

Roots and Coefficients of Polynomials

- 1 The roots of the equation $2x^3 - 5x^2 + 4x - 7 = 0$ are α , β and γ .
Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. [3]
- 2 The roots of the quadratic equation $x^2 - 6x + 4 = 0$ are α and β .
Find the value of $\alpha^2 + \beta^2$. [3]
- 3 The roots of the equation $x^3 - 2x^2 + 5x - 4 = 0$ are α , β and γ .
Find the value of $(\alpha + 2)(\beta + 2)(\gamma + 2)$. [3]
- 4 The roots of $x^3 + px^2 + qx + r = 0$ are α , β and γ , where $\alpha + \beta + \gamma = 5$, $\alpha\beta + \beta\gamma + \gamma\alpha = 2$ and $\alpha\beta\gamma = -6$.
Find the values of p , q and r . [3]
- 5 The roots of the equation $3x^2 + kx + 12 = 0$ are α and β , where $\alpha + \beta = 4$.
Find the value of k and the value of $\alpha\beta$. [3]
- 6 The roots of $x^3 - 2x^2 + 5x - 3 = 0$ are α , β and γ .
Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. [2]
- 7 The roots of the equation $2x^3 - x^2 + 6x + 4 = 0$ are α , β and γ .
Find the exact values of
(a) $\alpha^2 + \beta^2 + \gamma^2$, [3]
(b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. [2]
- 8 Given that 2 is a root of $x^3 - 6x^2 + 13x - 10 = 0$, find the other two roots. [4]
- 9 The roots of the equation $3x^2 - 15x + c = 0$ differ by 3.
Find the value of the constant c and the roots of the equation. [5]
- 10 Find a cubic equation with integer coefficients which has roots $\frac{1}{2}$, -3 and 4 . [4]
- 11 A cubic equation with real coefficients has roots 2 and $1 + i$.
Find the cubic equation, giving it in the form $x^3 + ax^2 + bx + c = 0$. [5]
- 12 A quartic equation with real coefficients has roots $1 + i$ and $3 - 2i$.
Find the quartic equation in the form $x^4 + ax^3 + bx^2 + cx + d = 0$. [5]
- 13 The roots of the equation $3x^2 - 5x - 2 = 0$ are α and β .
Without solving the equation, find the exact value of $(\alpha - \beta)^2$. [4]
- 14 The roots of $x^3 - 4x^2 + x + 5 = 0$ are α , β and γ .
Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [6]
- 15 The roots of $x^3 + 2x^2 - 5x - 1 = 0$ are α , β and γ .
Show that $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma$, and hence find its value. [6]
- 16 The roots of the equation $x^3 - 6x^2 + kx + 10 = 0$ are in arithmetic progression.
Find the roots and the value of k . [6]

- 17** The roots of the equation $x^3 - 7x^2 + kx - 8 = 0$ are in geometric progression. Find the roots and the value of k . [6]
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- 18** The roots of the equation $x^3 - 3x^2 + px + q = 0$ are α , $-\alpha$ and β , where p and q are constants. Show that $q = -3p$, and given that $p = -4$, find the roots. [6]
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- 19** The roots of $2x^3 + 3x^2 - 4x + 1 = 0$ are α , β and γ . Find the value of $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$. [5]
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- 20** Two of the roots of the equation $6x^3 - 33x^2 + kx - 18 = 0$ are reciprocals of each other. Find the value of k and solve the equation. [7]
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- 21** The roots of $x^3 - 2x^2 + 3x + 5 = 0$ are α , β and γ .
 (a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]
 (b) Hence explain why the equation has exactly one real root. [2]
 (c) Find the values of $\alpha^3 + \beta^3 + \gamma^3$ and $\alpha^4 + \beta^4 + \gamma^4$. [5]
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- 22** The roots of $x^3 - 4x^2 + 2x + 3 = 0$ are α , β and γ . Find a cubic equation with integer coefficients whose roots are $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$. [8]
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- 23** The roots of the equation $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$ are in arithmetic progression. Find all four roots. [8]
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- 24** The real numbers x , y and z satisfy $x + y + z = 2$, $x^2 + y^2 + z^2 = 14$ and $x^3 + y^3 + z^3 = 20$.
 (a) Show that $xy + yz + zx = -5$. [2]
 (b) Use the identity $x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ to find the value of xyz . [3]
 (c) Hence find all the solutions for x , y and z . [5]
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- 25** The roots of $x^4 - 3x^3 + 4x^2 - 5x - 3 = 0$ are α , β , γ and δ , where $\alpha\beta = -1$.
 (a) Find the value of $\gamma\delta$. [2]
 (b) Let $\alpha + \beta = s$ and $\gamma + \delta = t$. Show that $s = 2$ and $t = 1$. [6]
 (c) Hence find all four roots. [3]
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- 26** The roots of $2x^3 - 3x^2 + 4x - 5 = 0$ are α , β and γ . A student writes:
 “ $\Sigma\alpha = \frac{3}{2}$ and $\Sigma\alpha\beta = 2$, so $\alpha^2 + \beta^2 + \gamma^2 = (\Sigma\alpha)^2 + 2\Sigma\alpha\beta = \frac{9}{4} + 4 = \frac{25}{4}$.”
 (a) Identify the error in the student's working. [2]
 (b) Find the correct value of $\alpha^2 + \beta^2 + \gamma^2$, and state what it shows about the roots of the equation. [3]
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- 27** The roots of $x^3 + px + q = 0$ are α , β and γ .
 (a) Show that $\alpha + \beta + \gamma = 0$ and $\alpha^2 + \beta^2 + \gamma^2 = -2p$. [3]
 (b) By using the equation to replace α^3 , show that $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$. [4]
 (c) Show that $\alpha^5 + \beta^5 + \gamma^5 = 5pq$. [4]
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- 28** The cubic equation $x^3 + px + q = 0$ has roots α , β and γ .
 (a) Show that $3\alpha^2 + p = (\alpha - \beta)(\alpha - \gamma)$. [3]
 (b) Hence show that $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = -4p^3 - 27q^2$. [5]
 (c) Find the range of values of k for which $x^3 - 3x + k = 0$ has three distinct real roots. [4]

- 29** The roots of $x^3 - 2x^2 - x + 3 = 0$ are α , β and γ . Let $S_n = \alpha^n + \beta^n + \gamma^n$.
- (a) Show that $S_{n+3} = 2S_{n+2} + S_{n+1} - 3S_n$. [3]
- (b) Find S_1 , S_2 and S_3 , and hence find S_6 . [7]
- (c) Explain why S_n is an integer for every positive integer n . [2]
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- 30** The roots of the equation $x^3 - 6x^2 + kx - 8 = 0$ are all real.
- (a) By considering $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$, show that $(\alpha + \beta + \gamma)^2 \geq 3(\alpha\beta + \beta\gamma + \gamma\alpha)$ for real α , β , γ , and hence deduce that $k \leq 12$. [6]
- (b) Given that $k = 12$, find the three roots. [4]

Transformations of Polynomial Roots

- 1** The roots of $x^2 - 5x + 2 = 0$ are α and β .
Find a quadratic equation with roots 2α and 2β . [3]
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- 2** The roots of $x^3 - 2x^2 + 3x - 4 = 0$ are α , β and γ .
Find a cubic equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$. [3]
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- 3** The roots of $2x^2 - 3x + 7 = 0$ are α and β .
Find a quadratic equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. [3]
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- 4** The roots of $x^3 + 2x^2 - x + 6 = 0$ are α , β and γ .
Find a cubic equation with roots 3α , 3β and 3γ . [3]
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- 5** The roots of $x^2 - 4x + 1 = 0$ are α and β .
Find a quadratic equation with roots α^2 and β^2 . [4]
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- 6** The roots of $x^4 - 3x^2 + x + 2 = 0$ are α , β , γ and δ .
Find a quartic equation with roots $\alpha - 2$, $\beta - 2$, $\gamma - 2$ and $\delta - 2$. [3]
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- 7** The roots of $x^2 - 6x + 7 = 0$ are α and β .
Find a quadratic equation with roots $3\alpha + \beta$ and $3\beta + \alpha$. [6]
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- 8** The roots of $2x^4 - 3x^3 + 5x + 1 = 0$ are α , β , γ and δ .
Find a quartic equation with integer coefficients whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ and $\frac{1}{\delta}$. [3]
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- 9** The roots of $2x^2 - 5x + 1 = 0$ are α and β .
Find a quadratic equation with integer coefficients whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. [5]
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- 10** The roots of $x^3 - 4x^2 + x + 3 = 0$ are α , β and γ .
Find a cubic equation whose roots are $\alpha + \beta$, $\beta + \gamma$ and $\gamma + \alpha$. [4]
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- 11** The roots of $x^3 - 4x + 5 = 0$ are α , β and γ .
Find a cubic equation with integer coefficients and roots $2\alpha + 1$, $2\beta + 1$ and $2\gamma + 1$. [4]
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- 12** The roots of $x^2 - 4x + 2 = 0$ are α and β .
Find a quadratic equation with integer coefficients whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$. [6]

