

A LEVEL FURTHER MATHS • EXTREME PAPERS

MATHS



MODEL ANSWERS

EXTREME PAPER: FURTHER MECHANICS

SCORE

/90

PERCENTAGE

%

STEMUP

Paper: Further Mechanics (90 marks)

1 (11 marks) Edexcel | OCR A

(a) By symmetry A moves along its original line, with velocity w in its original direction (w may be negative), and B and C have equal speeds. The line of centres from A to B makes 30° with the path of A.

Let J be the impulse between A and B (and between A and C). If there is no impulse between B and C then B moves along the line of centres, so $J = m v$ where v is the speed of B. B1

Momentum of A along its path: $m u - m w = 2J \cos 30^\circ$, so $u - w = \sqrt{3} v$. M1

Restitution along the line of centres (A and B): $v - w \cos 30^\circ = e u \cos 30^\circ$. M1

Substituting $w = u - \sqrt{3} v$: $v + \frac{3}{2}v = \frac{\sqrt{3}}{2}(1 + e) u$, so $v = \frac{\sqrt{3}(1 + e)}{5} u$ and $w = u - \frac{3(1 + e)}{5} u = \frac{(2 - 3e)}{5} u$. A1

Check on the assumption: B and C each have a velocity component of magnitude $v \sin 30^\circ$ perpendicular to the path of A, B away from C and C away from B. So with no impulse between them they separate at once, which is consistent; a contact impulse could only push them apart and would not be needed. E1

Answer: A: velocity $\frac{(2 - 3e)}{5} u$; B: speed $\frac{\sqrt{3}(1 + e)}{5} u$ 5 marks

(b) A rebounds if and only if $w < 0$, that is $e > \frac{2}{3}$. A is brought to rest when $e = \frac{2}{3}$. A1

For $e = \frac{2}{3}$: $v = \frac{\sqrt{3}}{5} \times \frac{5}{3} u = \frac{u}{\sqrt{3}}$. KE after $= 2 \times \frac{1}{2} m \left(\frac{u^2}{3}\right) = \frac{1}{3} m u^2$. M1

KE before $= \frac{1}{2} m u^2$, so the fraction lost is $1 - \frac{2}{3} = \frac{1}{3}$. A1

Answer: A rebounds iff $e > 2/3$; fraction lost $= 1/3$ 3 marks

(c) Repeat with angle ϕ : $u - w = 2v \cos \phi$ and $v - w \cos \phi = e u \cos \phi$. Eliminating v : $w(1 + 2\cos^2\phi) = u(1 - 2e \cos^2\phi)$. M1 A1

A is brought to rest when $\cos^2\phi = \frac{1}{(2e)}$. The spheres B and C cannot overlap, so their centres are A1

at least $2r$ apart, which needs $\phi \geq 30^\circ$, that is $\cos^2\phi \leq \frac{3}{4}$; hence $\frac{1}{(2e)} \leq \frac{3}{4}$ and $e \geq \frac{2}{3}$. For $e = \frac{4}{5}$: $\cos^2\phi = \frac{5}{8}$.

Answer: $w = \frac{(1 - 2e \cos^2\phi)}{(1 + 2\cos^2\phi)} u$; need $e \geq 2/3$; $\cos^2\phi = 5/8$ 3 marks

2 (11 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Ascent: $m v \frac{dv}{dx} = -m g - \frac{m g v^2}{V^2}$, so $\frac{v}{(V^2 + v^2)} \frac{dv}{dx} = -\frac{g}{V^2}$. M1

Integrating from $x = 0$ ($v = U$) to $x = H$ ($v = 0$): $\frac{1}{2} [\ln(V^2 + v^2)]_U^0 = -\frac{g H}{V^2}$. M1

$\frac{1}{2} \ln\left(\frac{V^2 + U^2}{V^2}\right) = \frac{g H}{V^2}$, so $H = \frac{V^2}{(2g)} \ln\left(1 + \frac{U^2}{V^2}\right)$. A1

Answer: $H = \frac{V^2}{(2g)} \ln\left(1 + \frac{U^2}{V^2}\right)$ 3 marks

(b) Descent: $m v \frac{dv}{dx} = m g - \frac{m g v^2}{V^2}$, so $\int_0^w \frac{v}{(V^2 - v^2)} dv = \frac{g}{V^2} H$, giving $-\frac{1}{2} \ln\left(1 - \frac{w^2}{V^2}\right) = \frac{1}{2}$ M1

$\ln\left(1 + \frac{U^2}{V^2}\right)$.

