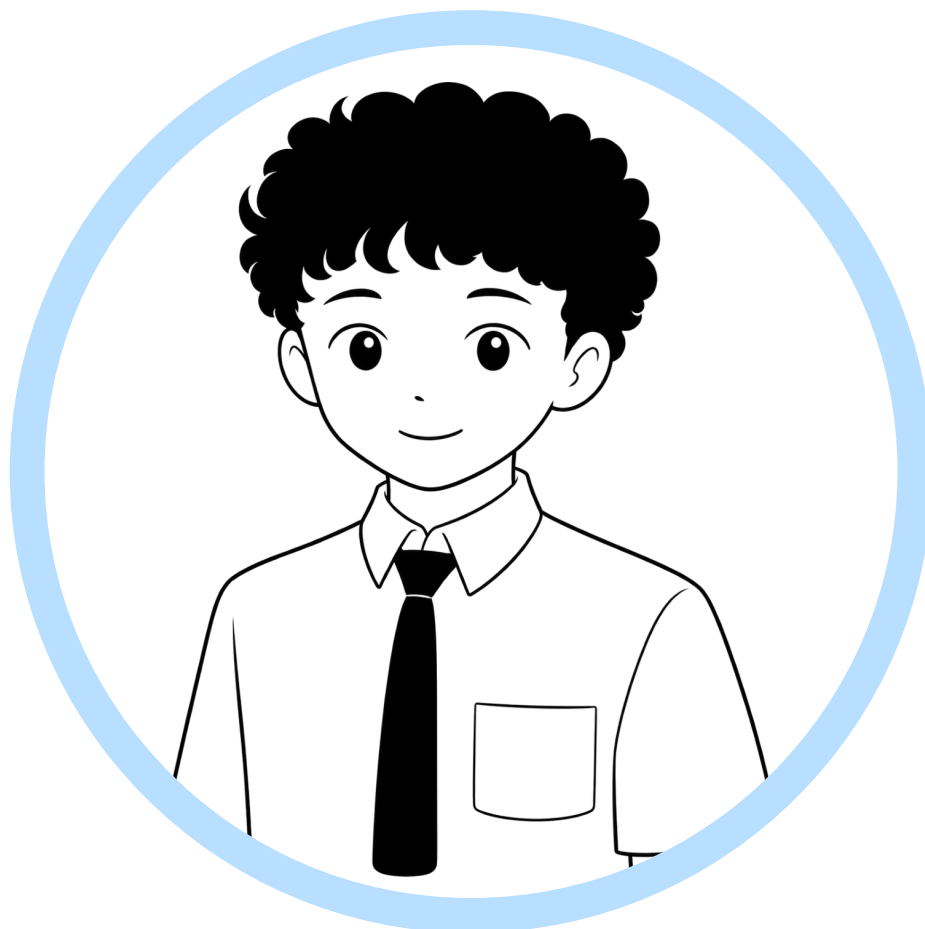


A LEVEL FURTHER MATHS • EXTREME PAPERS

MATHS



MODEL ANSWERS

EXTREME PAPER: FURTHER PURE MATHEMATICS

SCORE

/120

PERCENTAGE

%

STEMUP

Paper: Further Pure Mathematics (120 marks)

1 (8 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) $r = 0$: $\cos\theta = -\frac{1}{2}$, so $\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ M1

$r < 0$ exactly for $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, which traces the small loop A1

Answer: $\theta = 2\pi/3, 4\pi/3$; small loop for $2\pi/3 < \theta < 4\pi/3$ 2 marks

(b) $\text{Area} = \frac{1}{2} \int_{2\pi/3}^{4\pi/3} r^2 d\theta = \frac{1}{2} \int_{2\pi/3}^{4\pi/3} (1 + 4\cos\theta + 4\cos^2\theta) d\theta$ M1

$= \int_{2\pi/3}^{4\pi/3} \left(\frac{3}{2} + 2\cos\theta + \cos 2\theta\right) d\theta = \left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{2}\sin 2\theta\right]_{2\pi/3}^{4\pi/3}$ A1

