ so } C = -\frac{3}{2}$$

A1 A1 A1

$$\text{Answer: } -\frac{1}{2(r+1)} + \frac{2}{r+2} - \frac{3}{2(r+3)}$$

4 marks

$$(b) \text{ Write as } -\frac{1}{2}\left[\frac{1}{r+1} - \frac{1}{r+2}\right] + \frac{3}{2}\left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

M1 A1

$$\Sigma = -\frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+2}\right) + \frac{3}{2}\left(\frac{1}{3} - \frac{1}{n+3}\right)$$

M1 A1

$$= -\frac{1}{4} + \frac{1}{2(n+2)} + \frac{1}{2} - \frac{3}{2(n+3)} = \frac{1}{4} + \frac{1}{2(n+2)} - \frac{3}{2(n+3)}$$

A1

$$\text{Answer: } \sum_{r=1}^n \frac{r}{(r+1)(r+2)(r+3)} = \frac{1}{4} + \frac{1}{2(n+2)} - \frac{3}{2(n+3)}$$

5 marks

$$(c) \text{ As } n \rightarrow \infty \text{ both fractions tend to } 0$$

M1

$$\text{Sum to infinity} = \frac{1}{4}$$

A1

$$\text{Answer: } \frac{1}{4}$$

2 marks

30 (9 marks)

$$(a) \frac{r}{(r+1)!} = \frac{r(r+2)}{(r+2)!}$$

M1

$$\frac{r(r+2) - (r+1)}{(r+2)!} = \frac{r^2 + 2r - r - 1}{(r+2)!}$$

M1 A1

$$= \frac{r^2 + r - 1}{(r+2)!}$$

A1

$$\text{Answer: } \frac{r^2 + r - 1}{(r+2)!} \equiv \frac{r}{(r+1)!} - \frac{r+1}{(r+2)!}$$

4 marks

$$(b) \text{ Let } f(r) = \frac{r}{(r+1)!}: \text{ the term is } f(r) - f(r+1)$$

M1

$$\Sigma = f(1) - f(n+1) = \frac{1}{2} - \frac{n+1}{(n+2)!}$$

M1 A1

$$\text{As } n \rightarrow \infty, \frac{n+1}{(n+2)!} \rightarrow 0 \text{ (factorial grows faster than } n)$$

M1

$$\text{So the sum to infinity is } \frac{1}{2}$$

A1

$$\text{Answer: } \frac{1}{2} - \frac{n+1}{(n+2)!}; \frac{1}{2}$$

5 marks

Recurrence Relations**1 (3 marks)**

$$u_2 = 2(3) - 1 = 5$$

B1

$$u_3 = 2(5) - 1 = 9, u_4 = 2(9) - 1 = 17$$

B1 B1

$$\text{Answer: } 5, 9, 17$$

3 marks

2 (2 marks)

Arithmetic sequence with first term 2 and common difference 4

M1

$$u_n = 2 + 4(n - 1) = 4n - 2$$

A1

Answer: $u_n = 4n - 2$

2 marks

3 (2 marks)The limit L satisfies $L = \frac{1}{2}L + 3$

M1

$$L = 6$$

A1

Answer: 6

2 marks

4 (2 marks)Each term is 3 more than the previous term, so $u_{n+1} = u_n + 3$ with $u_1 = 2$

B1

$$u_n = 2 + 3(n - 1) = 3n - 1$$

B1

Answer: $u_{n+1} = u_n + 3$, $u_1 = 2$; $u_n = 3n - 1$

2 marks

5 (2 marks)

$$u_3 = 3(3) - 2(1) = 7$$

B1

$$u_4 = 3(7) - 2(3) = 15$$

B1

Answer: 7 and 15

2 marks

6 (3 marks) $(m - 2)(m - 3) = 0$, so $m = 2$ or $m = 3$

M1 A1

$$u_n = A \times 2^n + B \times 3^n$$

A1

Answer: $u_n = A \cdot 2^n + B \cdot 3^n$

3 marks

7 (5 marks)Particular solution (constant): $L = 3L + 2$, so $L = -1$

M1 A1

Complementary function: $u_n = A \times 3^{n-1}$; general solution $u_n = A \times 3^{n-1} - 1$

M1

$$u_1 = 1: A - 1 = 1, \text{ so } A = 2$$

M1

$$u_n = 2 \times 3^{n-1} - 1$$

A1

Answer: $u_n = 2 \times 3^{n-1} - 1$

5 marks

8 (4 marks)

$$u_n = u_1 + \sum_{k=1}^{n-1} 2k$$

M1

$$= 3 + 2 \times \frac{(n-1)n}{2}$$

M1 A1

$$= n^2 - n + 3$$

A1

Answer: $u_n = n^2 - n + 3$

4 marks

9 (5 marks)Auxiliary equation $m^2 - 5m + 6 = 0$, so $m = 2$ or 3 ; $u_n = A \times 2^n + B \times 3^n$

M1 A1

$$u_0 = 1: A + B = 1; u_1 = 0: 2A + 3B = 0$$

M1

$$A = 3 \text{ and } B = -2$$

A1

$$u_n = 3 \times 2^n - 2 \times 3^n$$

A1

Answer: $u_n = 3 \times 2^n - 2 \times 3^n$

5 marks

10 (5 marks)Auxiliary equation $m^2 - 6m + 9 = 0$, so $m = 3$ (repeated)

M1 A1

$$u_n = (A + Bn)3^n$$

M1

$$u_0 = 2 \text{ gives } A = 2; u_1 = 9 \text{ gives } 3(2 + B) = 9, \text{ so } B = 1$$

A1

$$u_n = (n + 2)3^n$$

A1

Answer: $u_n = (n + 2)3^n$

5 marks

11 (7 marks)

$$u_2 = 3k - 2$$

B1

$$u_3 = k(3k - 2) - 2 = 3k^2 - 2k - 2$$

M1

$$3k^2 - 2k - 2 = 19, \text{ so } (3k + 7)(k - 3) = 0$$

M1 A1

$$k = 3 \text{ or } k = -\frac{7}{3}$$

A1

$$\text{For } k = 3: u_4 = 3 \times 19 - 2 = 55$$

M1 A1

$$\text{Answer: } k = 3 \text{ or } k = -\frac{7}{3}; u_4 = 55$$

7 marks

12 (3 marks)

$$u_{n+1} = 2^{n+1} + 1$$

B1

$$2u_n - 1 = 2(2^n + 1) - 1 = 2^{n+1} + 1$$

M1

These are equal, so the recurrence relation is satisfied

A1

$$\text{Answer: } u_n = 2^n + 1 \text{ satisfies the recurrence relation } u_{n+1} = 2u_n - 1$$