- 13** The roots of the equation $x^2 + 3kx + 2k = 0$, where k is a non-zero constant, are α and β .
Find a quadratic equation, in terms of k , whose roots are $\frac{\alpha + \beta}{\alpha}$ and $\frac{\alpha + \beta}{\beta}$. [6]
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- 14** The roots of $x^4 - 2x^3 + 3x - 1 = 0$ are α , β , γ and δ .
(a) Find a quartic equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ and $\frac{1}{\delta}$. [3]
(b) Hence find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [3]
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- 15** The roots of $x^3 - 4x^2 + x + 3 = 0$ are α , β and γ .
Find a cubic equation whose roots are $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$. [5]
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- 16** The roots of $x^3 - 2x^2 - x + 3 = 0$ are α , β and γ .
Find a cubic equation with integer coefficients whose roots are $\frac{\alpha}{\beta\gamma}$, $\frac{\beta}{\gamma\alpha}$ and $\frac{\gamma}{\alpha\beta}$. [6]
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- 17** The roots of $x^3 - x^2 - x - 1 = 0$ are α , β and γ .
Find a cubic equation whose roots are α^3 , β^3 and γ^3 . [6]
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- 18** The cubic equation $y^3 - 3y^2 + 7y + 9 = 0$ has roots $2\alpha - 1$, $2\beta - 1$ and $2\gamma - 1$.
Find a cubic equation with integer coefficients whose roots are α , β and γ , and hence find the value of $\alpha^2 + \beta^2 + \gamma^2$. [6]
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- 19** The roots of $x^3 - x^2 - 1 = 0$ are α , β and γ .
(a) Find a cubic equation with roots $\frac{1}{(\alpha - 1)}$, $\frac{1}{(\beta - 1)}$ and $\frac{1}{(\gamma - 1)}$. [4]
(b) Hence find the value of $\sum \frac{1}{(\alpha - 1)^2}$. [3]
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- 20** The roots of $x^3 - 2x^2 + 3x - 1 = 0$ are α , β and γ .
Find a cubic equation with roots $\alpha^2 - \alpha$, $\beta^2 - \beta$ and $\gamma^2 - \gamma$. [6]
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- 21** The roots of $x^3 - 3x^2 + 2x + 4 = 0$ are α , β and γ .
Find a cubic equation with integer coefficients whose roots are $\frac{\beta + \gamma}{\alpha}$, $\frac{\gamma + \alpha}{\beta}$ and $\frac{\alpha + \beta}{\gamma}$. [8]
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- 22** The roots of $x^3 - 3x^2 + 3x + 3 = 0$ are α , β and γ .
(a) Show that the equation with roots $\frac{\alpha + 1}{\alpha - 1}$, $\frac{\beta + 1}{\beta - 1}$ and $\frac{\gamma + 1}{\gamma - 1}$ is $y^3 - 3y^2 + 3y + 1 = 0$. [6]
(b) By writing the equation in part (a) in terms of $(y - 1)$, find the value of $\sum (\frac{\alpha + 1}{\alpha - 1} - 1)^3$. [4]
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- 23** The roots of $x^3 - 4x^2 + 5x - 1 = 0$ are α , β and γ .
(a) Find a cubic equation whose roots are $\alpha^2 + 1$, $\beta^2 + 1$ and $\gamma^2 + 1$. [7]
(b) Hence show that $(\alpha^2 + 1)(\beta^2 + 1)(\gamma^2 + 1) = 25$. [2]
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- 24** The equation $2x^3 - 5x^2 + 4x - 7 = 0$ has roots α , β and γ . A student writes: "Let $y = \frac{1}{x}$. Then $2y^3 - 5y^2 + 4y - 7 = 0$ has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$."
(a) Explain the error in the student's working. [2]
(b) Find the correct equation, and use it to write down the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. [3]
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- 25** The roots of $x^3 - 2x^2 - x + 3 = 0$ are α , β and γ .
Find a cubic equation with integer coefficients whose roots are $\beta^2 + \gamma^2$, $\gamma^2 + \alpha^2$ and $\alpha^2 + \beta^2$. [9]

- 26** The roots of $x^3 - 3x^2 + 2x + 4 = 0$ are α , β and γ .
Find a cubic equation with integer coefficients whose roots are $\alpha + \frac{1}{\alpha}$, $\beta + \frac{1}{\beta}$ and $\gamma + \frac{1}{\gamma}$. [9]
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- 27** The roots of $x^3 + px^2 + qx + r = 0$ are α , β and γ .
(a) Show that the equation with roots α^2 , β^2 and γ^2 is $y^3 - (p^2 - 2q)y^2 + (q^2 - 2pr)y - r^2 = 0$. [6]
(b) Use this result to find the cubic equation with roots 1, 4 and 9 given that $p = -6$, $q = 11$ and $r = -6$. [3]
(c) Given that α , β and γ are all real, deduce that $p^2 \geq 2q$ and $q^2 \geq 2pr$. [2]
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- 28** The roots of $x^3 + px^2 + qx + r = 0$ are in geometric progression.
(a) Show that $q^3 = p^3r$. [6]
(b) Given that the roots of $x^3 - 14x^2 + qx - 64 = 0$ are in geometric progression, find q and solve the equation. [5]
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- 29** The cubic equation $x^3 - 6x^2 + 9x + k = 0$ has roots α , β and γ .
(a) By substituting $x = y + 2$, show that the equation can be written as $y^3 - 3y + (k + 2) = 0$. [4]
(b) Show that the equation has a repeated root when $k = 0$ or $k = -4$, and find the repeated root in each case. [5]
(c) Hence state, with a reason, the range of values of k for which the cubic has three distinct real roots. [3]
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- 30** The roots of the equation $x^3 - x - 1 = 0$ are α , β and γ .
(a) Find a cubic equation with roots α^2 , β^2 and γ^2 . [4]
(b) Hence find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]
(c) Let $S_n = \alpha^n + \beta^n + \gamma^n$. Show that $S_{n+3} = S_{n+1} + S_n$, and hence find S_5 . [5]

Polynomial Equations

- 1** Show that $x = 1$ is a root of $x^3 - 2x^2 - 5x + 6 = 0$ and solve the equation. [3]
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- 2** Solve the equation $x^2 - 4x + 13 = 0$. [2]
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- 3** Factorise $x^3 - 7x + 6$ completely and hence solve $x^3 - 7x + 6 = 0$. [3]
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- 4** Given that $x = -3$ is a root of $2x^3 + x^2 - 13x + 6 = 0$, find the other two roots. [3]
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- 5** Find a quadratic equation with real coefficients which has $2 + 3i$ as one of its roots. [3]
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- 6** Find all the roots, real and complex, of the equation $x^4 - 16 = 0$. [3]
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- 7** $f(x) = x^3 - 5x^2 + 17x - 13$.
Show that $f(1) = 0$ and hence solve the equation $f(x) = 0$ completely. [5]
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- 8** Given that $1 + 2i$ is a root of the equation $x^3 - 5x^2 + 11x - 15 = 0$, find the other two roots. [5]
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- 9** Given that $2i$ is a root of $x^4 - 2x^3 + 6x^2 - 8x + 8 = 0$, solve the equation completely. [5]
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- 10** The equation $x^3 + ax^2 + bx + 10 = 0$, where a and b are real, has a root $1 + i$.
Find the values of a and b , and the other roots. [5]