$= \pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} = \pi - \frac{3\sqrt{3}}{2}$ A1

Answer: $\pi - \frac{3\sqrt{3}}{2}$ 3 marks

(c) Key idea: the small loop lies wholly inside the large loop, because for $r < 0$ the points are on the side of O facing $\theta = 0$ (the large loop is traced for $-\frac{2\pi}{3} \leq \theta \leq \frac{2\pi}{3}$). So subtract. M1

Area bounded by the large loop $= \frac{1}{2} \int_{-2\pi/3}^{2\pi/3} r^2 d\theta = \left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{2}\sin 2\theta\right]_{-2\pi/3}^{2\pi/3} = 2\pi + 2\sqrt{3} -$ A1

$\frac{\sqrt{3}}{2} = 2\pi + \frac{3\sqrt{3}}{2}$

Required area $= (2\pi + \frac{3\sqrt{3}}{2}) - (\pi - \frac{3\sqrt{3}}{2}) = \pi + 3\sqrt{3}$ A1

Answer: $\pi + 3\sqrt{3}$ 3 marks

2 (11 marks) Edexcel | OCR A | OCR MEI

(a) From the first equation $y = 3x - x'$, so $y' = 3x' - x''$ M1

Substitute into the second: $3x' - x'' = x + (3x - x') + 4e^{2t}$ M1

$x'' - 4x' + 4x = -4e^{2t}$ A1

Answer: shown 3 marks

(b) Auxiliary equation $(m - 2)^2 = 0$: CF $= (A + Bt)e^{2t}$ M1 A1

Key idea: e^{2t} and te^{2t} are both in the CF, so the particular integral is kt^2e^{2t} . M1

$x = kt^2e^{2t}$: $x' = k(2t + 2t^2)e^{2t}$, $x'' = k(2 + 8t + 4t^2)e^{2t}$; the left side is $2ke^{2t}$, so $k = -2$ A1

$x(0) = 1$ gives $A = 1$. $x'(0) = 3x(0) - y(0) = 3$, and $x' = (2A + B)e^{2t} + \dots$, so $2A + B = 3$, $B = 1$ M1

$x = (1 + t - 2t^2)e^{2t}$ A1

Answer: $x = (1 + t - 2t^2)e^{2t}$ 6 marks

(c) $x' = (3 - 2t - 4t^2)e^{2t}$, so $y = 3x - x' = (5t - 2t^2)e^{2t}$ M1

$y = t(5 - 2t)e^{2t} = 0$ for $t > 0$ only when $t = \frac{5}{2}$ A1

Answer: $y = (5t - 2t^2)e^{2t}$; $t = 5/2$ 2 marks

3 (10 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) $\frac{dy}{dx} = \frac{2\arcsin x}{\sqrt{1-x^2}}$, so $\sqrt{1-x^2} y' = 2\arcsin x$ M1

Differentiate: $\sqrt{1-x^2} y'' - \frac{x}{\sqrt{1-x^2}} y' = \frac{2}{\sqrt{1-x^2}}$ M1

Multiply by $\sqrt{1-x^2}$: $(1-x^2)y'' - xy' = 2$ A1

Answer: shown 3 marks

(b) Differentiate the result of (a) k times using the Leibniz rule for products M1

$(1-x^2)y^{(k+2)} + k(-2x)y^{(k+1)} + \frac{k(k-1)}{2}(-2)y^{(k)} - [xy^{(k+1)} + ky^{(k)}] = 0$ (the right side 2 M1

differentiates to 0)

Collect: $-2kx - x = -(2k+1)x$ and $-k(k-1) - k = -k^2$, as required A1

Answer: shown**3 marks**

(c) $y(0) = 0$, $y'(0) = 0$, $y''(0) = 2$ (from (a)); at $x = 0$ the result of (b) gives $y^{(k+2)}(0) = k^2 y^{(k)}(0)$, so $y^{(4)}(0) = 4 \times 2 = 8$, $y^{(6)}(0) = 16 \times 8 = 128$ and the odd derivatives vanish