3 marks

13 (3 marks)

$$u_{n+1} = 3 \times 2^{n+1} + 5 = 2(3 \times 2^n) + 5$$

M1

$$3 \times 2^n = u_n - 5, \text{ so } u_{n+1} = 2(u_n - 5) + 5$$

M1

$$u_{n+1} = 2u_n - 5$$

A1

$$\text{Answer: } u_{n+1} = 2u_n - 5$$

3 marks

14 (6 marks)

$$n = 1: 2^2 - 3 = 1 = u_1, \text{ so the result is true for } n = 1$$

B1

$$\text{Assume } u_k = 2^{k+1} - 3 \text{ for some } k \geq 1$$

M1

$$u_{k+1} = 2u_k + 3 = 2(2^{k+1} - 3) + 3$$

M1 A1

$$= 2^{k+2} - 3$$

A1

True for $n = 1$, and if true for $n = k$ then true for $n = k + 1$, so true for all positive integers n

E1

$$\text{Answer: } u_n = 2^{n+1} - 3 \text{ for all positive integers } n$$

6 marks

15 (8 marks)

$$(a) 3y_n = y_{n+1} + 2$$

M1

$$y_n = \frac{y_{n+1} + 2}{3}$$

A1

$$\text{Answer: } y_n = \frac{y_{n+1} + 2}{3}$$

2 marks

$$(b) y_4 = \frac{82 + 2}{3} = 28$$

M1 A1

$$y_3 = 10, y_2 = 4, y_1 = 2$$

A1

$$\text{Answer: } y_4 = 28, y_3 = 10, y_2 = 4, y_1 = 2$$

3 marks

$$(c) \text{ The constant solution of } y = 3y - 2 \text{ is } y = 1, \text{ so } y_n = 1 + A \times 3^{n-1}$$

M1 A1

$$y_1 = 2 \text{ gives } A = 1, \text{ so } y_n = 3^{n-1} + 1$$

A1

$$\text{Answer: } y_n = 3^{n-1} + 1$$

3 marks

16 (8 marks)

$$\text{Complementary function: } m^2 - 5m + 6 = 0, \text{ so } A \times 2^n + B \times 3^n$$

M1 A1

$$\text{Try a particular solution } u_n = an + b: a(n + 2) + b - 5[a(n + 1) + b] + 6(an + b) = 4n$$

M1

$$2an - 3a + 2b = 4n, \text{ so } a = 2 \text{ and } b = 3$$

M1 A1

$$\text{General solution } u_n = A \times 2^n + B \times 3^n + 2n + 3$$

A1

$$u_0 = 1: A + B + 3 = 1; u_1 = 7: 2A + 3B + 5 = 7$$

M1

$$A = -8, B = 6, \text{ so } u_n = 6 \times 3^n - 8 \times 2^n + 2n + 3$$

A1

$$\text{Answer: } u_n = 6 \times 3^n - 8 \times 2^n + 2n + 3$$

8 marks

17 (7 marks)

Auxiliary equation $m^2 - 2m + 2 = 0$, so $m = 1 \pm i$

M1 A1

$1 \pm i = \sqrt{2}(\cos \frac{\pi}{4} \pm i \sin \frac{\pi}{4})$, so $u_n = 2^{n/2}(A \cos \frac{n\pi}{4} + B \sin \frac{n\pi}{4})$

M1 A1

$u_0 = 0$ gives $A = 0$

A1

$u_1 = 1: \sqrt{2} B \sin \frac{\pi}{4} = B = 1$

M1

$u_n = 2^{n/2} \sin \frac{n\pi}{4}$

A1

Answer: $u_n = 2^{n/2} \sin \frac{n\pi}{4}$

7 marks

18 (8 marks)

$B_{n+1} = 1.03B_n - 3000$

B1

Particular solution: $L = 1.03L - 3000$, so $L = 100\,000$

M1 A1

$B_n = A(1.03)^n + 100\,000$; $B_0 = 20\,000$ gives $A = -80\,000$

M1 A1

$B_n = 100\,000 - 80\,000(1.03)^n$

A1

$B_n \leq 0$ when $(1.03)^n \geq 1.25$, so $n \geq \frac{\ln 1.25}{\ln 1.03} = 7.55\dots$

M1

The loan is repaid after 8 years

A1

Answer: $B_n = 100\,000 - 80\,000(1.03)^n$; 8 years

8 marks

19 (9 marks)

(a) $(n+2)2^{n+2} - 4(n+1)2^{n+1} + 4n \times 2^n$

M1

$= 2^n[4(n+2) - 8(n+1) + 4n]$

M1 A1

$= 2^n[0] = 0$

A1

Answer: $u_n = n \times 2^n$ satisfies the recurrence relation $u_{n+2} - 4u_{n+1} + 4u_n = 0$

4 marks

(b) Auxiliary equation $(m-2)^2 = 0$, so $u_n = (A+Bn)2^n$

M1 A1

$u_0 = 3$ gives $A = 3$; $u_1 = 2$: $2(3+B) = 2$, so $B = -2$

M1 A1

$u_n = (3-2n)2^n$

A1

Answer: $u_n = (3-2n)2^n$

5 marks

20 (9 marks)

(a) $u_2 = \frac{1}{1-3} = -\frac{1}{2}$

B1

$u_3 = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$, $u_4 = \frac{1}{1-\frac{2}{3}} = 3$

B1 B1

Answer: $u_2 = -\frac{1}{2}$, $u_3 = \frac{2}{3}$, $u_4 = 3$

3 marks

(b) $u_4 = u_1$, so the sequence repeats with period 3

M1

$100 = 3 \times 33 + 1$, so $u_{100} = u_1 = 3$

A1

Answer: $u_{100} = 3$

2 marks

(c) Each block of three terms sums to $3 - \frac{1}{2} + \frac{2}{3} = \frac{19}{6}$

M1 A1

$\sum_{r=1}^{99} u_r = 33 \times \frac{19}{6} = \frac{209}{2}$, and adding $u_{100} = 3$ gives $\frac{215}{2}$

M1 A1

Answer: $\frac{215}{2}$

4 marks

21 (7 marks)

Define $u_0 = 0$ so that $u_2 = u_1 + u_0$

B1

Auxiliary equation $m^2 - m - 1 = 0$, so $m = \frac{1 \pm \sqrt{5}}{2}$

M1 A1

$u_n = A\varphi^n + B\psi^n$ where $\varphi = \frac{1 + \sqrt{5}}{2}$, $\psi = \frac{1 - \sqrt{5}}{2}$

M1

$u_0 = 0$ gives $B = -A$; $u_1 = 1$ gives $A(\varphi - \psi) = 1$

M1

$\varphi - \psi = \sqrt{5}$, so $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$

A1

$u_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)$

A1

Answer: $u_n = \frac{1}{\sqrt{5}}[(\frac{1 + \sqrt{5}}{2})^n - (\frac{1 - \sqrt{5}}{2})^n]$

7 marks

22 (10 marks)

(a) $u_1 = 1 < 2$

B1

Assume $u_k < 2$; then $u_{k+1} = \sqrt{2 + u_k} < \sqrt{4} = 2$

M1 A1

Hence true for all n by induction

E1

Answer: $u_n < 2$ for all positive integers n

4 marks

(b) $u_{n+1}^2 - u_n^2 = 2 + u_n - u_n^2 = (2 - u_n)(1 + u_n)$

M1 A1

This is positive since $0 < u_n < 2$, so $u_{n+1} > u_n$

M1 A1

Answer: $u_{n+1} > u_n$ for all n

4 marks

(c) $L = \sqrt{2 + L}$, so $L^2 - L - 2 = 0$

M1

$(L - 2)(L + 1) = 0$, and $L > 0$ so $L = 2$

A1

Answer: 2

2 marks

23 (6 marks)