So $1 - \frac{w^2}{V^2} = \frac{V^2}{(U^2 + V^2)}$, $w^2 = \frac{U^2 V^2}{(U^2 + V^2)}$. A1

Answer: $w = \frac{U V}{\sqrt{U^2 + V^2}}$ 2 marks

(c) Ascent: $\frac{dv}{dt} = -\frac{g(V^2 + v^2)}{V^2}$, so $T_{up} = \frac{V^2}{g} \int_0^U \frac{1}{(V^2 + v^2)} dv$.	M1
$= \frac{V}{g} \arctan\left(\frac{U}{V}\right) = \frac{V}{g} \theta$.	A1
Descent: $\frac{dv}{dt} = \frac{g(V^2 - v^2)}{V^2}$, so $T_{down} = \frac{V^2}{g} \int_0^w \frac{1}{(V^2 - v^2)} dv = \frac{V}{(2g)} \ln\left(\frac{(V+w)}{(V-w)}\right)$.	M1
From (b), $\frac{w}{V} = \frac{U}{\sqrt{U^2 + V^2}} = \sin \theta$, and $\frac{(1 + \sin \theta)}{(1 - \sin \theta)} = \frac{(1 + \sin \theta)^2}{\cos^2 \theta} = (\sec \theta + \tan \theta)^2$. So $T_{down} = \frac{V}{g} \ln(\sec \theta + \tan \theta)$.	A1
Answer: $T_{up} = \frac{V}{g} \theta$; $T_{down} = \frac{V}{g} \ln(\sec \theta + \tan \theta)$	4 marks
(d) Key idea: compare the two times as functions of θ . Let $f(\theta) = \ln(\sec \theta + \tan \theta) - \theta$ for $0 \leq \theta$	M1
$< \frac{\pi}{2}$, so $f(0) = 0$. $\frac{d}{d\theta} \ln(\sec \theta + \tan \theta) = \frac{(\sec \theta \tan \theta + \sec^2 \theta)}{(\sec \theta + \tan \theta)} = \sec \theta$.	
So $f'(\theta) = \sec \theta - 1 > 0$ for $0 < \theta < \frac{\pi}{2}$, hence $f(\theta) > 0$ and $T_{down} > T_{up}$ for every $U > 0$.	E1
Answer: $f(\theta) = \sec \theta - 1 > 0$, so $T_{down} > T_{up}$	2 marks
3 (11 marks) Edexcel AQA OCR A OCR MEI CIE	
(a) Take O as origin and the axis of symmetry as the x-axis. H is formed by rotating $y = \sqrt{a^2 - x^2}$, $0 \leq x \leq a$, about the x-axis; a slice of width δx has volume $\pi(a^2 - x^2) \delta x$.	M1
Volume $= \pi \int_0^a (a^2 - x^2) dx = \pi(a^3 - \frac{a^3}{3}) = \frac{2\pi a^3}{3}$.	A1
Volume $\times \bar{x} = \pi \int_0^a x(a^2 - x^2) dx = \pi[\frac{a^2 x^2}{2} - \frac{x^4}{4}]_0^a = \frac{\pi a^4}{4}$.	M1
$\bar{x} = \frac{\pi a^4}{4} \div \frac{2\pi a^3}{3} = \frac{3a}{8}$.	A1
Answer: $\frac{3a}{8}$ from O	4 marks
(b) Masses are proportional to volumes: H is $\frac{2}{3}\pi a^3$, C is $\frac{1}{3}\pi a^2 h$. The centre of mass of C is $\frac{h}{4}$ from its base. Moments about O, with x_G measured positive towards the apex of C:	M1
$(\frac{2}{3}\pi a^3 + \frac{1}{3}\pi a^2 h) x_G = \frac{1}{3}\pi a^2 h \cdot \frac{h}{4} - \frac{2}{3}\pi a^3 \cdot \frac{3a}{8}$.	A1
$\frac{\pi a^2(2a + h)}{3} x_G = \frac{\pi a^2(h^2 - 3a^2)}{12}$, so $x_G = -\frac{(3a^2 - h^2)}{(4(2a + h))}$: the distance quoted, on the side of H.	A1
Answer: distance $\frac{(3a^2 - h^2)}{(4(2a + h))}$ from O, on the side of H	3 marks
(c) For $h = a$, G is a distance $\frac{2a^2}{(12a)} = \frac{a}{6}$ from O, on the axis, on the side of H (so G is below O when φ is acute).	B1
The curved surface is part of a sphere of centre O, so the normal reaction at the contact point P acts along PO and OP = a is perpendicular to the plane. Three forces act (weight, normal reaction, friction), the last two at P, so the weight must act along the vertical through P: G is vertically above P.	M1
P is horizontally a $\sin \theta$ from O, on the up-slope side. G is horizontally $\frac{a}{6} \sin \varphi$ from O, also up-slope when G is below O on a tilted axis. Equating: $\frac{a}{6} \sin \varphi = a \sin \theta$, so $\sin \varphi = 6 \sin \theta$.	A1
This needs $\sin \varphi \leq 1$, so $\sin \theta \leq \frac{1}{6}$. Friction: $F = W \sin \theta$, $R = W \cos \theta$, so $\tan \theta \leq \frac{1}{5}$ is needed.	A1
At $\sin \theta = \frac{1}{6}$, $\tan \theta = \frac{1}{\sqrt{35}} < \frac{1}{5}$ ($35 > 25$), so friction is not the limit. Greatest $\theta = \arcsin(\frac{1}{6}) \approx 9.6^\circ$.	
Answer: $\sin \varphi = 6 \sin \theta$; $\theta_{max} = \arcsin(\frac{1}{6}) \approx 9.6^\circ$	4 marks