M1

$$y = x^2 + \frac{x^4}{3} + \frac{8x^6}{45} + \dots$$

A1

Answer: $x^2 + \frac{x^4}{3} + \frac{8x^6}{45}$

2 marks

(d) Coefficient c_n of x^{2n} : $c_{n+1}/c_n = \frac{(2n)^2}{(2n+2)(2n+1)}$; the stated formula gives the same ratio (using

M1

$C(2n+2, n+1) = C(2n, n) \frac{(2n+2)(2n+1)}{(n+1)^2}$) and $c_1 = 1$ agrees

$$\text{At } x = \frac{1}{2}: \left(\frac{\pi}{6}\right)^2 = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n^2 C(2n, n)} \cdot \frac{1}{2^{2n}} = \frac{1}{2} \Sigma, \text{ so } \Sigma = \frac{\pi^2}{18}$$

A1

Answer: $\frac{\pi^2}{18}$

2 marks

4 (10 marks) Edexcel (FP1 option) | AQA (Discrete option) | OCR A (Additional Pure) | OCR MEI (Extra Pure)

(a) x is odd, so $x - 1$ and $x + 1$ are consecutive even numbers, one of which is a multiple of 4: $(x - 1)(x + 1)$ is a multiple of 8

M1

x is not a multiple of 3, but one of $x - 1$, x , $x + 1$ is, so it is $x - 1$ or $x + 1$: $(x - 1)(x + 1)$ is a multiple of 3

A1

8 and 3 are coprime, so 24 divides $x^2 - 1$

A1**Answer: proved****3 marks**

(b) For elements a, b : $(ab)^2 = e$, so $ab = (ab)^{-1}$

M1

$(ab)^{-1} = b^{-1}a^{-1} = ba$, because every element is its own inverse. So $ab = ba$

A1**Answer: proved****2 marks**

(c) Key idea: count pairs of non-identity elements.

Let a, b be distinct non-identity elements of U . By (a) and (b), $\{1, a, b, ab\}$ is closed with $ab \notin \{1, a, b\}$, so it is a subgroup of order 4 (and it is the smallest subgroup containing a and b)

B1

Every subgroup of order 4 has 3 non-identity elements and so contains $C(3, 2) = 3$ such pairs; there are $C(7, 2) = 21$ pairs altogether

M1

$21 \div 3 = 7$ subgroups

A1**Answer: 7****3 marks**

(d) By Lagrange, subgroup orders divide 8, so are 1, 2, 4 or 8. Order 1 and 8: one each. Order 2: each non-identity element generates one (seven in total)

M1

Total = $1 + 7 + 7 + 1 = 16$

A1**Answer: 16****2 marks**

5 (11 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) $\frac{d}{dx}(\cos^n x \sin nx) = -n \cos^{n-1} x \sin x \sin nx + n \cos^n x \cos nx$

M1

$$= n \cos^{n-1} x (\cos x \cos nx - \sin x \sin nx) = n \cos^{n-1} x \cos((n+1)x)$$

A1**Answer: shown****2 marks**

(b) Integrate (a) over $[0, \frac{\pi}{2}]$: $n \int_0^{\pi/2} \cos^{n-1} x \cos((n+1)x) dx = [\cos^n x \sin nx]_0^{\pi/2} = 0$

B1

$$\cos x \cos nx = \frac{1}{2} [\cos((n+1)x) + \cos((n-1)x)], \text{ so } C_n = \frac{1}{2} \int_0^{\pi/2} \cos^{n-1} x \cos((n+1)x) dx +$$

M1

$$\frac{1}{2} \int_0^{\pi/2} \cos^{n-1} x \cos((n-1)x) dx$$

The first integral is 0 by the above, so $C_n = \frac{1}{2} C_{n-1}$

A1

$$C_0 = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}$$

B1

$$C_n = \frac{\pi}{2^{n+1}}$$

A1

Answer: $C_n = \pi/2^{n+1}$ **5 marks**

$$(c) \frac{d}{dx}(\cos^n x \cos nx) = -n \cos^{n-1} x \sin x \cos nx - n \cos^n x \sin nx = -n \cos^{n-1} x \sin((n+1)x)$$