$v_{n+1} = \frac{1}{u_{n+1}} = \frac{3 - u_n}{u_n} = 3v_n - 1$

M1 A1

Particular solution $L = 3L - 1$, so $L = \frac{1}{2}$; $v_n = A \times 3^{n-1} + \frac{1}{2}$

M1 A1

$v_1 = 1$ gives $A = \frac{1}{2}$

M1

$v_n = \frac{1}{2}(3^{n-1} + 1)$, so $u_n = \frac{2}{3^{n-1} + 1}$

A1

Answer: $u_n = \frac{2}{3^{n-1} + 1}$

6 marks

24 (7 marks)

(a) $A \times 2^n + B \times 2^n = (A + B)2^n$ has only one arbitrary constant, so it cannot match two initial conditions

E1 E1

Answer: Only one independent constant

2 marks

(b) $u_n = (A + Bn)2^n$

M1 A1

$u_0 = 1$ gives $A = 1$; $u_1 = 4$: $2(1 + B) = 4$, so $B = 1$

M1 A1

$u_n = (n + 1)2^n$

A1

Answer: $u_n = (n + 1)2^n$

5 marks

25 (9 marks)Complementary function: $m^2 - 3m + 2 = 0$, so $A + B \times 2^n$ M1 A1 2^n is part of the complementary function, so try $u_n = kn \times 2^n$ M1 A1

$$k(n+2)2^{n+2} - 3k(n+1)2^{n+1} + 2kn \times 2^n = 2^n$$
 M1

$$k \times 2^n [4(n+2) - 6(n+1) + 2n] = 2k \times 2^n, \text{ so } k = \frac{1}{2}$$
 A1

$$u_n = A + B \times 2^n + n \times 2^{n-1}; u_0 = 1 \text{ and } u_1 = 2 \text{ give } A = 1, B = 0$$
 M1 A1

$$u_n = 1 + n \times 2^{n-1}$$
 A1

Answer: $u_n = 1 + n \times 2^{n-1}$ 9 marks

26 (7 marks)Complementary function $A \times 2^n$ B1Try a particular solution $u_n = an + b$: $a(n+1) + b = 2an + 2b + n$ M1 n : $a = 2a + 1$, so $a = -1$; constant: $a + b = 2b$, so $b = -1$ A1 A1

$$u_n = A \times 2^n - n - 1$$
 M1

$$u_1 = 1: 2A - 2 = 1, \text{ so } A = \frac{3}{2}$$
 M1

$$u_n = 3 \times 2^{n-1} - n - 1$$
 A1

Answer: $u_n = 3 \times 2^{n-1} - n - 1$ 7 marks

27 (10 marks)**(a)** $n = 0$ and $n = 1$ are true: $\sin 0 = 0$ and $\sin \theta$ B1Assume $u_k = \sin k\theta$ and $u_{k+1} = \sin(k+1)\theta$ M1

$$u_{k+2} = 2 \cos \theta \sin(k+1)\theta - \sin k\theta$$
 M1

$$2 \cos \theta \sin(k+1)\theta = \sin(k+2)\theta + \sin k\theta$$
 M1 A1

So $u_{k+2} = \sin(k+2)\theta$; true for all n by induction A1 E1

Answer: $u_n = \sin n\theta$ for all integers $n \geq 0$ 7 marks

(b) Terms: $0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$ B1 B1The sequence is periodic with period 6 B1

Answer: $0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$; period 6 3 marks

28 (12 marks)

(a) $b_n = \frac{a_{n+1} - 3a_n}{2}$ M1 A1

Then $b_{n+1} = a_n + 2b_n$ gives $\frac{a_{n+2} - 3a_{n+1}}{2} = a_n + a_{n+1} - 3a_n$ M1 A1

$$a_{n+2} = 5a_{n+1} - 4a_n$$
 A1

Answer: $a_{n+2} = 5a_{n+1} - 4a_n$ 5 marks

(b) Auxiliary equation $m^2 - 5m + 4 = 0$, so $m = 1$ or 4 ; $a_n = A + B \times 4^n$ M1 A1

$$a_0 = 1 \text{ and } a_1 = 3: A + B = 1, A + 4B = 3, \text{ so } B = \frac{2}{3}, A = \frac{1}{3}$$
 M1 A1

$$a_n = \frac{1}{3}(2 \times 4^n + 1)$$
 A1

$$b_n = \frac{a_{n+1} - 3a_n}{2} = \frac{1}{3}(4^n - 1)$$
 M1 A1

Answer: $a_n = \frac{1}{3}(2 \times 4^n + 1), b_n = \frac{1}{3}(4^n - 1)$ 7 marks

29 (11 marks)**(a)** Length 1: 0, 1 so $a_1 = 2$; length 2: 00, 01, 10 so $a_2 = 3$ **B1 B1**A valid string of length $n + 2$ ends in 0 (preceded by any valid string of length $n + 1$: a_{n+1} ways)**M1 E1 E1**or ends in 01 (preceded by any valid string of length n : a_n ways)So $a_{n+2} = a_{n+1} + a_n$ **A1****Answer: $a_1 = 2$ and $a_2 = 3$** **6 marks****(b)** $a_3 = 5$, $a_4 = 8$, $a_5 = 13$, $a_6 = 21$, $a_7 = 34$ **M1 A1** $a_8 = 55$, $a_9 = 89$, $a_{10} = 144$ **A1****Answer: 144****3 marks****(c)** Each of the $2^{10} = 1024$ sequences is equally likely; 1 represents a head**M1**Probability = $\frac{144}{1024} = \frac{9}{64}$ **A1****Answer: $\frac{9}{64}$** **2 marks****30 (10 marks)****(a)** $\frac{1}{\alpha} = \frac{2}{3 + \sqrt{5}} = \frac{2(3 - \sqrt{5})}{9 - 5} = \frac{3 - \sqrt{5}}{2}$ **M1 A1** $\alpha + \frac{1}{\alpha} = \frac{3 + \sqrt{5}}{2} + \frac{3 - \sqrt{5}}{2} = 3$ **A1****Answer: $\alpha + \frac{1}{\alpha} = 3$** **3 marks****(b)** $n = 1$: $\alpha + \alpha^{-1} = 3 = u_1$ **B1**Assume $u_k = \alpha^{2^{k-1}} + \alpha^{-2^{k-1}}$ **M1** $u_{k+1} = (\alpha^{2^{k-1}} + \alpha^{-2^{k-1}})^2 - 2$ **M1** $= \alpha^{2^k} + 2 + \alpha^{-2^k} - 2 = \alpha^{2^k} + \alpha^{-2^k}$ **A1**True for $n = 1$ and $n = k$ implies $n = k + 1$, so true for all n **E1****Answer: $u_n = \alpha^{2^{n-1}} + \alpha^{-2^{n-1}}$ for all positive integers n** **5 marks****(c)** $u_2 = 9 - 2 = 7$, $u_3 = 49 - 2 = 47$ **M1** $u_4 = 47^2 - 2 = 2207$ **A1****Answer: 2207****2 marks****Maclaurin and Taylor Series****1 (2 marks)**Replace x by $2x$: $1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6}$ **M1** $= 1 + 2x + 2x^2 + \frac{4}{3}x^3$ **A1****Answer: $1 + 2x + 2x^2 + \frac{4}{3}x^3$** **2 marks****2 (2 marks)** $\sin u = u - \frac{u^3}{6}$ with $u = 2x$ **M1** $= 2x - \frac{4}{3}x^3$ **A1****Answer: $2x - \frac{4}{3}x^3$** **2 marks**