- 11** $f(x) = x^3 - 5x^2 + 8x - 4$.
Show that the equation $f(x) = 0$ has a repeated root, and solve the equation. [5]
- 12** $x^3 + ax^2 + bx - 6$ has factors $(x - 1)$ and $(x + 2)$.
Find the values of a and b and the third factor. [5]
- 13** Solve the equation $x^4 + 3x^2 - 4 = 0$, giving all real and complex roots. [4]
- 14** The equation $x^3 + ax^2 + bx + 15 = 0$, where a and b are real, has a root $2 - i$.
Find the values of a and b , and the other roots. [7]
- 15** Show that $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$, and hence solve the equation $x^4 + 4 = 0$. [5]
- 16** Solve the equation $2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0$.
[Hint: divide by x^2 and let $z = x + \frac{1}{x}$.] [6]
- 17** The equation $x^4 + ax^3 + 21x^2 + cx + 30 = 0$, where a and c are real, has roots $1 + 2i$ and 3 .
Find the values of a and c and the remaining roots. [7]
- 18** Factorise $x^3 + 8$, and hence find all three roots of $x^3 + 8 = 0$, giving them in the form $a + bi$ where a and b are real and exact. [5]
- 19** $f(x) = x^3 - 3x^2 + k$, where k is a constant.
Given that the equation $f(x) = 0$ has a repeated root, find the two possible values of k and solve the equation in each case. [6]
- 20** By substituting $u = x^2 - 2x$, or otherwise, solve the equation $x^4 - 4x^3 + 7x^2 - 6x + 2 = 0$. [6]
- 21** The cubic equation $z^3 - 6z^2 + (12 - i)z + c = 0$, where c is a complex constant, has a root $z = 3$.
Find c and the other two roots, and state why the non-real roots are not a conjugate pair. [9]
- 22** Solve the equation $3x^4 - 10x^3 + 14x^2 - 10x + 3 = 0$, giving the non-real roots in the form $a + bi$ with a and b exact. [7]
- 23** $x^2 + 4x + 13$ is a factor of $x^4 + 4x^3 + ax^2 + bx + 39$, where a and b are real constants.
Find the values of a and b and solve the equation $x^4 + 4x^3 + ax^2 + bx + 39 = 0$. [8]
- 24** $f(x) = 2x^3 - 7x^2 + 7x - 2$.
(a) Show that if α is a root of $f(x) = 0$, then so is $\frac{1}{\alpha}$. [4]
(b) Hence solve the equation $f(x) = 0$. [4]
- 25** A student is asked to find a cubic equation with real coefficients that has roots 2 and $1 + i$. The student writes: " $(x - 2)(x - 1 - i) = x^2 - (3 + i)x + 2 + 2i$, so the equation is $x^2 - (3 + i)x + 2 + 2i = 0$."
(a) Give two errors in the student's answer. [2]
(b) Find a correct cubic equation. [3]
- 26** Factorise $x^6 - 1$ into linear and irreducible quadratic factors with real coefficients, and hence find all six roots of $x^6 = 1$ in exact form. [7]

- 27** The cubic equation $x^3 + px^2 + qx + r = 0$ has real coefficients p , q and r , and has a root of the form ki , where k is real and non-zero.
- (a) Show that $r = pq$. [5]
- (b) Given that $p = 2$ and $q = 9$, solve $x^3 + px^2 + qx + r = 0$. [4]
- (c) Show that $x^3 + 2x^2 + 9x + 18$ has the factor $x^2 + 9$ and state the complete factorisation. [2]
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- 28** The cubic equation $x^3 - 3x - 1 = 0$ can be solved using the triple angle formula for cosine.
- (a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [4]
- (b) By writing $x = 2\cos\theta$, show that $x^3 - 3x = 2\cos 3\theta$. [2]
- (c) Hence solve $x^3 - 3x - 1 = 0$, giving the roots in the form $2\cos \alpha$, where α is a multiple of π . [5]
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- 29** The equation $x^4 - 4x^3 + 6x^2 - 4x + 17 = 0$ has four complex roots.
- (a) Show that the equation can be written as $(x - 1)^4 = -16$. [3]
- (b) Let $y = x - 1$. Show that $y^4 + 16 = (y^2 + 4y + 8)(y^2 - 4y + 8)$, and hence solve the original equation. [5]
- (c) The four roots are plotted on an Argand diagram. Show that they are the vertices of a square and find its area. [3]
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- 30** The polynomial $p(x)$ leaves remainder 7 when divided by $(x - 1)$, and remainder $2x + 3$ when divided by $(x^2 + 1)$.
- (a) Find the remainder when $p(x)$ is divided by $(x - 1)(x^2 + 1)$. [7]
- (b) Find the remainder when x^{100} is divided by $x^2 + 1$. [3]

Rational Functions

- 1** Find the equations of the vertical and horizontal asymptotes of the curve $y = \frac{2x + 1}{x - 3}$. [2]
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- 2** Find the coordinates of the points where the curve $y = \frac{x + 4}{x - 2}$ crosses the coordinate axes. [2]
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- 3** Write $\frac{3x - 5}{x + 1}$ in the form $A + \frac{B}{x + 1}$, where A and B are constants, and hence write down the equations of the asymptotes of the curve $y = \frac{3x - 5}{x + 1}$. [3]
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- 4** Write down the equations of the three asymptotes of the curve $y = \frac{x^2}{x^2 - 4}$. [2]
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- 5** Find the equation of the oblique asymptote of the curve $y = \frac{x^2 + 1}{x}$. [2]
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- 6** Solve the equation $\frac{1}{x - 1} = \frac{2}{x + 3}$. [2]
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- 7** Sketch the curve $y = \frac{x + 1}{x - 2}$, showing clearly the equations of the asymptotes and the coordinates of any points where the curve crosses the axes. [5]
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- 8** The curve C has equation $y = \frac{x^2 + x - 6}{x - 1}$.
Find the equations of the asymptotes of C , and the coordinates of the points where C crosses the axes. [6]

- 9** The curve C has equation $y = \frac{x^2 + bx + c}{x - 2}$, where b and c are constants. The oblique asymptote of C is $y = x + 5$ and C passes through the point $(0, -2)$. Find the values of b and c. [6]
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- 10** The curve C has equation $y = \frac{x^2}{(x - 1)(x - 4)}$. Write down the equations of the asymptotes of C, and find the coordinates of the point where C crosses its horizontal asymptote. [5]
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- 11** Describe a sequence of two transformations which maps the curve $y = \frac{1}{x}$ onto the curve $y = \frac{3x - 1}{x + 2}$, after first writing $\frac{3x - 1}{x + 2}$ in the form $A + \frac{B}{x + 2}$. [5]
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- 12** A curve has equation $y = \frac{3x + b}{x + c}$. Its asymptotes are $x = 2$ and $y = 3$, and it passes through the point $(0, 1)$. Find the values of b and c. [3]
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- 13** Sketch the curve $y = \frac{1}{x^2 - 9}$, stating the equations of the asymptotes and the coordinates of the point where the curve crosses the y-axis, which is also a stationary point. [5]
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- 14** The curve C has equation $y = \frac{x^2}{x^2 - 1}$. Find the equations of the asymptotes of C, and sketch C, showing the coordinates of any stationary point and of any points where C meets the axes. [7]
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- 15** The curve C has equation $y = \frac{x^2 - 3x}{x + 1}$, $x \neq -1$. Find the coordinates of the turning points of C. [6]
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- 16** Find the set of values of y which the function $y = \frac{x}{x^2 + 1}$ can take. [5]
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- 17** The curves C_1 and C_2 have equations $y = \frac{2}{x}$ and $y = \frac{x + 1}{x + 3}$ respectively. Sketch C_1 and C_2 on the same axes, showing the equations of the asymptotes and the coordinates of any points where each curve crosses the axes. Find the coordinates of the points where the curves meet, and hence solve the inequality $\frac{2}{x} \geq \frac{x + 1}{x + 3}$. [10]
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- 18** Solve the inequality $\frac{x + 1}{x - 2} \leq \frac{x}{x + 3}$. [5]
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- 19** The curve C has equation $y = \frac{x^2 - x - 2}{x^2 - 4x + 3}$. Find the equations of the asymptotes of C, and the coordinates of the point where C crosses its horizontal asymptote. [6]
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- 20** The curve C has equation $y = \frac{3x + 3}{x - 1}$ and the line l has equation $y = x + 1$. Find the coordinates of the points where C and l meet, and hence solve the inequality $\frac{3x + 3}{x - 1} < x + 1$. [6]
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- 21** The curve C has equation $y = \frac{(x+2)(x-3)}{(x+1)(x-2)}$.
Find the equations of the asymptotes of C and the coordinates of the points where C meets the axes. Show that C does not meet its horizontal asymptote, and sketch C. [9]
-
- 22** The curve C has equation $y = \frac{x^2 + 2x - 3}{x - 2}$.
Find the equations of the asymptotes of C, the coordinates of the turning points of C (determining their nature), and sketch C. [10]
-
- 23** The curve C has equation $y = \frac{x^2}{x-1}$, $x \neq 1$.
(a) Show that there are no points on C for which $0 < y < 4$. [5]
(b) Hence find the coordinates of the turning points of C. [3]
-
- 24** A manufacturer's average cost per item, in pounds, when n items are produced is $A = \frac{n^2 + 60n + 3600}{n}$, where $n > 0$.
(a) Show that $A = n + 60 + \frac{3600}{n}$. [2]
(b) Find the number of items for which the average cost is least, and the least average cost. [6]
(c) Show that $A - 180 = \frac{(n-60)^2}{n}$, and explain what this shows. [3]
-
- 25** A student is asked to solve $\frac{x}{x-1} \geq 2$ and writes:
"Multiply both sides by $(x-1)$: $x \geq 2x - 2$, so $x \leq 2$."
(a) Explain why the student's method is not valid, and show that the answer $x \leq 2$ is wrong by giving a value of x which satisfies $x \leq 2$ but not the inequality. [3]
(b) Solve the inequality correctly. [4]
-
- 26** The curve C has equation $y = \frac{2x+1}{x-2}$, $x \neq 2$.
Find the exact values of the constant k for which the line $y = k - x$ is a tangent to C. [7]
-
- 27** The curve C has equation $y = \frac{x^2 + px + q}{x-2}$, $x \neq 2$, where p and q are constants. C crosses the y -axis at $(0, -\frac{3}{2})$ and has a stationary point at $x = 5$.
(a) Find the values of p and q . [6]
(b) Find the coordinates of both stationary points and state their nature. [3]
(c) Write down the equations of the asymptotes and the set of values which y cannot take. [3]
-
- 28** The curve C has equation $y = \frac{(x-1)(x-4)}{(x-2)(x-3)}$.
(a) Write down the equations of the asymptotes of C, and show that C does not cross its horizontal asymptote. [4]
(b) Show that C has exactly one turning point, and find its coordinates. [4]
(c) Hence find the set of values which y can take. [4]
-
- 29** The curve C has equation $y = \frac{x^2 + 3}{x+1}$, $x > -1$.
(a) Show that $y = x - 1 + \frac{4}{x+1}$, and write down the equation of the oblique asymptote of C. [3]
(b) The region R is bounded by C, the x -axis and the lines $x = 1$ and $x = 3$. Find the exact area of R. [6]
-