4 (10 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) When the string goes slack, $T = 0$ and the speed v satisfies, resolving towards O, $m g \sin \alpha = \frac{m v^2}{r}$, so $v^2 = g r \sin \alpha$. M1

Energy from the lowest point (a rise of $r(1 + \sin \alpha)$): $\frac{1}{2} m u^2 = \frac{1}{2} m v^2 + m g r (1 + \sin \alpha)$. M1

$u^2 = g r \sin \alpha + 2 g r (1 + \sin \alpha) = g r (2 + 3 \sin \alpha)$. A1

Answer: $u^2 = g r (2 + 3 \sin \alpha)$ 3 marks

(b) Take O as origin, x to the right, y upwards, and write $s = \sin \alpha$, $c = \cos \alpha$, $k = \sqrt{c^2 + 2}$. At slackening P is at $(r c, r s)$ with velocity $v(-s, c)$ and $v^2 = g r s$. So $x = r c - v t s$ and $y = r s + v t c - \frac{1}{2} g t^2$. M1

$y = 0$ gives $\frac{1}{2} g t^2 - v c t - r s = 0$, so $t = \sqrt{r s/g}(c + k)$ for the positive root. M1

$x = -r: r c - v t s = -r$. Now $v t s = r s^2(c + k)$, so $c - s^2(c + k) = -1$. Using $c - s^2 c = c^3: c^3 + 1 = (1 - c^2) k$. M1

Divide by $(1 + c)$: $c^2 - c + 1 = (1 - c) k$. Square: $(c^2 - c + 1)^2 = (1 - c)^2(c^2 + 2)$, i.e. $c^4 - 2c^3 + 3c^2 - 2c + 1 = c^4 - 2c^3 + 3c^2 - 4c + 2$, so $2c = 1$. Check $c = \frac{1}{2}$: both sides of the unsquared equation equal $\frac{3}{4}$. A1

So $\alpha = 60^\circ$, $\sin \alpha = \frac{\sqrt{3}}{2}$ and $u^2 = g r (2 + \frac{3\sqrt{3}}{2}) = \frac{(4 + 3\sqrt{3})}{2} g r$. A1

Answer: $\cos \alpha = 1/2; u^2 = \frac{(4 + 3\sqrt{3})}{2} g r$ 5 marks

(c) Energy from slackening to B (a fall of $r \sin \alpha$): $\text{speed}^2 = g r \sin \alpha + 2 g r \sin \alpha = 3 g r \sin \alpha = \frac{3\sqrt{3}}{2} g r$, so the speed is $\sqrt{3\sqrt{3} g r/2}$. M1

The horizontal component is constant: $v_x^2 = v^2 \sin^2 \alpha = g r \sin^3 \alpha$, so $\tan^2(\text{angle}) = \frac{3 g r \sin \alpha}{(g r \sin^3 \alpha)} - 1 = \frac{3}{\sin^2 \alpha} - 1 = 3$. The velocity is directed downwards at 60° to the horizontal. A1