3 (2 marks)

$$\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} \text{ with } u = 3x$$

M1

$$= 3x - \frac{9}{2}x^2 + 9x^3$$

A1

$$\text{Answer: } 3x - \frac{9}{2}x^2 + 9x^3$$

2 marks

4 (2 marks)

$$\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} \text{ with } u = 3x$$

M1

$$= 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4$$

A1

$$\text{Answer: } 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4$$

2 marks

5 (3 marks)

$$f' = \frac{1}{1+x}, f'' = -\frac{1}{(1+x)^2}, f''' = \frac{2}{(1+x)^3}$$

M1

$$f(0) = 0, f'(0) = 1, f''(0) = -1, f'''(0) = 2$$

A1

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

A1

$$\text{Answer: } x - \frac{x^2}{2} + \frac{x^3}{3}$$

3 marks

6 (3 marks)

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

B1

$$x = 0.1: 1 - 0.1 + 0.005 - 0.000167$$

M1

$$= 0.9048$$

A1

$$\text{Answer: } 0.9048$$

3 marks

7 (4 marks)

$$(1 + x + \frac{x^2}{2} + \dots)(x - \frac{x^3}{6} + \dots)$$

M1

$$x^2: x \times x = x^2; x^3: \frac{x^2}{2} \times x - \frac{x^3}{6}$$

M1 A1

$$= x + x^2 + \frac{1}{3}x^3$$

A1

$$\text{Answer: } x + x^2 + \frac{1}{3}x^3$$

4 marks

8 (5 marks)

$$\ln(\cosh x) = \ln(1 + u) \text{ where } u = \frac{x^2}{2} + \frac{x^4}{24}$$

M1

$$\ln(1 + u) = u - \frac{u^2}{2} + \dots$$

M1

$$= \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8}$$

M1 A1

$$= \frac{x^2}{2} - \frac{x^4}{12}$$

A1

$$\text{Answer: } \frac{x^2}{2} - \frac{x^4}{12}$$

5 marks

9 (4 marks)

$$(1 + x + \frac{x^2}{2} + \frac{x^3}{6})(1 - \frac{x^2}{2}) \quad \text{M1}$$

$$x^2: \frac{1}{2} - \frac{1}{2} = 0; x^3: \frac{1}{6} - \frac{1}{2} \quad \text{M1 A1}$$

$$= 1 + x - \frac{1}{3}x^3 \quad \text{A1}$$

Answer: $1 + x - \frac{1}{3}x^3$ 4 marks

10 (5 marks)

$$f' = \sec^2 x, f'' = 2 \sec^2 x \tan x \quad \text{M1 A1}$$

$$f''' = 4 \sec^2 x \tan^2 x + 2 \sec^4 x, \text{ so } f'''(0) = 0 + 2 = 2 \quad \text{M1 A1}$$

$$f(0) = 0, f'(0) = 1, f''(0) = 0: \tan x = x + \frac{2}{6}x^3 = x + \frac{1}{3}x^3 \quad \text{A1}$$

Answer: $x + \frac{1}{3}x^3$ 5 marks

11 (7 marks)

$$(a) \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) \quad \text{M1}$$

$$(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots) - (-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots) \quad \text{M1 A1}$$

The even powers cancel and the odd powers double A1

Answer: $\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots$ for $|x| < 1$ 4 marks

$$(b) \frac{1+x}{1-x} = 3 \text{ when } x = \frac{1}{2} \quad \text{B1}$$

$$2\left(\frac{1}{2}\right) + \frac{2}{3}\left(\frac{1}{8}\right) + \frac{2}{5}\left(\frac{1}{32}\right) = 1 + \frac{1}{12} + \frac{1}{80} \quad \text{M1}$$

$$\approx 1.096 \quad \text{A1}$$

Answer: 1.096 3 marks

12 (5 marks)

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24}, \text{ so } \ln(\cos x) = \ln(1+u) \text{ with } u = -\frac{x^2}{2} + \frac{x^4}{24} \quad \text{M1 A1}$$

$$\ln(1+u) = u - \frac{u^2}{2} + \dots \quad \text{M1}$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} \quad \text{M1}$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} \quad \text{A1}$$

Answer: $\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} + \dots$ 5 marks

13 (6 marks)

$$1 - x - 2x^2 = (1 - 2x)(1 + x) \quad \text{B1}$$

$$\ln(1 - 2x) = -2x - 2x^2 - \frac{8}{3}x^3 \quad \text{M1 A1}$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} \quad \text{M1}$$

$$\text{Sum: } -x - \frac{5}{2}x^2 - \frac{7}{3}x^3 \quad \text{A1}$$

$$\ln(1 - 2x) \text{ needs } |2x| < 1 \text{ and } \ln(1 + x) \text{ needs } |x| < 1, \text{ so } |x| < \frac{1}{2} \quad \text{B1}$$

Answer: $-x - \frac{5}{2}x^2 - \frac{7}{3}x^3; |x| < \frac{1}{2}$ 6 marks

14 (5 marks)

$$\text{The } n\text{th term is } \frac{(-1)^{n+1}}{n \cdot 2^n} = \frac{(-1)^{n+1}}{n} \left(\frac{1}{2}\right)^n$$

M1 A1

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \text{ is valid for } -1 < x \leq 1, \text{ so } x = \frac{1}{2} \text{ may be used}$$

M1 B1

$$\text{Sum} = \ln\left(1 + \frac{1}{2}\right) = \ln \frac{3}{2}$$

A1

$$\text{Answer: } \ln \frac{3}{2}$$

5 marks

15 (6 marks)

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3, \sin 3x = 3x - \frac{9}{2}x^3$$

B1 B1

$$x^2: 2 \times 3 = 6; x^3: 2 \times 3 = 6 - \frac{9}{2}$$

M1 A1

$$x^4: \frac{4}{3} \times 3 - 2 \times \frac{9}{2} = 4 - 9$$

M1

$$= 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4$$

A1

$$\text{Answer: } 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4$$

6 marks

16 (4 marks)

$$\int_0^{0.1} (3x + 6x^2 + \frac{3}{2}x^3 - 5x^4) dx = \left[\frac{3}{2}x^2 + 2x^3 + \frac{3}{8}x^4 - x^5\right]_0^{0.1}$$