- 30** The curve C has equation $y = \frac{1}{x-1}$, $x \neq 1$, and the line l has equation $y = k(x-2)$, where k is a non-zero constant.
- (a) Show that the x-coordinates of any points of intersection satisfy $kx^2 - 3kx + 2k - 1 = 0$. [2]
- (b) Find the set of values of k for which l meets C at two distinct points. [4]
- (c) Given that l is a tangent to C, find the coordinates of the point of contact. [4]

Partial Fractions

- 1** Express $\frac{5}{(x-1)(x+4)}$ in partial fractions. [3]
- 2** Express $\frac{3x+7}{(x+1)(x+3)}$ in partial fractions. [3]
- 3** Express $\frac{6}{x^2-9}$ in partial fractions. [3]
- 4** Find the constants A and B such that $\frac{7x-1}{(x-1)(2x+1)} \equiv \frac{A}{x-1} + \frac{B}{2x+1}$. [3]
- 5** Express $\frac{1}{(x+2)(x+5)}$ in partial fractions, and hence find $\int_0^1 \frac{1}{(x+2)(x+5)} dx$, giving your answer as a single logarithm. [5]
- 6** Express $\frac{2}{x(x+2)}$ in partial fractions, and hence find the value of $\frac{2}{x(x+2)}$ when $x = 2$, checking that your partial fractions give the same value. [3]
- 7** Express $\frac{x^2+10x+7}{(x-1)(x+2)^2}$ in partial fractions. [5]
- 8** Express $\frac{4x^2+3x-7}{(x+1)(x-2)(x+3)}$ in partial fractions. [4]
- 9** Express $\frac{3x^2+x+8}{(x+1)(x^2+4)}$ in partial fractions. [5]
- 10** Express $\frac{x^2+3x+1}{x(x+1)}$ in the form $A + \frac{B}{x} + \frac{C}{x+1}$. [4]
- 11** Find the constants A, B and C such that $\frac{4x^2-3x-4}{x(x-1)(x+2)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$. [4]
- 12** Express $\frac{x+8}{(x-1)(x+2)}$ in partial fractions, and hence show that $\int_2^4 \frac{x+8}{(x-1)(x+2)} dx = \ln 12$. [6]
- 13** $f(x) = \frac{5x+1}{(x-1)(x+2)}$, $x > 1$.
Express f(x) in partial fractions and hence show that f is a decreasing function of x. [6]
- 14** Given that $\frac{x^2-3x+14}{(x+1)(x-2)^2} \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$, find the values of A, B and C. [5]
- 15** Express $\frac{x^3+2x^2-x+4}{(x-1)(x+2)}$ in partial fractions. [5]

- 16** Express $\frac{5-x}{(1-2x)(1+x)}$ in partial fractions, and hence find the series expansion of $\frac{5-x}{(1-2x)(1+x)}$ in ascending powers of x up to and including the term in x^3 . State the range of values of x for which the expansion is valid. [7]
- 17** Express $\frac{3x+2}{(x+1)(x^2+1)}$ in partial fractions, and hence show that $\int_0^1 \frac{3x+2}{(x+1)(x^2+1)} dx = \frac{5\pi}{8} - \frac{1}{4} \ln 2$. [7]
- 18** Express $\frac{3x^3-2x^2+9x+1}{(x^2+1)(x^2+4)}$ in the form $\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$. [5]
- 19** Express $\frac{x^2+3x-4}{x(x-1)(x+4)}$ in the form $\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+4}$, finding A , B and C . Comment on your values of B and C . [4]
- 20** Express $\frac{2}{y(y^2-1)}$ in partial fractions and hence show that $\int_2^3 \frac{2}{y(y^2-1)} dy = \ln \frac{32}{27}$. [6]
- 21** Express $\frac{x^3+x^2+4x+3}{(x-1)^2(x+2)}$ in partial fractions. [7]
- 22** Express $\frac{3x^2+x-2}{x^2(x+2)}$ in partial fractions and hence find the exact value of $\int_1^2 \frac{3x^2+x-2}{x^2(x+2)} dx$. [8]
- 23** Let $f(x) = \frac{1}{(1-x)(1-2x)}$.
- (a) Express $\frac{1}{(1-x)(1-2x)}$ in partial fractions. [3]
- (b) Hence show that the coefficient of x^n in the expansion of $\frac{1}{(1-x)(1-2x)}$ in ascending powers of x is $2^{n+1} - 1$. [4]
- 24** Let $f(x) = \frac{1}{x^4-1}$.
- (a) Express $\frac{1}{x^4-1}$ in partial fractions. [6]
- (b) Hence find $\int \frac{1}{x^4-1} dx$. [3]
- 25** A student writes $\frac{3x}{(x-1)(x+2)^2} \equiv \frac{A}{x-1} + \frac{B}{(x+2)^2}$.
- (a) Explain why this form is incorrect. [2]
- (b) Find the correct partial fractions. [5]
- 26** Express $\frac{x^3+5x^2+2x+2}{(x+2)(x^2+1)}$ in the form $A + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$, and hence find the exact value of $\int_0^1 \frac{x^3+5x^2+2x+2}{(x+2)(x^2+1)} dx$. [10]

- 27 The distinct real numbers a , b and c are constants, and $f(x) = \frac{1}{(x-a)(x-b)(x-c)}$.
- (a) Show that $f(x) = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$, where $A = \frac{1}{(a-b)(a-c)}$, and write down similar expressions for B and C . [3]
- (b) Show that $A + B + C = 0$. [3]
- (c) Hence find the partial fractions of $\frac{1}{(x-1)(x-2)(x-4)}$ and verify that the numerators sum to zero. [4]
-
- 28 The function f is given by $f(x) = \frac{2x^3 + x^2 + 5x + 3}{(x^2 + 1)^2}$.
- (a) Find the constants A , B , C and D such that $f(x) \equiv \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$. [5]
- (b) Given that $\int \frac{1}{(x^2 + 1)^2} dx = \frac{1}{2} \left(\frac{x}{x^2 + 1} + \arctan x \right) + c$, find the exact value of $\int_0^1 f(x) dx$. [7]
-
- 29 Let $f(x) = \frac{1}{(1-x)^2(1-2x)}$, where $|x| < \frac{1}{2}$.
- (a) Express $\frac{1}{(1-x)^2(1-2x)}$ in partial fractions. [5]
- (b) Hence show that, for $|x| < \frac{1}{2}$, the coefficient of x^n in the expansion of $\frac{1}{(1-x)^2(1-2x)}$ in ascending powers of x is $2^{n+2} - n - 3$. [5]
- (c) Find the coefficient of x^{10} . [2]
-
- 30 Consider the function $f(x) = \frac{1}{x^4 + 4}$.
- (a) Show that $\frac{1}{x^4 + 4} \equiv \frac{Ax + B}{x^2 + 2x + 2} + \frac{Cx + D}{x^2 - 2x + 2}$, and find A , B , C and D . [7]
- (b) Hence show that $\int \frac{1}{x^4 + 4} dx = \frac{1}{16} \ln \left(\frac{x^2 + 2x + 2}{x^2 - 2x + 2} \right) + \frac{1}{8} (\arctan(x + 1) + \arctan(x - 1)) + c$. [5]

Inequalities and Modulus Functions

- 1 Solve the inequality $|2x - 3| < 5$. [2]
- 2 Solve the inequality $|x + 4| \geq 3$. [2]
- 3 Solve the inequality $(x - 1)(x + 3)(x - 5) < 0$. [3]
- 4 Solve the equation $|3x - 1| = 8$. [2]
- 5 Sketch the graph of $y = |2x - 4|$, showing the coordinates of the points where it meets the axes. [3]
- 6 Solve the inequality $x^2 - 2x - 8 \geq 0$. [2]
- 7 Solve the equation $|x - 3| = 2x + 1$. [4]
- 8 Solve the inequality $|x + 1| < |x - 3|$. [3]
- 9 Solve the inequality $|x^2 - 4| < 5$. [4]

