

A LEVEL FURTHER MATHS • EXTREME PAPERS

MATHS



EXAM PAPER

EXTREME PAPER: FURTHER PURE MATHEMATICS

SCORE

/120

PERCENTAGE

%

STEMUP

Paper: Further Pure Mathematics (120 marks)

1 Edexcel | AQA | OCR A | OCR MEI | CIE

The curve C has polar equation $r = 1 + 2\cos\theta$ for $0 \leq \theta < 2\pi$. It consists of a large loop with a small loop inside it, both passing through the pole O .

- Find the values of θ in $[0, 2\pi)$ at which C passes through O , and state the range of θ for which the small loop is traced. [2]
- Find the exact area enclosed by the small loop. [3]
- Find the exact area of the region that is inside the large loop but outside the small loop. [3]

2 Edexcel | OCR A | OCR MEI

For $t \geq 0$ the variables x and y satisfy the coupled differential equations

$$\frac{dx}{dt} = 3x - y, \quad \frac{dy}{dt} = x + y + 4e^{2t},$$

with $x = 1$ and $y = 0$ when $t = 0$.

- Show that $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = -4e^{2t}$. [3]
- Find x in terms of t . [6]
- Find y in terms of t , and show that $y = 0$ for exactly one positive value of t , stating it. [2]

3 Edexcel | AQA | OCR A | OCR MEI | CIE

Let $y = (\arcsin x)^2$ for $-1 < x < 1$, and write $y^{(k)}$ for the k th derivative of y with respect to x .

- Show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$. [3]
- Show that, for every integer $k \geq 1$,
 $(1 - x^2)y^{(k+2)} - (2k + 1)x y^{(k+1)} - k^2 y^{(k)} = 0$. [3]
- Hence find the Maclaurin series for y up to and including the term in x^6 . [2]
- Show that the coefficient of x^{2n} in the series is $\frac{2^{2n-1}}{n^2 C(2n, n)}$, where $C(2n, n)$ is the binomial coefficient, and hence find the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2 C(2n, n)}$. You may assume the series is valid at $x = \frac{1}{2}$. [2]

4 Edexcel (FP1 option) | AQA (Discrete option) | OCR A (Additional Pure) | OCR MEI (Extra Pure)

The set $U = \{1, 5, 7, 11, 13, 17, 19, 23\}$ is a group under multiplication modulo 24. You may assume this. The identity is 1.

- Prove that $x^2 \equiv 1 \pmod{24}$ for every x in U . [3]
- Prove that any group in which $g^2 = e$ for every element g is abelian. [2]
- Show that U has exactly seven subgroups of order 4. [3]
- Find the total number of subgroups of U , justifying your answer. [2]

5 Edexcel | AQA | OCR A | OCR MEI | CIE

For integers $n \geq 0$ let $C_n = \int_0^{\pi/2} \cos^n x \cos nx \, dx$ and $T_n = \int_0^{\pi/2} \cos^n x \sin nx \, dx$.

- Show that, for $n \geq 1$, $\frac{d}{dx}(\cos^n x \sin nx) = n \cos^{n-1} x \cos((n+1)x)$. [2]
- Hence show that $C_n = \frac{1}{2}C_{n-1}$ for $n \geq 1$, and deduce a formula for C_n . [5]
- Show that $T_n = \frac{1}{2}T_{n-1} + \frac{1}{2n}$ for $n \geq 1$, and hence find the exact value of T_4 . [4]

6 Edexcel | AQA | OCR A | OCR MEI | CIE

The lines l_1 and l_2 have vector equations

$$l_1: \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad l_2: \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

- (a) Show that the shortest distance between l_1 and l_2 is 3. [4]
- (b) Points P and P' lie on l_1 with $PP' = 4$, and points Q and Q' lie on l_2 with $QQ' = 6$. Show that the volume of the tetrahedron $PP'QQ'$ does not depend on where on the lines these segments are placed, and find this volume. [3]
- (c) The common perpendicular of l_1 and l_2 meets l_1 at P_0 and l_2 at Q_0 . The points P and P' are the points of l_1 at distance 2 from P_0 either side of it, and Q and Q' are the points of l_2 at distance 3 from Q_0 either side of it. Find the position vector of the centre, and the exact radius, of the sphere through P, P', Q and Q'. [5]

7 Edexcel | AQA | OCR A | OCR MEI | CIE

The matrix A is given by $A = \begin{pmatrix} 7 & -6 \\ 3 & -2 \end{pmatrix}$.