M1 A1

$$= 0.015 + 0.002 + 0.0000375 - 0.00001$$

M1

$$= 0.01703$$

A1

$$\text{Answer: } 0.01703$$

4 marks

17 (6 marks)

$$y'(0) = 0 + 1 = 1$$

B1

$$y'' = 1 + y', \text{ so } y''(0) = 2$$

M1 A1

$$y''' = y'', \text{ so } y'''(0) = 2; y^{(4)} = y''', \text{ so } y^{(4)}(0) = 2$$

M1 A1

$$y = 1 + x + \frac{2}{2}x^2 + \frac{2}{6}x^3 + \frac{2}{24}x^4 = 1 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4$$

A1

$$\text{Answer: } 1 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4$$

6 marks

18 (6 marks)

$$\sin x = x - \frac{x^3}{6}; \text{ let } u = x - \frac{x^3}{6}$$

B1

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24}$$

M1

$$u^2 = x^2 - \frac{x^4}{3}, u^3 = x^3, u^4 = x^4$$

M1 A1

$$\text{Collect terms: } x^3: -\frac{1}{6} + \frac{1}{6} = 0; x^4: -\frac{1}{6} + \frac{1}{24} = -\frac{1}{8}$$

M1

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

A1

$$\text{Answer: } 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

6 marks

19 (4 marks)

$$(1+x)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) \quad \text{M1}$$

$$= x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 \quad \text{A1}$$

$$\frac{(1+x)\ln(1+x) - x}{x^2} = \frac{1}{2} - \frac{1}{6}x + \dots \quad \text{M1}$$

$$\text{The limit is } \frac{1}{2} \quad \text{A1}$$

$$\text{Answer: } x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4; \text{ limit } \frac{1}{2} \quad \text{4 marks}$$

20 (6 marks)

$$f' = \frac{1}{2\sqrt{x}}, f'' = -\frac{1}{4x^{3/2}}, f''' = \frac{3}{8x^{5/2}} \quad \text{M1 A1}$$

$$f(4) = 2, f'(4) = \frac{1}{4}, f''(4) = -\frac{1}{32}, f'''(4) = \frac{3}{256} \quad \text{A1}$$

$$\sqrt{x} = 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} \quad \text{A1}$$

$$x = 4.4: 2 + 0.1 - 0.0025 + 0.000125 = 2.097625 \quad \text{M1}$$

$$= 2.0976 \text{ (4 d.p.)} \quad \text{A1}$$

$$\text{Answer: } 2.0976 \quad \text{6 marks}$$

21 (12 marks)

$$\text{(a) } \frac{dy}{dx} = \frac{\cos x}{1 + \sin x} \quad \text{M1 A1}$$

$$\frac{d^2y}{dx^2} = \frac{-\sin x(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2} = -\frac{1 + \sin x}{(1 + \sin x)^2} \quad \text{M1 A1}$$

$$= -\frac{1}{1 + \sin x}, \text{ so } (1 + \sin x) \frac{d^2y}{dx^2} = -1 \quad \text{A1}$$

$$\text{Answer: } (1 + \sin x) \frac{d^2y}{dx^2} = -1 \quad \text{5 marks}$$

$$\text{(b) Differentiate: } (1 + \sin x)y''' + \cos x y'' = 0 \quad \text{M1 A1}$$

$$\text{Again: } (1 + \sin x)y^{(4)} + 2\cos x y''' - \sin x y'' = 0 \quad \text{M1 A1}$$

$$y(0) = 0, y'(0) = 1, y''(0) = -1, y'''(0) = 1, y^{(4)}(0) = -2 \quad \text{M1 A1}$$

$$y = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} \quad \text{A1}$$

$$\text{Answer: } x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} \quad \text{7 marks}$$

22 (7 marks)

$$f' = \sec^2 x, f'' = 2\sec^2 x \tan x, f''' = 4\sec^2 x \tan^2 x + 2\sec^4 x \quad \text{M1 A1 A1}$$

$$\text{At } \frac{\pi}{4}: \tan = 1, \sec^2 = 2: f = 1, f' = 2, f'' = 4, f''' = 8 + 8 = 16 \quad \text{M1 A1}$$

$$\tan x = 1 + 2h + 2h^2 + \frac{8}{3}h^3, \text{ where } h = x - \frac{\pi}{4} \quad \text{M1 A1}$$

$$\text{Answer: } 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 \quad \text{7 marks}$$

23 (7 marks)

$$1 + e^x = 2(1 + v) \text{ with } v = \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{12} + \frac{x^4}{48}$$

M1 A1

$$\ln(1 + e^x) = \ln 2 + v - \frac{v^2}{2} + \frac{v^3}{3} - \frac{v^4}{4}$$

M1

$$v^2 = \frac{x^2}{4} + \frac{x^3}{4} + \frac{7x^4}{48}, v^3 = \frac{x^3}{8} + \frac{3x^4}{16}, v^4 = \frac{x^4}{16}$$

M1 A1

$$x^3: \frac{1}{12} - \frac{1}{8} + \frac{1}{24} = 0$$

M1

$$x^4: \frac{1}{48} - \frac{7}{96} + \frac{1}{16} - \frac{1}{64} = -\frac{1}{192}$$

A1

$$\text{Answer: } \ln(1 + e^x) = \ln 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$$

7 marks

24 (8 marks)

$$(a) f' = \cos x, f'' = -\sin x, f''' = -\cos x$$

M1

$$\text{At } \frac{\pi}{6}: f = \frac{1}{2}, f' = \frac{\sqrt{3}}{2}, f'' = -\frac{1}{2}, f''' = -\frac{\sqrt{3}}{2}$$

A1

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}h - \frac{h^2}{4} - \frac{\sqrt{3}}{12}h^3, \text{ with } h = x - \frac{\pi}{6}$$

M1 A1

$$\text{Answer: } \frac{1}{2} + \frac{\sqrt{3}}{2}h - \frac{h^2}{4} - \frac{\sqrt{3}}{12}h^3$$

4 marks

$$(b) 31^\circ = 30^\circ + 1^\circ, \text{ so } h = 0.0174533$$

B1

$$0.5 + 0.866025(0.0174533) - 0.25(0.00030462) - 0.144338(0.0000053166)$$

M1

$$= 0.5 + 0.015115 - 0.0000762 - 0.0000008$$

M1

$$= 0.51504$$

A1

$$\text{Answer: } 0.51504$$

4 marks

25 (6 marks)

$$(a) \text{ The third term has not been treated like the second: } (2x)^3 \text{ should replace } x^3$$

E1 E1

$$\text{Answer: } x^3 \text{ should be } (2x)^3$$

2 marks

$$(b) \ln(1 + 2x) = 2x - 2x^2 + \frac{8}{3}x^3$$

M1 A1

$$\text{Valid for } -\frac{1}{2} < x \leq \frac{1}{2}$$

B1 B1

$$\text{Answer: } 2x - 2x^2 + \frac{8}{3}x^3; -\frac{1}{2} < x \leq \frac{1}{2}$$