- 10** Solve the equation $|x^2 - 3x| = 4$. [4]
- 11** Sketch the graph of $y = |x^2 - x - 6|$, showing the coordinates of the points where it meets the axes and of the local maximum point. [6]
- 12** Find the set of values of x for which both $2x + 5 > 1$ and $x^2 - x - 12 < 0$. [4]
- 13** Solve the inequality $\frac{3}{x+2} < 2$. [4]
- 14** Solve the equation $|x - 1| + |x + 2| = 5$. [6]
- 15** Solve the inequality $|2x + 1| > |x - 4|$. [5]
- 16** A rectangle has sides of length $(x + 2)$ cm and $(x + 6)$ cm. Its area is at least 60 cm^2 and its perimeter is at most 44 cm.
Find the set of possible values of x . [7]
- 17** Solve the inequality $|3x - 2| \geq x + 1$. [6]
- 18** Solve the inequality $\frac{x^2 - 1}{x - 3} \geq 0$. [6]
- 19** Solve the inequality $x^2 - 3|x| - 4 < 0$. [5]
- 20** On the same axes, sketch the graphs of $y = |2x - 3|$ and $y = x + 1$, showing the coordinates of where each meets the axes. Hence, or otherwise, solve $|2x - 3| < x + 1$. [6]
- 21** Solve the inequality $|x^2 - 3x| \leq 2x$. [7]
- 22** Solve the inequality $\frac{|x - 2|}{x + 1} < \frac{1}{2}$. [7]
- 23** The triangle inequality is a fundamental result about moduli.
(a) Prove that $|a + b| \leq |a| + |b|$ for all real a and b . [4]
(b) Hence show that $|x - 1| + |x - 4| \geq 3$ for all real x , and state the values of x for which equality holds. [4]
- 24** Let a , b and c be real numbers.
(a) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$ for all real numbers a , b and c . [3]
(b) Given that $a + b + c = 6$, deduce that $ab + bc + ca \leq 12$. [3]
- 25** A student is asked to solve $x^2 > 4x$ and writes: "Divide both sides by x : $x > 4$."
(a) Explain the error, and show by substituting a suitable value that the student's answer is incomplete. [3]
(b) Solve the inequality correctly. [2]
- 26** Solve the inequality $|x + 3| < 2|x - 1|$. [6]
- 27** Consider the function $f(x) = |x^2 - 4x + 3|$.
(a) Sketch the graph of $y = |x^2 - 4x + 3|$, showing the coordinates of the points where it meets the axes and of the local maximum point. [5]
(b) Hence find the set of values of k for which the equation $|x^2 - 4x + 3| = k$ has (i) exactly 4 real solutions, (ii) exactly 3 real solutions. [3]

28 Prove that, for all positive real numbers a , b and c ,
 $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$,
 and state when equality holds. [8]

29 Solve the inequality $|x^2 - 4| \geq 2|x - 1|$, giving your answers in exact form. [9]

30 Solve the inequality $\frac{|2x - 1|}{x + 2} \leq 1$. [8]

Summation of Series

1 Evaluate $\sum_{r=1}^{20} r$. [2]

2 Evaluate $\sum_{r=1}^{10} r^2$. [2]

3 Evaluate $\sum_{r=1}^8 r^3$. [2]

4 Show that $\sum_{r=1}^n (2r + 3) = n(n + 4)$. [2]

5 Evaluate $\sum_{r=1}^{12} (3r + 1)$. [2]

6 Evaluate $\sum_{r=11}^{20} r$. [2]

7 Show that $\sum_{r=1}^n (r^2 + 3r) = \frac{n(n + 1)(n + 5)}{3}$. [4]

8 Find a fully simplified expression for $\sum_{r=1}^n (6r^2 - 2r + 1)$. [4]

9 Show that $\sum_{r=1}^n r(r + 2) = \frac{n(n + 1)(2n + 7)}{6}$, and hence find the value of $\sum_{r=1}^{15} r(r + 2)$. [5]

10 Find a simplified expression, in factorised form, for $\sum_{r=1}^n r(r^2 + 1)$. [4]

11 Show that $\sum_{r=n+1}^{2n} r = \frac{n(3n + 1)}{2}$. [3]

12 Find the sum of the series $1 \times 3 + 2 \times 4 + 3 \times 5 + \dots + 20 \times 22$. [4]

13 Find the smallest positive integer n such that $1 + 2 + 3 + \dots + n > 1000$. [4]

- 14** Show that $\sum_{r=1}^n r(r+1)(r+2) = \frac{1}{4} n(n+1)(n+2)(n+3)$. [5]
-
- 15** Show that the sum of the squares of the first n odd positive integers, $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2$, is $\frac{1}{3} n(2n-1)(2n+1)$. [5]
-
- 16** Consider the sum $\sum_{r=1}^n (3r^2 - r)$.
- (a) Show that $\sum_{r=1}^n (3r^2 - r) = n^2(n+1)$. [3]
- (b) Hence find the value of n for which $\sum_{r=1}^n (3r^2 - r) = 1100$. [3]
-
- 17** It is given that $\sum_{r=1}^n r(r+p)(r+q) = \frac{1}{12} n(n+1)(n+2)(3n+13)$ for all positive integers n , where p and q are positive integers with $p < q$. Find the values of p and q . [8]
-
- 18** Show that $\sum_{r=1}^n r^2(r+1) = \frac{1}{12} n(n+1)(n+2)(3n+1)$. [5]
-
- 19** Show that $2^2 + 4^2 + 6^2 + \dots + (2n)^2 = \frac{2}{3} n(n+1)(2n+1)$.
Hence, using the result for the sum of the squares of the first n odd numbers, find the value of $(2^2 + 4^2 + \dots + 40^2) - (1^2 + 3^2 + \dots + 39^2)$.
[The sum of the squares of the first n odd numbers is $\frac{1}{3} n(2n-1)(2n+1)$.] [7]
-
- 20** A display of tins is arranged in layers. The r th layer from the top is a square arrangement containing r^2 tins. Find the total number of tins in a display with 12 layers, and the least number of layers needed for the display to contain more than 1000 tins. [6]
-
- 21** Find, in fully factorised form, an expression for the sum of the first n terms of the series $5 \times 2 + 7 \times 5 + 9 \times 8 + 11 \times 11 + \dots$ [7]
-
- 22** Find an expression, in terms of n , for $\sum_{r=1}^n (r^2 + 3^r)$. [5]
-
- 23** Find the sum of the first n terms of the series $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \dots$, giving your answer in fully factorised form. [7]
-
- 24** The r th term of a series is $u_r = 3r^2 - 5r + 2$.
Show that $\sum_{r=n+1}^{2n} u_r = n^2(7n-3)$. [6]
-
- 25** A student writes: " $\sum_{r=1}^n (r+1)^2 = \frac{n(n+1)(2n+1)}{6} + 1$."
- (a) Explain the error. [2]
- (b) Find the correct sum, in the form $\frac{n}{6}(an^2 + bn + c)$. [4]
-

- 26** Show that $\sum_{r=1}^{n-1} (2r+1)(r+2) = \frac{1}{6}(n-1)(4n^2 + 13n + 12)$. [7]
-
- 27** Consider the alternating sum $1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2$.
 (a) Show that $(2r-1)^2 - (2r)^2 = 1 - 4r$. [2]
 (b) Hence show that $1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2 = -n(2n+1)$. [4]
-
- 28** Show that $1 \times n + 2 \times (n-1) + 3 \times (n-2) + \dots + n \times 1 = \frac{1}{6}n(n+1)(n+2)$.
 Hence find the value of $1 \times 20 + 2 \times 19 + 3 \times 18 + \dots + 20 \times 1$. [8]
-
- 29** Show that the sum of the cubes of the first n odd positive integers is $n^2(2n^2 - 1)$, and hence find the sum of the cubes of the odd integers from 21 to 39 inclusive. [10]
-
- 30** Cannonballs are stacked in a triangular pyramid. The r th layer from the top is a triangular arrangement containing $\frac{r(r+1)}{2}$ cannonballs.
 (a) Show that a pyramid with n layers contains $\frac{1}{6}n(n+1)(n+2)$ cannonballs. [5]
 (b) A pyramid contains 220 cannonballs. Find the number of layers. [3]
 (c) Find the number of cannonballs in layers 6 to 10 inclusive of the pyramid in part (b). [3]
-