M1

So $n \int_0^{\pi/2} \cos^{n-1} x \sin((n+1)x) dx = -[\cos^n x \cos nx]_0^{\pi/2} = 1$, since $\cos^n x \cos nx$ is 0 at $\frac{\pi}{2}$ and 1 at 0

A1

$$\cos x \sin nx = \frac{1}{2}[\sin((n+1)x) + \sin((n-1)x)], \text{ so } T_n = \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{2} T_{n-1}$$

M1

$$T_0 = 0, \text{ so } T_1 = \frac{1}{2}, T_2 = \frac{1}{2}, T_3 = \frac{5}{12}, T_4 = \frac{1}{3}$$

A1**Answer: $\frac{1}{3}$** **4 marks****6 (12 marks) Edexcel | AQA | OCR A | OCR MEI | CIE**

$$(a) d_1 \times d_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \text{ magnitude } 9$$

M1 A1

$$A = (1, 0, -1) \text{ on } l_1 \text{ and } B = (0, 1, 4) \text{ on } l_2: \mathbf{AB} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$$

B1

$$\text{Distance} = |\mathbf{AB} \cdot \mathbf{n}|/|\mathbf{n}| = |(-1)(-6) + 1 \cdot 6 + 5 \cdot 3|/9 = 27/9 = 3$$

A1**Answer: 3****4 marks**

$$(b) \text{ Key idea: Volume} = \frac{1}{6} |[\mathbf{PP'}, \mathbf{QQ'}, \mathbf{PQ}]|, \text{ and } \mathbf{PP'} = \pm \frac{4}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \mathbf{QQ'} = \pm 2 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

M1

$[\mathbf{PP'}, \mathbf{QQ'}, \mathbf{PQ}] = \pm \frac{8}{3} (d_1 \times d_2) \cdot \mathbf{PQ}$; moving P along l_1 or Q along l_2 changes $\mathbf{PQ}$ by a multiple of d_1 or d_2 , both perpendicular to $d_1 \times d_2$, so the value is unchanged

A1

$$\text{Using } \mathbf{PQ} = \mathbf{AB}: (d_1 \times d_2) \cdot \mathbf{AB} = 27, \text{ so volume} = \frac{1}{6} \times \frac{8}{3} \times 27 = 12$$

A1**Answer: 12****3 marks**

$$(c) P_0 = \begin{pmatrix} 1 + 2\lambda \\ \lambda \\ -1 + 2\lambda \end{pmatrix}, Q_0 = \begin{pmatrix} \mu \\ 1 + 2\mu \\ 4 - 2\mu \end{pmatrix}; \mathbf{P_0 Q_0} \text{ perpendicular to } d_1 \text{ and } d_2 \text{ gives } 9 - 9\lambda = 0 \text{ and } 9\mu - 9 = 0$$

M1

$$\lambda = \mu = 1: P_0 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, Q_0 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

A1

$$\text{Key idea: with unit vectors } u_1, u_2 \text{ along the lines and } e = \mathbf{P_0 Q_0}/3 = \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \text{ the centre is}$$

M1

equidistant from P, P' so lies in the plane through P_0 perpendicular to l_1 ; similarly for l_2 . Hence it is on the common perpendicular: $C = P_0 + se$, with $0 < s < 3$ along $P_0 Q_0$.

$$|CP|^2 = 4 + s^2, |CQ|^2 = 9 + (3-s)^2; \text{ equal when } 4 + s^2 = 9 + 6s - s^2, \text{ so } s = \frac{7}{2}$$

M1

$$C = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \frac{7}{2} \cdot \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 13 \\ 23 \\ 16 \end{pmatrix}, \text{ radius}^2 = 4 + \frac{49}{9} = \frac{85}{9}, \text{ radius } \frac{\sqrt{85}}{3}$$