4 marks

26 (7 marks)

$$\text{At } x = 0: y''(0) = -y(0) = -1$$

M1 A1

$$\text{Differentiate: } y''' + x y'' + 2y' = 0, \text{ so } y'''(0) = 0$$

M1 A1

$$\text{Differentiate again: } y^{(4)} + x y''' + 3y'' = 0, \text{ so } y^{(4)}(0) = 3$$

M1 A1

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{8}$$

A1

$$\text{Answer: } 1 - \frac{x^2}{2} + \frac{x^4}{8}$$

7 marks

27 (11 marks)

$$(a) \frac{dy}{dx} = \frac{1}{1+x^2} e^{\arctan x}$$

M1 A1

$$\text{So } (1+x^2) \frac{dy}{dx} = y$$

A1

$$\text{Answer: } (1+x^2) \frac{dy}{dx} = y$$

3 marks

$$(b) \text{ Differentiate: } (1+x^2)y'' + 2xy' = y'$$

M1 A1

$$\text{Again: } (1+x^2)y''' + (4x-1)y'' + 2y' = 0, \text{ and once more: } (1+x^2)y^{(4)} + (6x-1)y''' + 6y'' = 0$$

M1 A1

$$y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = -1, y^{(4)}(0) = -7$$

M1 A1 A1

$$y = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24}$$

A1

Answer: $1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24}$

8 marks

28 (9 marks)

(a) $n = 1$: $f' = \frac{1}{1+x}$, matches $(-1)^0 0!/(1+x)$

B1

Assume true for $n = k$; differentiate: $f^{(k+1)} = (-1)^{k-1}(k-1)!(-k)(1+x)^{-k-1}$

M1 A1

$= \frac{(-1)^k k!}{(1+x)^{k+1}}$, which is the formula for $n = k + 1$

A1

Hence true for all n by induction

E1

Answer: $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ for all positive integers n

5 marks

(b) $f^{(n)}(0) = (-1)^{n-1}(n-1)!$

M1 A1

Term in x^n is $\frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^{n-1}(n-1)!}{n!}x^n$

M1

$= \frac{(-1)^{n-1}}{n}x^n$

A1

Answer: the Maclaurin series of $f(x)$ has general term $\frac{(-1)^{n-1}x^n}{n}$

4 marks

29 (10 marks)

(a) $e^x - e^{-x} = 2x + \frac{x^3}{3} + \dots$, so numerator $= \frac{x^3}{3} + \dots$

M1 A1

$x - \sin x = \frac{x^3}{6} + \dots$

B1

Limit $= \frac{x^3}{3 \div x^3/6} = 2$

M1 A1

Answer: 2

5 marks

(b) $\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} + \dots$

B1

$2 \ln(\cos x) + x^2 = -\frac{x^4}{6} + \dots$

M1 A1

Limit $= -\frac{1}{6}$

M1 A1

Answer: $-\frac{1}{6}$

5 marks

30 (9 marks)

(a) Replace x by $-x^2$ in e^x : $1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}$

M1 A1

Answer: $1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}$

2 marks

(b) $\int_0^{1/2} = [x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}]_0^{1/2}$

M1 A1

$= \frac{1}{2} - \frac{1}{24} + \frac{1}{320} - \frac{1}{5376}$

M1

$= 0.4613$

A1

Answer: 0.4613

4 marks

(c) The terms alternate in sign and decrease in magnitude, so the error is smaller than the first omitted term

E1

$\frac{1}{2^9/216} = \frac{1}{110\,592} \approx 9 \times 10^{-6} < 10^{-5}$

M1 A1

Answer: Error $< 9 \times 10^{-6}$

3 marks

Further Binomial Expansions

1 (2 marks)

$$(1+x)^{-1} = 1 - x + x^2 - x^3$$

M1 A1

$$\text{Answer: } 1 - x + x^2 - x^3$$

2 marks

2 (3 marks)

$$1 + \left(-\frac{1}{2}\right)(-2x) + \frac{(-1)}{2(-3/2)/2}(-2x)^2 + \frac{(-1)}{2(-3/2)(-5/2)/6}(-2x)^3$$

M1 A1

$$= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3$$

A1

$$\text{Answer: } 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3$$

3 marks

3 (3 marks)

$$1 + (-2)(3x) + \frac{(-2)(-3)}{2}(3x)^2 + \frac{(-2)(-3)(-4)}{6}(3x)^3$$

M1 A1

$$= 1 - 6x + 27x^2 - 108x^3$$

A1

$$\text{Answer: } 1 - 6x + 27x^2 - 108x^3$$

3 marks

4 (4 marks)

$$1 + \frac{1}{3}(4x) + \frac{(1)}{3(-2/3)/2}(4x)^2$$

M1 A1

$$= 1 + \frac{4}{3}x - \frac{16}{9}x^2$$

A1

$$\text{Valid for } |4x| < 1, \text{ i.e. } |x| < \frac{1}{4}$$

B1

$$\text{Answer: } 1 + \frac{4}{3}x - \frac{16}{9}x^2; |x| < \frac{1}{4}$$

4 marks

5 (4 marks)

$$\sqrt{4+x} = 2\left(1 + \frac{x}{4}\right)^{1/2}$$

M1

$$= 2\left[1 + \frac{1}{2}\left(\frac{x}{4}\right) - \frac{1}{8}\left(\frac{x}{4}\right)^2\right] = 2 + \frac{x}{4} - \frac{x^2}{64}$$

M1 A1

$$\text{Valid for } |x| < 4$$

B1

$$\text{Answer: } 2 + \frac{x}{4} - \frac{x^2}{64}; |x| < 4$$

4 marks

6 (4 marks)

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

M1 A1

$$x = 0.02: 1 + 0.01 - 0.00005 + 0.0000005$$

M1

$$= 1.009950$$

A1

$$\text{Answer: } 1.009950$$

4 marks

7 (5 marks)

$$(8+3x)^{1/3} = 2\left(1 + \frac{3x}{8}\right)^{1/3}$$

M1

$$= 2\left[1 + \frac{1}{3}\left(\frac{3x}{8}\right) - \frac{1}{9}\left(\frac{3x}{8}\right)^2\right]$$

M1 A1

$$= 2 + \frac{x}{4} - \frac{x^2}{32}$$

A1

$$\text{Valid for } |x| < \frac{8}{3}$$

B1

$$\text{Answer: } 2 + \frac{x}{4} - \frac{x^2}{32}; |x| < \frac{8}{3}$$

5 marks

8 (5 marks)

$$1 + 3x + \frac{(-3)(-4)}{2}x^2 + \frac{(-3)(-4)(-5)}{6}(-x)^3 = 1 + 3x + 6x^2 + 10x^3$$

M1 A1

$$\text{Coefficient of } x^n \text{ is } \frac{(n+1)(n+2)}{2}$$

M1 A1

$$n = 3: \frac{4 \times 5}{2} = 10$$

B1

$$\text{Answer: } 1 + 3x + 6x^2 + 10x^3; \frac{(n+1)(n+2)}{2}$$

5 marks

9 (4 marks)

$$(2 - x)^{-2} = \frac{1}{4}\left(1 - \frac{x}{2}\right)^{-2}$$

M1

$$= \frac{1}{4}\left[1 + 2\left(\frac{x}{2}\right) + 3\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right)^3\right]$$