Method of Differences

- 1** Show that $\frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r+1}$. [2]
-
- 2** Evaluate $\sum_{r=1}^5 (\frac{1}{r} - \frac{1}{r+1})$ by writing out the terms. [2]
-
- 3** Let $f(r) = r^2$. Show that $\sum_{r=1}^n [f(r+1) - f(r)] = n^2 + 2n$. [2]
-
- 4** Use the method of differences to show that $\sum_{r=1}^n \ln(\frac{r+1}{r}) = \ln(n+1)$. [2]
-
- 5** Express $\frac{1}{r(r+2)}$ in partial fractions. [2]
-
- 6** Given that $\sum_{r=1}^n \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}$, find the sum to infinity of the series $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$ [2]
-
- 7** Use the result $\frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r+1}$ to show that $\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$. [4]
-
- 8** Use the identity $\frac{1}{r(r+2)} \equiv \frac{1}{2}(\frac{1}{r} - \frac{1}{r+2})$ to find $\sum_{r=1}^n \frac{1}{r(r+2)}$, and state the sum to infinity. [6]
-
- 9** Show that $\frac{1}{(2r-1)(2r+1)} \equiv \frac{1}{2}(\frac{1}{2r-1} - \frac{1}{2r+1})$, and hence find $\sum_{r=1}^n \frac{1}{(2r-1)(2r+1)}$. [5]
-

- 10** Show that, for $n \geq 2$, $\sum_{r=2}^n \frac{1}{r^2 - 1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$. [6]
-
- 11** Show that $\frac{1}{\sqrt{r} + \sqrt{r+1}} \equiv \sqrt{r+1} - \sqrt{r}$, and hence find $\sum_{r=1}^n \frac{1}{\sqrt{r} + \sqrt{r+1}}$. [5]
-
- 12** Let r be a positive integer.
 (a) Show that $(r+1)! - r! = r \times r!$. [2]
 (b) Hence find $\sum_{r=1}^n r \times r!$. [2]
-
- 13** Find the exact value of $\sum_{r=11}^{20} \frac{1}{r(r+1)}$. [4]
-
- 14** Consider the series $\sum_{r=1}^n \frac{1}{r(r+1)(r+2)}$.
 (a) Show that $\frac{1}{r(r+1)(r+2)} \equiv \frac{1}{2} \left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \right]$. [3]
 (b) Hence find $\sum_{r=1}^n \frac{1}{r(r+1)(r+2)}$, and the sum to infinity of the series. [5]
-
- 15** Let $f(r) = (r-2)2^r$. Show that $f(r+1) - f(r) = r \times 2^r$, and hence find $\sum_{r=1}^n r \times 2^r$. [6]
-
- 16** Show that $\frac{2r+1}{r^2(r+1)^2} \equiv \frac{1}{r^2} - \frac{1}{(r+1)^2}$, and hence find $\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2}$. [5]
-
- 17** Find $\sum_{r=1}^n \frac{1}{r(r+3)}$ in terms of n . [6]
-
- 18** Show that $\frac{r}{(r+1)!} \equiv \frac{1}{r!} - \frac{1}{(r+1)!}$, and hence find $\sum_{r=1}^n \frac{r}{(r+1)!}$. [5]
-
- 19** By considering $(r+1)^3 - r^3$, show that $\sum_{r=1}^n r^2 = \frac{1}{6} n(n+1)(2n+1)$. You may use $\sum_{r=1}^n r = \frac{1}{2} n(n+1)$. [6]
-
- 20** Show that $\sum_{r=1}^n \frac{1}{r(r+1)} > 0.99$ when $n \geq 100$, and find the least value of n for which this is true. [5]
-
- 21** Show that $\frac{r+2}{r(r+1)2^r} \equiv \frac{1}{r^2 2^{r-1}} - \frac{1}{(r+1)2^r}$, and hence find $\sum_{r=1}^n \frac{r+2}{r(r+1)2^r}$ and the sum to infinity. [7]
-
- 22** Consider the series $\sum_{r=1}^n \frac{1}{r(r+1)(r+2)(r+3)}$.
 (a) Show that $\frac{1}{r(r+1)(r+2)} - \frac{1}{(r+1)(r+2)(r+3)} = \frac{3}{r(r+1)(r+2)(r+3)}$. [3]
 (b) Hence find $\sum_{r=1}^n \frac{1}{r(r+1)(r+2)(r+3)}$. [4]
-

23

A student writes: " $\frac{1}{r(r+2)} = \frac{1}{2}(\frac{1}{r} - \frac{1}{r+2})$, so $\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{1}{2}(1 - \frac{1}{n+2})$, because all the middle terms cancel."

(a) Explain the error. [2]

(b) Find the correct sum, and check it for $n = 2$. [4]

24

This question derives the sum of the cubes of the first n positive integers.

(a) Show that $(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$. [2]

(b) Hence show that $\sum_{r=1}^n r^3 = \frac{1}{4} n^2(n+1)^2$. [7]

25

Show that $\ln(1 - \frac{1}{(r+1)^2}) = \ln r + \ln(r+2) - 2 \ln(r+1)$, and hence find $\sum_{r=1}^n \ln(1 - \frac{1}{(r+1)^2})$

in the form $\ln(\frac{n+2}{a(n+1)})$, stating the value of a . Find also the sum to infinity of the series. [7]

26

Using the result $\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$, show that $\sum_{r=n+1}^{2n} \frac{1}{r(r+1)} = \frac{n}{(n+1)(2n+1)}$. [5]

27

Consider the series whose r th term is $\arctan(\frac{1}{r^2 + r + 1})$.

(a) Show that $\arctan(r+1) - \arctan r = \arctan(\frac{1}{r^2 + r + 1})$. [4]

(b) Hence find $\sum_{r=1}^n \arctan(\frac{1}{r^2 + r + 1})$ and the sum to infinity of the series. [5]

28

Let θ be a real number which is not a multiple of 2π .

(a) Show that $2 \sin(\frac{\theta}{2}) \sin r\theta = \cos((r - \frac{1}{2})\theta) - \cos((r + \frac{1}{2})\theta)$. [2]

(b) Hence show that $\sum_{r=1}^n \sin r\theta = \frac{\sin(\frac{n\theta}{2}) \sin(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})}$. [5]

(c) Find the exact value of $\sin 30^\circ + \sin 60^\circ + \sin 90^\circ + \sin 120^\circ + \sin 150^\circ$. [4]

29

Consider the series $\sum_{r=1}^n \frac{r}{(r+1)(r+2)(r+3)}$.

(a) Express $\frac{r}{(r+1)(r+2)(r+3)}$ in partial fractions. [4]

(b) Hence show that $\sum_{r=1}^n \frac{r}{(r+1)(r+2)(r+3)} = \frac{1}{4} + \frac{1}{2(n+2)} - \frac{3}{2(n+3)}$. [5]

(c) Find the sum to infinity of the series. [2]

30

Consider the series $\sum_{r=1}^n \frac{r^2 + r - 1}{(r+2)!}$.

(a) Show that $\frac{r^2 + r - 1}{(r+2)!} \equiv \frac{r}{(r+1)!} - \frac{r+1}{(r+2)!}$. [4]

(b) Hence find $\sum_{r=1}^n \frac{r^2 + r - 1}{(r+2)!}$, and show that the sum to infinity is $\frac{1}{2}$. [5]

Recurrence Relations

- 1 A sequence is defined by $u_1 = 3$ and $u_{n+1} = 2u_n - 1$.
Find u_2 , u_3 and u_4 . [3]
- 2 A sequence is defined by $u_1 = 2$ and $u_{n+1} = u_n + 4$. Find an expression for u_n in terms of n . [2]
- 3 A sequence is defined by $u_1 = 10$ and $u_{n+1} = \frac{1}{2}u_n + 3$. Given that the sequence converges, find its limit. [2]
- 4 Find a recurrence relation of the form $u_{n+1} = u_n + d$, with first term, for the sequence 2, 5, 8, 11, ... and hence write down u_n . [2]
- 5 A sequence is defined by $u_1 = 1$, $u_2 = 3$ and $u_{n+2} = 3u_{n+1} - 2u_n$.
Find u_3 and u_4 . [2]
- 6 The recurrence relation $u_{n+2} = 5u_{n+1} - 6u_n$ has auxiliary equation $m^2 - 5m + 6 = 0$. Solve this equation and write down the general solution of the recurrence relation. [3]
- 7 A sequence is defined by $u_1 = 1$ and $u_{n+1} = 3u_n + 2$. Find an expression for u_n in terms of n . [5]
- 8 A sequence is defined by $u_1 = 3$ and $u_{n+1} = u_n + 2n$. Find an expression for u_n in terms of n . [4]
- 9 A sequence is defined by $u_0 = 1$, $u_1 = 0$ and $u_{n+2} = 5u_{n+1} - 6u_n$. Find an expression for u_n in terms of n . [5]
- 10 A sequence is defined by $u_0 = 2$, $u_1 = 9$ and $u_{n+2} = 6u_{n+1} - 9u_n$. Find an expression for u_n in terms of n . [5]
- 11 A sequence is defined by $u_1 = 3$ and $u_{n+1} = ku_n - 2$, where k is a constant.
Given that $u_3 = 19$, find the two possible values of k , and find u_4 when k is positive. [7]
- 12 Show that $u_n = 2^n + 1$ satisfies the recurrence relation $u_{n+1} = 2u_n - 1$. [3]
- 13 A sequence has n th term $u_n = 3 \times 2^n + 5$. Find a recurrence relation of the form $u_{n+1} = a u_n + b$ which the sequence satisfies. [3]
- 14 A sequence is defined by $u_1 = 1$ and $u_{n+1} = 2u_n + 3$.
Prove by induction that $u_n = 2^{n+1} - 3$ for all positive integers n . [6]
- 15 A sequence is defined by $y_{n+1} = 3y_n - 2$ for $n \geq 1$, and $y_5 = 82$.
(a) Express y_n in terms of y_{n+1} . [2]
(b) Hence find y_4 , y_3 , y_2 and y_1 . [3]
(c) Find y_n in terms of n . [3]
- 16 A sequence is defined by $u_0 = 1$, $u_1 = 7$ and $u_{n+2} = 5u_{n+1} - 6u_n + 4n$.
Find an expression for u_n in terms of n . [8]
- 17 A sequence is defined by $u_0 = 0$, $u_1 = 1$ and $u_{n+2} = 2u_{n+1} - 2u_n$.
Find an expression for u_n in terms of n , giving your answer in terms of trigonometric functions. [7]