Answer: speed $\sqrt{3\sqrt{3} g r/2}$; 60° below the horizontal 2 marks

5 (11 marks) Edexcel | AQA | OCR A | OCR MEI

(a) Hanging part has weight $\frac{M g x}{L}$; part on the table has mass $\frac{M(L - x)}{L}$, so friction is $\frac{1}{2} \cdot \frac{M g (L - x)}{L}$. M1

The chain is inextensible and moves as a whole: $M \frac{d^2 x}{dt^2} = \frac{M g x}{L} - \frac{M g (L - x)}{(2L)}$. A1

$\frac{d^2 x}{dt^2} = \frac{g}{(2L)}(3x - L) = \frac{3g}{(2L)}(x - \frac{L}{3})$. A1

Answer: shown 3 marks

(b) Let $\omega^2 = \frac{3g}{(2L)}$ and $y = x - \frac{L}{3}$. Then $\frac{d^2 y}{dt^2} = \omega^2 y$ with $y = \frac{L}{6}$ and $\frac{dy}{dt} = 0$ at $t = 0$. General solution $y = A \cosh \omega t + B \sinh \omega t$. M1

$A = \frac{L}{6}$, $B = 0$, so $x = \frac{L}{3} + \frac{L}{6} \cosh \omega t$. A1

Last link leaves when $x = L$: $\frac{L}{6} \cosh \omega t = \frac{2L}{3}$, so $\cosh \omega t = 4$. M1

$t = \frac{1}{\omega} \operatorname{arcosh} 4 = \sqrt{2L/(3g)} \ln(4 + \sqrt{15})$. A1

Answer: $t = \sqrt{2L/(3g)} \ln(4 + \sqrt{15})$ 4 marks

(c) Net force along the chain when x is hanging: $F = \frac{M g x}{L} - \frac{M g (L - x)}{(2L)}$. Work-energy: $\frac{1}{2} M v^2$ M1

$$= \int_{L/2}^L F \, dx.$$

$$\text{Gravity term: } \int_{L/2}^L \frac{M g x}{L} \, dx = \frac{M g}{(2L)} \left(L^2 - \frac{L^2}{4} \right) = \frac{3 M g L}{8}.$$

A1

$$\text{Friction term: } \int_{L/2}^L \frac{M g (L - x)}{(2L)} \, dx = \frac{M g}{(2L)} \cdot \frac{(L)^2}{2} = \frac{M g L}{16}.$$

A1

$$\frac{1}{2} M v^2 = \frac{3 M g L}{8} - \frac{M g L}{16} = \frac{5 M g L}{16}, \text{ so } v = \sqrt{5 g L/8}.$$

A1

$$\text{Answer: } v = \sqrt{5 g L/8}$$

4 marks

6 (12 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

$$\text{(a) } 0 < x < l: \text{ AP is slack, BP has extension } 2l - x, \text{ so } m \frac{d^2x}{dt^2} = \frac{\lambda}{l}(2l - x): \text{ SHM about } x = 2l,$$

M1

$$\omega_1^2 = \frac{\lambda}{(m l)}.$$

$$l < x < 2l: \text{ both taut: } m \frac{d^2x}{dt^2} = -\frac{\lambda}{l}(x - l) + \frac{\lambda}{l}(2l - x) = -\frac{2\lambda}{l}(x - \frac{3l}{2}): \text{ SHM about } x = \frac{3l}{2}, \omega_2^2$$

A1

$$= \frac{2\lambda}{(m l)}, \text{ so } \omega_2 = \sqrt{2} \omega_1.$$

$$2l < x < 3l: \text{ BP slack: } m \frac{d^2x}{dt^2} = -\frac{\lambda}{l}(x - l): \text{ SHM about } x = l \text{ with angular speed } \omega_1.$$

A1

$$\text{Answer: outer ranges: centres } 2l \text{ and } l, \text{ angular speed } \omega_1; \text{ middle: centre } \frac{3l}{2}, \omega_2 = \sqrt{2} \omega_1$$

3 marks

$$\text{(b) Energy from } x = 0 \text{ (BP extension } 2l, \text{ AP slack) to } x = l \text{ (BP extension } l): \frac{1}{2} m v^2 = \frac{\lambda(2l)^2}{(2l)} -$$