A1**Answer: centre (13/9, 23/9, 16/9), radius $\sqrt{85}/3$** **5 marks**

7 (11 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

$$(a) \begin{vmatrix} 7-\lambda & -6 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = 0 \quad \text{M1}$$

$$\lambda = 1 \text{ and } \lambda = 4 \quad \text{A1}$$

$$\lambda = 1: \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \lambda = 4: \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{A1}$$

$$\text{Answer: } 1 \text{ with } \begin{pmatrix} 1 \\ 1 \end{pmatrix}; 4 \text{ with } \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{3 marks}$$

$$(b) P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, P^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}, A^n = PD^nP^{-1} \quad \text{M1}$$

$$= \begin{pmatrix} 1 & 2 \cdot 4^n \\ 1 & 4^n \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \quad \text{M1}$$

$$A^n = \begin{pmatrix} 2 \cdot 4^n - 1 & 2 - 2 \cdot 4^n \\ 4^n - 1 & 2 - 4^n \end{pmatrix} \quad \text{A1}$$

$$\text{Answer: } \begin{pmatrix} 2 \cdot 4^n - 1 & 2 - 2 \cdot 4^n \\ 4^n - 1 & 2 - 4^n \end{pmatrix} \quad \text{3 marks}$$

(c) Key idea: X commutes with $X^2 = A$, so X preserves each eigenspace of A. B1

If $Av = \lambda v$ then $A(Xv) = X(Av) = \lambda(Xv)$, so Xv lies in the same one-dimensional eigenspace: $Xv = \mu v$. M1

$$\text{Then } \mu^2 = \lambda: \mu = \pm 1 \text{ on } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mu = \pm 2 \text{ on } \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \text{ So } X = P \text{diag}(\mu_1, \mu_2) P^{-1} \quad \text{M1}$$

Exactly four matrices (any combination of signs), each valid since $X^2 = P \text{diag}(1, 4) P^{-1} = A$ A1

$$(1, 2): \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}; (1, -2): \begin{pmatrix} -5 & 6 \\ -3 & 4 \end{pmatrix}; (-1, 2): \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}; (-1, -2): \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix} \quad \text{A1}$$

$$\text{Answer: } \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -5 & 6 \\ -3 & 4 \end{pmatrix}, \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}, \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix} \quad \text{5 marks}$$

8 (11 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

$$(a) \text{ Let } A = \arctan(r+1), B = \arctan(r-1). \tan(A-B) = \frac{(r+1) - (r-1)}{1 + (r+1)(r-1)} \quad \text{M1}$$

$$= \frac{2}{r^2} \quad \text{A1}$$

Since A is in $(0, \frac{\pi}{2})$ and B is in $[0, \frac{\pi}{2})$ with $A > B$, we have $0 < A - B < \frac{\pi}{2}$, so $A - B =$ B1

$\arctan(\frac{2}{r^2})$ with no multiple of π

Answer: shown 3 marks

$$(b) \text{ Sum (a) from } r = 1 \text{ to } n: \Sigma = \sum_{r=1}^n [\arctan(r+1) - \arctan(r-1)] \quad \text{M1}$$

Cancelling in pairs leaves $\arctan(n+1) + \arctan n - \arctan 1 - \arctan 0$ A1

$$= \arctan n + \arctan(n+1) - \frac{\pi}{4} \quad \text{A1}$$

$$\text{Answer: } \arctan n + \arctan(n+1) - \frac{\pi}{4} \quad \text{3 marks}$$

$$(c) \text{ As } n \rightarrow \infty, \arctan n \rightarrow \frac{\pi}{2} \text{ and } \arctan(n+1) \rightarrow \frac{\pi}{2}, \text{ so the sum is } \frac{3\pi}{4} \quad \text{B1}$$

$$\text{Answer: } \frac{3\pi}{4} \quad \text{1 mark}$$

$$(d) \text{ Arg } P_n = \sum_{r=1}^n \arctan(\frac{2}{r^2}) = \theta_n = \arctan n + \arctan(n+1) - \frac{\pi}{4}, \text{ with } 0 < \theta_n < \frac{3\pi}{4} \quad \text{M1}$$