M1 A1

$$= \frac{1}{4} + \frac{x}{4} + \frac{3x^2}{16} + \frac{x^3}{8}$$

A1

$$\text{Answer: } \frac{1}{4} + \frac{x}{4} + \frac{3x^2}{16} + \frac{x^3}{8}$$

4 marks

10 (4 marks)

$$1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16}$$

M1 A1

$$x = 0.1: 1 - 0.05 + 0.00375 - 0.0003125$$

M1

$$= 0.9534$$

A1

$$\text{Answer: } 0.9534$$

4 marks

11 (3 marks)

$$1 + \left(-\frac{3}{2}\right)(2x) + \frac{(-3)}{2\left(-5/2\right)/2}(2x)^2$$

M1 A1

$$= 1 - 3x + \frac{15}{2}x^2$$

A1

$$\text{Answer: } 1 - 3x + \frac{15}{2}x^2$$

3 marks

12 (4 marks)

$$\text{Coefficient} = \frac{(1)}{3(-2/3)(-5/3)/6}(-3)^3$$

M1 A1

$$= \frac{10}{162} \times (-27)$$

M1

$$= -\frac{5}{3}$$

A1

$$\text{Answer: } -\frac{5}{3}$$

4 marks

13 (4 marks)

$$(2 + 3x)^{-1} = \frac{1}{2}\left(1 + \frac{3x}{2}\right)^{-1}$$

M1

$$= \frac{1}{2}\left[1 - \frac{3x}{2} + \frac{9x^2}{4}\right] = \frac{1}{2} - \frac{3x}{4} + \frac{9x^2}{8}$$

M1 A1

$$|x| < \frac{2}{3}$$

B1

$$\text{Answer: } \frac{1}{2} - \frac{3x}{4} + \frac{9x^2}{8}; |x| < \frac{2}{3}$$

4 marks

14 (6 marks)

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}, (1-x)^{-1} = 1 + x + x^2 + x^3$$

B1 B1

$$x^2: 1 + \frac{1}{2} - \frac{1}{8} = \frac{11}{8}; x^3: 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16}$$

M1 A1

$$= 1 + \frac{3}{2}x + \frac{11}{8}x^2 + \frac{23}{16}x^3$$

A1

Valid for $|x| < 1$

B1

$$\text{Answer: } 1 + \frac{3}{2}x + \frac{11}{8}x^2 + \frac{23}{16}x^3; |x| < 1$$

6 marks

15 (7 marks)

$$5 + x = A(1 + 2x) + B(1 - x); x = 1 \text{ gives } A = 2; x = -\frac{1}{2} \text{ gives } B = 3$$

M1 A1 A1

$$2(1 - x)^{-1} = 2(1 + x + x^2 + x^3)$$

M1

$$3(1 + 2x)^{-1} = 3(1 - 2x + 4x^2 - 8x^3)$$

M1

$$= 5 - 4x + 14x^2 - 22x^3$$

A1

$$\text{Valid for } |x| < \frac{1}{2}$$

B1

$$\text{Answer: } 5 - 4x + 14x^2 - 22x^3; |x| < \frac{1}{2}$$

7 marks

16 (5 marks)

$$(4 - x)^{-1/2} = \frac{1}{2}(1 - \frac{x}{4})^{-1/2}$$

M1

$$= \frac{1}{2}[1 + \frac{x}{8} + \frac{3x^2}{128}] = \frac{1}{2} + \frac{x}{16} + \frac{3x^2}{256}$$

M1 A1

$$\text{Multiply by } (1 + 2x): \frac{1}{2} + (\frac{1}{16} + 1)x + (\frac{3}{256} + \frac{1}{8})x^2$$

M1

$$= \frac{1}{2} + \frac{17x}{16} + \frac{35x^2}{256}$$

A1

$$\text{Answer: } \frac{1}{2} + \frac{17x}{16} + \frac{35x^2}{256}$$

5 marks

17 (4 marks)

$$(1 - x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4$$

M1 A1

$$x = 0.1: 1 + 0.005 + 0.0000375$$

M1

$$= 1.00504$$

A1

$$\text{Answer: } 1.00504$$

4 marks

18 (4 marks)

$$(1 + x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81}$$

M1 A1

$$x = 0.06: 1 + 0.02 - 0.0004 + 0.0000133$$

M1

$$= 1.01961$$

A1

$$\text{Answer: } 1.01961$$

4 marks

19 (5 marks)

$$\text{Coefficient of } x^2 \text{ is } \frac{(-2)(-3)}{2}a^2 = 3a^2$$

M1 A1

$$3a^2 = 12, \text{ so } a = 2$$

A1

$$\text{Coefficient of } x^3 = \frac{(-2)(-3)(-4)}{6}(2)^3 = -32$$

M1 A1

$$\text{Answer: } a = 2; -32$$

5 marks

20 (6 marks)

$$na = -6$$

M1 A1

$$\frac{n(n-1)}{2}a^2 = 24, \text{ so } n(n-1)a^2 = 48$$

M1

$$\text{Divide by } (na)^2 = 36: \frac{n-1}{n} = \frac{4}{3}, \text{ so } n = -3$$

M1 A1

$$a = 2$$

A1

Answer: $a = 2, n = -3$

6 marks

21 (7 marks)

$$(a) 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{(-1)}{2(-3/2)/2}(-x^2)^2$$

M1 A1

$$= 1 + \frac{x^2}{2} + \frac{3x^4}{8}$$

A1

$$\text{Answer: } 1 + \frac{x^2}{2} + \frac{3x^4}{8}$$

3 marks

$$(b) \arcsin x = \int_0^x (1-t^2)^{-1/2} dt, \text{ since } \arcsin 0 = 0$$

M1

$$= x + \frac{x^3}{6} + \frac{3x^5}{40}$$

M1 A1 A1

$$\text{Answer: } x + \frac{x^3}{6} + \frac{3x^5}{40}$$

4 marks

22 (7 marks)

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

B1

$$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16}$$

M1 A1

$$x: \frac{1}{2} + \frac{1}{2} = 1; x^2: \frac{3}{8} + \frac{1}{4} - \frac{1}{8} = \frac{1}{2}$$

M1 A1

$$x^3: \frac{5}{16} + \frac{3}{16} - \frac{1}{16} + \frac{1}{16} = \frac{1}{2}$$

M1

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{2}$$

A1

$$\text{Answer: } 1 + x + \frac{x^2}{2} + \frac{x^3}{2}$$

7 marks

23 (6 marks)

$$(a) \frac{7}{5}\left(1 + \frac{1}{49}\right)^{1/2} = \frac{7}{5} \times \sqrt{50/49}$$

M1

$$= \frac{7}{5} \times \frac{5\sqrt{2}}{7} = \sqrt{2}$$

A1

$$\text{Answer: } \sqrt{2} = \frac{7}{5}\left(1 + \frac{1}{49}\right)^{1/2}$$

2 marks

$$(b) x = \frac{1}{49}: (1+x)^{1/2} \approx 1 + \frac{1}{98} - \frac{1}{8}\left(\frac{1}{49}\right)^2$$