- (a) Find the eigenvalues of A and a corresponding eigenvector for each. [3]
- (b) Hence find A^n , for positive integers n, as a single matrix. [3]
- (c) Find all real 2×2 matrices X such that $X^2 = A$. [5]

8 Edexcel | AQA | OCR A | OCR MEI | CIE

Throughout, $\arctan$ takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

- (a) Show that, for every integer $r \geq 1$, $\arctan(r+1) - \arctan(r-1) = \arctan(\frac{2}{r^2})$. [3]
- (b) Hence find $\sum_{r=1}^n \arctan(\frac{2}{r^2})$ in terms of n. [3]
- (c) Write down the sum to infinity of the series in (b). [1]
- (d) Let $P_n = (1^2 + 2i)(2^2 + 2i)(3^2 + 2i) \dots (n^2 + 2i)$. Find all positive integers n for which P_n is purely imaginary. [4]

9 Edexcel | AQA | OCR A | OCR MEI | CIE

In this question n is a positive integer and $C(n, r)$ denotes the binomial coefficient " n choose r ".

- (a) By expanding $(1 + e^{i\theta})^n$, show that $\sum_{r=0}^n C(n, r) \cos r\theta = 2^n \cos^n(\frac{\theta}{2}) \cos(\frac{n\theta}{2})$, for $-\pi < \theta < \pi$. [3]
- (b) Let $\omega = e^{2\pi i/3}$ and let $S_n = C(n, 0) + C(n, 3) + C(n, 6) + \dots$, the sum of the binomial coefficients $C(n, r)$ with r a multiple of 3. [4]
- By considering $(1 + 1)^n$, $(1 + \omega)^n$ and $(1 + \omega^2)^n$, show that $S_n = \frac{1}{3}(2^n + 2\cos(\frac{n\pi}{3}))$.
- (c) Find S_{2026} exactly as a single fraction. [2]
- (d) Differentiate the result in (a) with respect to θ to show that [3]
- $$\sum_{r=1}^n r C(n, r) \sin r\theta = n \cdot 2^{n-1} \cos^{n-1}(\frac{\theta}{2}) \sin(\frac{(n+1)\theta}{2}).$$

10 Edexcel | AQA | OCR A | OCR MEIFor integers $n \geq 2$ define

$$I_n = \int_0^{\infty} \frac{1}{(x + \sqrt{x^2 + 1})^n} dx \text{ and } K_n = \int_1^{\infty} \frac{1}{(x + \sqrt{x^2 - 1})^n} dx.$$

(a) Use a hyperbolic substitution to show that $I_n = \frac{n}{n^2 - 1}$. [5]

(b) Show that $K_n = \frac{1}{n^2 - 1}$. [3]

(c) Let $r = x + \sqrt{x^2 - 1}$ for $x \geq 1$.

(i) Show that $\sum_{n=2}^N \frac{1}{n^2 - 1} = \frac{3}{4} - \frac{2N + 1}{2N(N + 1)}$.

(ii) By summing the geometric series $\sum_{n=2}^{\infty} r^{-n}$ and integrating term by term (which you

may assume is valid), deduce the exact value of $\int_1^{\infty} \frac{1}{r(r - 1)} dx$. [4]

11 Edexcel | AQA | OCR A | OCR MEI

Let $I = \int_0^{\infty} \frac{1}{1 + x^4} dx$ and $J = \int_0^{\infty} \frac{x^2}{1 + x^4} dx$. Both integrals converge.

(a) Use the substitution $x = \frac{1}{t}$ to show that $I = J$. [3]

(b) Hence, by considering $I + J$ and the substitution $u = x - \frac{1}{x}$, find the exact value of I . [6]

(c) Hence, or otherwise, find the exact value of $\int_0^{\infty} \frac{1}{x^4 + x^2 + 1} dx$. [3]