P_n is purely imaginary exactly when $\theta_n = \frac{\pi}{2}$ (the only multiple of $\frac{\pi}{2}$ of the right form in this range, and this gives $\text{Im } P_n > 0$) M1

So $\arctan n + \arctan(n+1) = \frac{3\pi}{4}$, hence $\tan$ of it = -1 : $\frac{2n+1}{1-n(n+1)} = -1$, so $n^2 - n - 2 = 0$

A1

$n = 2$ ($n = -1$ rejected). Check: $(1+2i)(4+2i) = 10i$

A1

Answer: $n = 2$ only

4 marks

9 (12 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) $(1 + e^{i\theta})^n = \sum_{r=0}^n C(n, r)e^{ir\theta}$; the real part of the right side is $\sum C(n, r)\cos r\theta$

M1

$1 + e^{i\theta} = e^{i\theta/2}(e^{-i\theta/2} + e^{i\theta/2}) = 2\cos(\frac{\theta}{2})e^{i\theta/2}$, so $(1 + e^{i\theta})^n = 2^n\cos^n(\frac{\theta}{2})e^{in\theta/2}$

M1

Real part: $2^n\cos^n(\frac{\theta}{2})\cos(\frac{n\theta}{2})$, as required

A1

Answer: shown

3 marks

(b) $1 + \omega^r + \omega^{2r} = 3$ if 3 divides r , and $= 0$ otherwise (geometric series with $\omega^3 = 1$)

B1

So $3S_n = \sum_{r=0}^n C(n, r)(1 + \omega^r + \omega^{2r}) = 2^n + (1 + \omega)^n + (1 + \omega^2)^n$

M1

$1 + \omega = 1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\pi/3}$, and $1 + \omega^2 = e^{-i\pi/3}$

M1

So the last two terms sum to $2\cos(\frac{n\pi}{3})$, giving the result

A1

Answer: shown

4 marks

(c) $\frac{2026\pi}{3} = 675\pi + \frac{\pi}{3}$, so $\cos(\frac{2026\pi}{3}) = -\cos(\frac{\pi}{3}) = -\frac{1}{2}$

M1

$S_{2026} = \frac{1}{3}(2^{2026} - 1)$

A1

Answer: $\frac{2^{2026} - 1}{3}$

2 marks

(d) Differentiating the left side of (a) gives $-\sum r C(n, r) \sin r\theta$

M1

Right side, product rule: $2^n[-\frac{n}{2}\cos^{n-1}(\frac{\theta}{2})\sin(\frac{\theta}{2})\cos(\frac{n\theta}{2}) - \frac{n}{2}\cos^n(\frac{\theta}{2})\sin(\frac{n\theta}{2})]$

M1

Negate and factor: $n \cdot 2^{n-1}\cos^{n-1}(\frac{\theta}{2})[\sin(\frac{\theta}{2})\cos(\frac{n\theta}{2}) + \cos(\frac{\theta}{2})\sin(\frac{n\theta}{2})] =$

A1

$n \cdot 2^{n-1}\cos^{n-1}(\frac{\theta}{2})\sin(\frac{(n+1)\theta}{2})$

Answer: shown

3 marks

10 (12 marks) Edexcel | AQA | OCR A | OCR MEI

(a) $x = \sinh u$, $dx = \cosh u du$, and $x + \sqrt{x^2 + 1} = \sinh u + \cosh u = e^u$

B1

$I_n = \int_0^\infty e^{-nu}\cosh u du = \frac{1}{2}\int_0^\infty (e^{-(n-1)u} + e^{-(n+1)u}) du$

M1

Integrate up to a limit R : $\frac{1}{2}[\frac{1 - e^{-(n-1)R}}{n-1} + \frac{1 - e^{-(n+1)R}}{n+1}]$

M1

As $R \rightarrow \infty$ the exponentials tend to 0 because $n \geq 2$ ($n-1 > 0$)