- 18** A loan of £20 000 is taken out. At the end of each year interest of 3% is added to the outstanding balance, and then a repayment of £3000 is made. Let B_n be the balance, in pounds, after n repayments, so $B_0 = 20\,000$.
Find a recurrence relation for B_n , solve it, and find the number of years needed for the loan to be fully repaid. [8]
-
- 19** Consider the recurrence relation $u_{n+2} - 4u_{n+1} + 4u_n = 0$.
(a) Show that $u_n = n \times 2^n$ satisfies the recurrence relation $u_{n+2} - 4u_{n+1} + 4u_n = 0$. [4]
(b) Hence find the solution of this recurrence relation with $u_0 = 3$ and $u_1 = 2$. [5]
-
- 20** A sequence is defined by $u_1 = 3$ and $u_{n+1} = \frac{1}{1 - u_n}$.
(a) Find u_2 , u_3 and u_4 . [3]
(b) Hence write down the value of u_{100} . [2]
(c) Find the value of $\sum_{r=1}^{100} u_r$. [4]
-
- 21** The Fibonacci sequence is defined by $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$.
Show that $u_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$. [7]
-
- 22** A sequence is defined by $u_1 = 1$ and $u_{n+1} = \sqrt{2 + u_n}$.
(a) Prove by induction that $u_n < 2$ for all positive integers n . [4]
(b) Show that $u_{n+1} > u_n$ for all n . [4]
(c) Given that the sequence converges, find its limit. [2]
-
- 23** A sequence is defined by $u_1 = 1$ and $u_{n+1} = \frac{u_n}{3 - u_n}$.
By letting $v_n = \frac{1}{u_n}$, or otherwise, find an expression for u_n in terms of n . [6]
-
- 24** A student is asked to solve $u_{n+2} - 4u_{n+1} + 4u_n = 0$ with $u_0 = 1$ and $u_1 = 4$. The student writes: "The auxiliary equation is $(m - 2)^2 = 0$, so $m = 2$ and $u_n = A \times 2^n + B \times 2^n$."
(a) Explain why this general solution is incorrect. [2]
(b) Find the correct solution. [5]
-
- 25** A sequence is defined by $u_0 = 1$, $u_1 = 2$ and $u_{n+2} = 3u_{n+1} - 2u_n + 2^n$.
Find an expression for u_n in terms of n . [9]
-
- 26** A sequence is defined by $u_1 = 1$ and $u_{n+1} = 2u_n + n$.
Find an expression for u_n in terms of n . [7]
-
- 27** A sequence is defined by $u_0 = 0$, $u_1 = \sin \theta$ and $u_{n+2} = 2 \cos \theta \times u_{n+1} - u_n$, where $0 < \theta < \pi$.
(a) Prove by induction that $u_n = \sin n\theta$ for all integers $n \geq 0$. [7]
(b) For $\theta = \frac{\pi}{3}$, write down the first seven terms of the sequence and describe its long-term behaviour. [3]
-
- 28** The sequences (a_n) and (b_n) are defined by $a_0 = 1$, $b_0 = 0$ and $a_{n+1} = 3a_n + 2b_n$, $b_{n+1} = a_n + 2b_n$.
(a) Show that $a_{n+2} = 5a_{n+1} - 4a_n$. [5]
(b) Hence find expressions for a_n and b_n in terms of n . [7]

- 29** Let a_n be the number of binary strings of length n (strings of 0s and 1s) which contain no two consecutive 1s.
- (a) Show that $a_1 = 2$ and $a_2 = 3$, and explain why $a_{n+2} = a_{n+1} + a_n$. [6]
- (b) Find a_{10} . [3]
- (c) A fair coin is tossed 10 times. Find the probability that no two consecutive tosses are both heads. [2]
-
- 30** A sequence is defined by $u_1 = 3$ and $u_{n+1} = u_n^2 - 2$. Let $\alpha = \frac{3 + \sqrt{5}}{2}$.
- (a) Show that $\alpha + \frac{1}{\alpha} = 3$. [3]
- (b) Prove by induction that $u_n = \alpha^{2^{n-1}} + \alpha^{-2^{n-1}}$ for all positive integers n . [5]
- (c) Find u_4 . [2]

Maclaurin and Taylor Series

- 1** Use the standard series for e^x to write down the Maclaurin series of e^{2x} up to and including the term in x^3 . [2]
-
- 2** Write down the Maclaurin series of $\sin 2x$ up to and including the term in x^3 . [2]
-
- 3** Write down the series expansion of $\ln(1 + 3x)$ in ascending powers of x up to and including the term in x^3 . [2]
-
- 4** Write down the Maclaurin series of $\cos 3x$ up to and including the term in x^4 . [2]
-
- 5** $f(x) = \ln(1 + x)$. Find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$, and hence write down the Maclaurin series of $f(x)$ up to the term in x^3 . [3]
-
- 6** Write down the expansion of e^{-x} up to and including the term in x^3 , and hence find an approximation to $e^{-0.1}$ to 4 decimal places. [3]
-
- 7** By multiplying standard series, find the Maclaurin series of $e^x \sin x$ up to and including the term in x^3 . [4]
-
- 8** Given that $\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots$, find the Maclaurin series of $\ln(\cosh x)$ up to and including the term in x^4 . [5]
-
- 9** Find the Maclaurin series of $e^x \cos x$ up to and including the term in x^3 . [4]
-
- 10** $f(x) = \tan x$. Find $f'(x)$ and $f''(x)$, and show that $f'''(0) = 2$. Hence find the Maclaurin series of $\tan x$ up to the term in x^3 . [5]
-
- 11** The Maclaurin series for $\ln(1 + x)$ and $\ln(1 - x)$ can be combined to give a series which converges quickly.
- (a) Using the series for $\ln(1 + x)$ and $\ln(1 - x)$, show that $\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots$ for $|x| < 1$. [4]
- (b) By choosing $x = \frac{1}{2}$, estimate $\ln 3$ using the first three terms. [3]
-
- 12** Use the series for $\cos x$ and $\ln(1 + u)$ to show that $\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} + \dots$ [5]

- 13** Find the Maclaurin series of $\ln(1 - x - 2x^2)$ up to and including the term in x^3 , and state the range of values of x for which the expansion is valid. [6]
-
- 14** Use the Maclaurin series for $\ln(1 + x)$ to find the exact sum to infinity of the series
 $\frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \dots$ [5]
-
- 15** Find the Maclaurin series of $e^{2x} \sin 3x$ up to and including the term in x^4 . [6]
-
- 16** Using the series $e^{2x} \sin 3x = 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 + \dots$, find an approximation to $\int_0^{0.1} e^{2x} \sin 3x \, dx$, giving your answer to 4 significant figures. [4]
-
- 17** Given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$, find the Maclaurin series for y up to and including the term in x^4 . [6]
-
- 18** Find the Maclaurin series of $e^{\sin x}$ up to and including the term in x^4 . [6]
-
- 19** Find the series expansion of $(1 + x)\ln(1 + x)$ up to and including the term in x^4 , and hence evaluate $\lim_{x \rightarrow 0} \frac{(1 + x)\ln(1 + x) - x}{x^2}$. [4]
-
- 20** $f(x) = \sqrt{x}$. Find the Taylor series of $f(x)$ in ascending powers of $(x - 4)$ up to and including the term in $(x - 4)^3$. Hence estimate $\sqrt{4.4}$ to 4 decimal places. [6]
-
- 21** $y = \ln(1 + \sin x)$.
 (a) Show that $(1 + \sin x) \frac{d^2y}{dx^2} = -1$. [5]
 (b) By differentiating this result twice, find the Maclaurin series of y up to and including the term in x^4 . [7]
-
- 22** $f(x) = \tan x$.
 Find the Taylor series of $f(x)$ in ascending powers of $(x - \frac{\pi}{4})$ up to and including the term in $(x - \frac{\pi}{4})^3$. [7]
-
- 23** Show that $\ln(1 + e^x) = \ln 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$ [7]
-
- 24** $f(x) = \sin x$.
 (a) Find the Taylor series of $f(x)$ in ascending powers of $(x - \frac{\pi}{6})$ up to and including the term in $(x - \frac{\pi}{6})^3$. [4]
 (b) Use your series to estimate $\sin 31^\circ$ to 5 decimal places, taking $\frac{\pi}{180} = 0.0174533$. [4]
-
- 25** A student writes: " $\ln(1 + 2x) = 2x - \frac{(2x)^2}{2} + \frac{x^3}{3} = 2x - 2x^2 + \frac{x^3}{3}$."
 (a) Identify the error in the student's working. [2]
 (b) Find the correct expansion up to the term in x^3 and state the values of x for which the full series is valid. [4]
-