M1 A1

$$= 1 + 0.010204 - 0.0000521 = 1.010152$$

M1

$$\sqrt{2} \approx 1.4 \times 1.010152 = 1.41421$$

A1

Answer: 1.41421

4 marks

24 (6 marks)

$$(a) \text{ The 2 must be factored out first, giving } 2^{-1}\left(1 + \frac{x}{2}\right)^{-1}$$

E1

$$\text{The binomial theorem was applied to } (1+x), \text{ not to } \left(1 + \frac{x}{2}\right)$$

E1

Answer: Did not factor out 2

2 marks

$$(b) \frac{1}{2}\left(1 - \frac{x}{2} + \frac{x^2}{4}\right)$$

M1 A1

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}; \text{ valid for } |x| < 2$$

A1 B1

Answer: $\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}; |x| < 2$

4 marks

25 (7 marks)

$$\frac{1}{1+x} + \frac{1}{2-x} = \frac{(2-x) + (1+x)}{(1+x)(2-x)} = \frac{3}{(1+x)(2-x)}$$

M1 A1

$$(1+x)^{-1} = 1 - x + x^2 - x^3$$

B1

$$(2-x)^{-1} = \frac{1}{2} \left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} \right)$$

M1

$$\text{Sum} = \frac{3}{2} - \frac{3x}{4} + \frac{9x^2}{8} - \frac{15x^3}{16}; \text{divide by 3}$$

M1

$$\frac{1}{2} - \frac{x}{4} + \frac{3x^2}{8} - \frac{5x^3}{16}$$

A1

Valid for $|x| < 1$

B1

Answer: $\frac{1}{2} - \frac{x}{4} + \frac{3x^2}{8} - \frac{5x^3}{16}; |x| < 1$

7 marks

26 (7 marks)

(a) $1 + \frac{1}{2}(4x) + \frac{3}{8}(16x^2) + \frac{5}{16}(64x^3)$

M1 A1

$$= 1 + 2x + 6x^2 + 20x^3$$

A1

Answer: $1 + 2x + 6x^2 + 20x^3$

3 marks

(b) $(1 + 2x + 6x^2 + 20x^3)^2$; x : 4; x^2 : $4 + 12 = 16$

M1 A1

$$x^3: 40 + 24 = 64$$

A1

$$(1 - 4x)^{-1} = 1 + 4x + 16x^2 + 64x^3, \text{ which agrees}$$

B1

Answer: $(1 + 2x + 6x^2 + 20x^3)^2 = 1 + 4x + 16x^2 + 64x^3 + \dots$, which is the expansion of $(1 - 4x)^{-1}$

4 marks

27 (8 marks)

$$\text{Coefficient} = \frac{(1 - 4x)^{-1/2}}{2(3/2)(5/2)\dots(\frac{2n-1}{2})/n!} 4^n$$

M1 A1

$$= \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{n!} \times 2^n$$

M1 A1

$$1 \times 3 \times \dots \times (2n-1) = \frac{(2n)!}{2 \times 4 \times \dots \times 2n} = \frac{(2n)!}{2^n n!}$$

M1 A1

$$\text{Coefficient} = \frac{(2n)!}{2^n n! n!} \times 2^n = \frac{(2n)!}{(n!)^2}$$

A1 A1

Answer: the coefficient of x^n in the expansion of $(1 - 4x)^{-1/2}$ is $\frac{(2n)!}{(n!)^2}$

8 marks

28 (10 marks)

(a) $(1 - x)^{-1/2} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \frac{35x^4}{128}$

M1 A1

$$(1 + x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128}$$

A1

Answer: $(1 - x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4$; $(1 + x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4$

3 marks

(b) $x^2: \frac{3}{8} + \frac{3}{8} - \frac{1}{4} = \frac{1}{2}$

M1 A1

$$x^4: 2\left(\frac{35}{128}\right) + \left(\frac{3}{8}\right)^2 - 2\left(\frac{1}{2}\right)\left(\frac{5}{16}\right)$$

M1 A1

$$= \frac{35}{64} + \frac{9}{64} - \frac{20}{64} = \frac{24}{64} = \frac{3}{8}$$

A1

Answer: the coefficients of x^2 and x^4 in the product are $\frac{1}{2}$ and $\frac{3}{8}$

5 marks

(c) $(1 - x)^{-1/2}(1 + x)^{-1/2} = (1 - x^2)^{-1/2}$

M1

Odd powers cancel in the product and the even coefficients $\frac{1}{2}$ and $\frac{3}{8}$ match $1 + \frac{x^2}{2} + \frac{3x^4}{8}$

A1

Answer: The product is $(1 - x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \dots$, which matches the expansion of $(1 - x^2)^{-1/2}$ 2 marks

29 (9 marks)

(a) $(1 - y)^{-1/2} = 1 + \frac{1}{2}y + \frac{3}{8}y^2 + \frac{5}{16}y^3 + \dots$ M1 A1

$\frac{1 \times 3}{2 \times 4} = \frac{3}{8}$ and $\frac{1 \times 3 \times 5}{2 \times 4 \times 6} = \frac{5}{16}$, so the series matches with $y = \frac{1}{4}$ M1 A1

Sum = $(1 - \frac{1}{4})^{-1/2} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ M1 A1

Answer: $y = \frac{1}{4}$; sum $\frac{2\sqrt{3}}{3}$ 6 marks

(b) $1 + 0.125 + 0.0234375 + 0.0048828 = 1.15332$ M1 A1

Exact sum 1.15470, so the four-term sum is too small by about 0.0014 (0.12%) A1

Answer: 1.1533 against 1.1547 3 marks

30 (12 marks)

(a) $(1 - t)^{-1/2} = 1 + \frac{t}{2} + \frac{3t^2}{8} + \dots$ with $t = \frac{v^2}{c^2}$ M1 A1

$K \approx mc^2(\frac{v^2}{2c^2} + \frac{3v^4}{8c^4})$

$= \frac{1}{2}mv^2 + \frac{3mv^4}{8c^2} = \frac{1}{2}mv^2(1 + \frac{3v^2}{4c^2})$ A1

Answer: for v much smaller than c , $K \approx \frac{1}{2}mv^2(1 + \frac{3v^2}{4c^2})$ 3 marks

(b) Ratio = $\frac{3v^2}{4c^2} = \frac{3}{4}(0.01)$ M1 A1

= 0.75% A1

Answer: Second term : first term = 0.75% 3 marks

(c) $K = mc^2[(1 - 0.36)^{-1/2} - 1] = mc^2(1.25 - 1) = 0.25mc^2$ M1 A1

Approximation: $\frac{1}{2}(0.36) + \frac{3}{8}(0.1296) = 0.18 + 0.0486 = 0.2286$ M1 A1

Error = $\frac{0.25 - 0.2286}{0.25} = 8.56\%$ M1 A1

Answer: $K = 0.25mc^2$; error 8.56% 6 marks