M1

$I_n = \frac{1}{2}(\frac{1}{n-1} + \frac{1}{n+1}) = \frac{n}{n^2 - 1}$

A1

Answer: shown

5 marks

(b) $x = \cosh u$ ($u \geq 0$), $dx = \sinh u du$, $x + \sqrt{x^2 - 1} = e^u$; limits $u = 0$ to ∞

M1

$K_n = \int_0^\infty e^{-nu}\sinh u du = \frac{1}{2}\int_0^\infty (e^{-(n-1)u} - e^{-(n+1)u}) du$

M1

$= \frac{1}{2}(\frac{1}{n-1} - \frac{1}{n+1}) = \frac{1}{n^2 - 1}$

A1

Answer: shown

3 marks

(c) (i) $\frac{1}{n^2 - 1} = \frac{1}{2}(\frac{1}{n-1} - \frac{1}{n+1})$; the sum telescopes to $\frac{1}{2}(1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}) = \frac{3}{4} -$

M1 A1

$\frac{2N+1}{2N(N+1)}$

$$(ii) \text{ For } x > 1, r > 1 \text{ and } \sum_{n=2}^{\infty} r^{-n} = \frac{r^{-2}}{1 - r^{-1}} = \frac{1}{r(r-1)}$$

B1

$$\text{Integrating term by term, the integral} = \sum_{n \geq 2} K_n = \lim_{N \rightarrow \infty} \left[\frac{3}{4} - \frac{2N+1}{2N(N+1)} \right] = \frac{3}{4}$$

A1

$$\text{Answer: } \frac{3}{4}$$

4 marks

11 (12 marks) Edexcel | AQA | OCR A | OCR MEI

$$(a) x = \frac{1}{t}, dx = -\frac{1}{t^2} dt; \text{ limits } x: 0 \rightarrow \infty \text{ become } t: \infty \rightarrow 0$$

M1

$$\frac{1}{1+x^4} = \frac{t^4}{t^4+1}, \text{ so } I = \int_{\infty}^0 \frac{t^4}{1+t^4} \cdot \left(-\frac{1}{t^2}\right) dt$$

M1

$$= \int_0^{\infty} \frac{t^2}{1+t^4} dt = J$$

A1

Answer: shown

3 marks

$$(b) \text{ Key idea: } I + J = 2I = \int_0^{\infty} \frac{1+x^2}{1+x^4} dx$$

M1

$$\text{Divide numerator and denominator by } x^2: \frac{1+x^{-2}}{x^2+x^{-2}}$$

M1

$$u = x - \frac{1}{x}, du = \left(1 + \frac{1}{x^2}\right) dx$$

M1

$$u^2 + 2 = x^2 + \frac{1}{x^2}; \text{ as } x \text{ goes from } 0 \text{ to } \infty, u \text{ goes from } -\infty \text{ to } \infty \text{ increasing}$$

A1

$$2I = \int_{-\infty}^{\infty} \frac{1}{u^2+2} du = \left[\frac{1}{\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right) \right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{2}}$$

M1

$$I = \frac{\pi}{2\sqrt{2}} = \frac{\sqrt{2}\pi}{4}$$

A1

$$\text{Answer: } \frac{\pi}{2\sqrt{2}}$$

6 marks

$$(c) \text{ The substitution } x = \frac{1}{t} \text{ sends } \frac{1}{x^4+x^2+1} dx \text{ to } \frac{t^2}{t^4+t^2+1} dt, \text{ so the integral equals } \int_0^{\infty}$$

B1

$$\frac{x^2}{x^4+x^2+1} dx$$

$$\text{Twice the integral} = \int_0^{\infty} \frac{1+x^{-2}}{x^2+1+x^{-2}} dx; \text{ with } u = x - \frac{1}{x}: x^2+1+\frac{1}{x^2} = u^2+3$$

M1

$$= \int_0^{\infty} \frac{1}{u^2+3} du = \frac{\pi}{\sqrt{3}}, \text{ so the integral is } \frac{\pi}{2\sqrt{3}}$$

A1

$$\text{Answer: } \frac{\pi}{2\sqrt{3}}$$

3 marks