- 26** Given that $y(0) = 1$, $y'(0) = 0$ and $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, find the Maclaurin series of y up to and including the term in x^4 . [7]
-
- 27** $y = e^{\arctan x}$.
 (a) Show that $(1 + x^2)\frac{dy}{dx} = y$. [3]
 (b) Hence find the Maclaurin series of y up to and including the term in x^4 . [8]
-
- 28** $f(x) = \ln(1 + x)$.
 (a) Prove by induction that $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ for all positive integers n . [5]
 (b) Hence show that the Maclaurin series of $f(x)$ has general term $\frac{(-1)^{n-1}x^n}{n}$. [4]
-
- 29** Use Maclaurin series to evaluate the limits
 (a) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ [5]
 (b) $\lim_{x \rightarrow 0} \frac{2 \ln(\cos x) + x^2}{x^4}$ [5]
-
- 30** $f(x) = e^{-x^2}$.
 (a) Find the Maclaurin series of $f(x)$ up to and including the term in x^6 . [2]
 (b) Hence find an approximation to $\int_0^{1/2} e^{-x^2} dx$, giving your answer to 4 decimal places. [4]
 (c) The next term in the integrated series is $+\frac{x^9}{216}$. Explain why the error in your approximation is less than 0.00001. [3]

Further Binomial Expansions

- 1** Write down the expansion of $(1 + x)^{-1}$ in ascending powers of x up to and including the term in x^3 . [2]
-
- 2** Expand $(1 - 2x)^{-1/2}$ in ascending powers of x up to and including the term in x^3 . [3]
-
- 3** Expand $(1 + 3x)^{-2}$ in ascending powers of x up to and including the term in x^3 . [3]
-
- 4** Expand $(1 + 4x)^{1/3}$ up to and including the term in x^2 , and state the range of x for which the expansion is valid. [4]
-
- 5** Expand $\sqrt{4 + x}$ in ascending powers of x up to and including the term in x^2 , and state the range of validity. [4]
-
- 6** Expand $\sqrt{1 + x}$ up to and including the term in x^3 , and use $x = 0.02$ to find an approximation to $\sqrt{1.02}$, to 6 decimal places. [4]
-
- 7** Expand $(8 + 3x)^{1/3}$ in ascending powers of x up to and including the term in x^2 , and state the range of validity. [5]
-
- 8** Find the first four terms in the expansion of $(1 - x)^{-3}$ and state the coefficient of x^n in terms of n .
 Verify your formula for $n = 3$. [5]

- 9** Expand $(2 - x)^{-2}$ in ascending powers of x up to and including the term in x^3 . [4]
- 10** Expand $(1 + x)^{-1/2}$ up to and including the term in x^3 , and use $x = 0.1$ to estimate $\frac{1}{\sqrt{1.1}}$ to 4 decimal places. [4]
- 11** Find the first three terms in the expansion of $(1 + 2x)^{-3/2}$ in ascending powers of x . [3]
- 12** Find the coefficient of x^3 in the expansion of $(1 - 3x)^{1/3}$. [4]
- 13** Expand $(2 + 3x)^{-1}$ up to and including the term in x^2 , and state the set of values of x for which the expansion is valid. [4]
- 14** Expand $\frac{\sqrt{1+x}}{1-x}$ in ascending powers of x up to and including the term in x^3 , and state the range of validity. [6]
- 15** Express $\frac{5+x}{(1-x)(1+2x)}$ in partial fractions and hence find its expansion in ascending powers of x up to and including the term in x^3 . State the set of values of x for which the expansion is valid. [7]
- 16** Find the expansion of $(1 + 2x)(4 - x)^{-1/2}$ in ascending powers of x up to and including the term in x^2 . [5]
- 17** Expand $(1 - x^2)^{-1/2}$ up to and including the term in x^4 . By taking $x = 0.1$, estimate $\frac{1}{\sqrt{0.99}}$ to 5 decimal places. [4]
- 18** Use the binomial expansion of $(1 + x)^{1/3}$ up to and including the term in x^3 , with $x = 0.06$, to find an approximation to the cube root of 1.06, giving your answer to 5 decimal places. [4]
- 19** The expansion of $(1 + ax)^{-2}$ contains the term $12x^2$, where $a > 0$. Find the value of a and the coefficient of x^3 . [5]
- 20** In the expansion of $(1 + ax)^n$, where n is a negative integer, the first three terms are $1 - 6x + 24x^2$. Find the values of a and n . [6]
- 21** Consider the function $(1 - x^2)^{-1/2}$.
 (a) Expand $(1 - x^2)^{-1/2}$ up to and including the term in x^4 . [3]
 (b) Hence, by integration, find the series for $\arcsin x$ up to and including the term in x^5 . [4]
- 22** Expand $\sqrt{\frac{1+x}{1-x}}$ in ascending powers of x up to and including the term in x^3 , by writing it as $(1 + x)^{1/2}(1 - x)^{-1/2}$. [7]
- 23** The binomial expansion can be used to approximate square roots.
 (a) Show that $\sqrt{2} = \frac{7}{5}(1 + \frac{1}{49})^{1/2}$. [2]
 (b) Use the binomial expansion of $(1 + x)^{1/2}$ up to and including the term in x^2 to find an approximation to $\sqrt{2}$, giving your answer to 5 decimal places. [4]

- 24** A student writes: " $(2 + x)^{-1} = 2^{-1} + (-1)x + \frac{(-1)(-2)}{2}x^2 = \frac{1}{2} - x + x^2$."
- (a) Explain the error. [2]
- (b) Find the correct expansion up to the term in x^2 , and its range of validity. [4]
-
- 25** Show that $\frac{1}{(1+x)(2-x)} = \frac{1}{3}\left(\frac{1}{1+x} + \frac{1}{2-x}\right)$, and hence expand it in ascending powers of x up to and including the term in x^3 . State the set of values of x for which the expansion is valid. [7]
-
- 26** Consider the expansion of $(1 - 4x)^{-1/2}$.
- (a) Expand $(1 - 4x)^{-1/2}$ up to and including the term in x^3 . [3]
- (b) By squaring your answer to part (a), verify that the first four terms agree with the expansion of $(1 - 4x)^{-1}$. [4]
-
- 27** Prove that the coefficient of x^n in the expansion of $(1 - 4x)^{-1/2}$ is $\frac{(2n)!}{(n!)^2}$. [8]
-
- 28** Consider the binomial series for $(1 - x)^{-1/2}$ and $(1 + x)^{-1/2}$.
- (a) Expand $(1 - x)^{-1/2}$ and $(1 + x)^{-1/2}$ in ascending powers of x up to and including the term in x^4 . [3]
- (b) By multiplying these series, show that the coefficients of x^2 and x^4 in the product are $\frac{1}{2}$ and $\frac{3}{8}$. [5]
- (c) Explain how this confirms the expansion of $(1 - x^2)^{-1/2}$. [2]
-
- 29** Consider the series $1 + \frac{1}{2} \cdot \frac{1}{4} + \frac{1 \times 3}{2 \times 4} \left(\frac{1}{4}\right)^2 + \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \left(\frac{1}{4}\right)^3 + \dots$
- (a) Show that $1 + \frac{1}{2} \cdot \frac{1}{4} + \frac{1 \times 3}{2 \times 4} \left(\frac{1}{4}\right)^2 + \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \left(\frac{1}{4}\right)^3 + \dots$ is the binomial expansion of $(1 - y)^{-1/2}$ for a value of y to be found, and find the exact sum of the series. [6]
- (b) Find the sum of the first four terms and compare it with the exact sum. [3]
-
- 30** In special relativity, a particle of rest mass m moving at speed v has kinetic energy $K = mc^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]$, where c is the speed of light.
- (a) By expanding, show that for v much smaller than c , $K \approx \frac{1}{2}mv^2\left(1 + \frac{3v^2}{4c^2}\right)$. [3]
- (b) When $v = 0.1c$, find the ratio of the second term, $\frac{3mv^4}{8c^2}$, to the first term, $\frac{1}{2}mv^2$, in the two-term approximation for K . Give your answer as a percentage. [3]
- (c) When $v = 0.6c$, find K exactly as a multiple of mc^2 , and find the percentage error in the two-term approximation in part (a), to 3 significant figures. [6]