M1

$$\frac{\lambda l^2}{(2l)} = \frac{3\lambda l}{2}.$$

$$v^2 = \frac{3\lambda l}{m} = 3 \omega_1^2 l^2, v = \sqrt{3} \omega_1 l.$$

A1

$$\text{In the middle range } v^2 = \omega_2^2(a^2 - (x - \frac{3l}{2})^2); \text{ at } x = l, x - \frac{3l}{2} = -\frac{l}{2}. \text{ So } 3 \omega_1^2 l^2 = 2 \omega_1^2(a^2 - \frac{l^2}{4}).$$

M1

$$a^2 = \frac{3l^2}{2} + \frac{l^2}{4} = \frac{7l^2}{4}, \text{ so } a = \frac{\sqrt{7}}{2} l.$$

A1

$$\text{Answer: } v = \sqrt{3} \omega_1 l = \sqrt{3\lambda l/m}; a = \frac{\sqrt{7}}{2} l$$

4 marks

$$\text{(c) Range } 0 < x < l: x - 2l = -2l \cos \omega_1 t \text{ (} x = 0 \text{ and at rest at } t = 0). x = l \text{ when } \cos \omega_1 t_1 = \frac{1}{2}.$$

M1

$$t_1 = \frac{\pi}{(3 \omega_1)}.$$

A1

$$\text{Middle range: } u = x - \frac{3l}{2} = a \sin \psi, \text{ from } u = -\frac{l}{2} \text{ to } u = \frac{l}{2}; \sin \psi_0 = \frac{(l/2)/a}{1} = \frac{1}{\sqrt{7}}. \text{ Time } t_2 = \frac{2\psi_0}{\omega_2}.$$

M1

$$t_2 = \frac{2}{(\sqrt{2} \omega_1)} \arcsin\left(\frac{1}{\sqrt{7}}\right) = \frac{\sqrt{2}}{\omega_1} \arcsin\left(\frac{1}{\sqrt{7}}\right).$$

A1

$$\text{The motion is symmetrical about } x = \frac{3l}{2}: \text{ P reaches } x = 3l \text{ at rest after } t_1 + t_2 + t_1 \text{ and returns, so } T$$

A1

$$= 4t_1 + 2t_2 = \sqrt{m l/\lambda} \left[\frac{4\pi}{3} + 2\sqrt{2} \arcsin\left(\frac{1}{\sqrt{7}}\right) \right] \approx 5.29 \sqrt{m l/\lambda}.$$

$$\text{Answer: } T = \sqrt{m l/\lambda} \left[\frac{4\pi}{3} + 2\sqrt{2} \arcsin\left(\frac{1}{\sqrt{7}}\right) \right] \approx 5.29 \sqrt{m l/\lambda}$$

5 marks

7 (12 marks) Edexcel | OCR A | OCR MEI | CIE

(a) Triangle OAP is isosceles ($OA = OP = a$): $AP = 2a \sin(\frac{\theta}{2})$. Extension $= 2a \sin(\frac{\theta}{2}) - a$. M1 A1

$$V = m g a \cos \theta + \frac{3 m g}{(2a)} (2a \sin(\frac{\theta}{2}) - a)^2 = m g a [\cos \theta + \frac{3}{2} (2 \sin(\frac{\theta}{2}) - 1)^2]. \text{ Expand: } \frac{3}{2} (4 \sin^2(\frac{\theta}{2}) - 4 \sin(\frac{\theta}{2}) + 1) = 3(1 - \cos \theta) - 6 \sin(\frac{\theta}{2}) + \frac{3}{2}, \text{ so } V = m g a [\frac{9}{2} - 6 \sin(\frac{\theta}{2}) - 2 \cos \theta].$$

Answer: shown 3 marks

(b) For $0 < \theta < \frac{\pi}{3}$ the string is slack and $V = m g a \cos \theta$, so $V' = -m g a \sin \theta \neq 0$: no equilibrium there. B1

$$\text{For } \theta \geq \frac{\pi}{3}: V' = m g a [-3 \cos(\frac{\theta}{2}) + 2 \sin \theta] = m g a \cos(\frac{\theta}{2}) (4 \sin(\frac{\theta}{2}) - 3).$$
M1

$V' = 0$: $\cos(\frac{\theta}{2}) = 0$ gives $\theta = \pi$; or $\sin(\frac{\theta_0}{2}) = \frac{3}{4}$, i.e. $\theta_0 \approx 97.2^\circ > 60^\circ$, which is valid. Exactly two equilibria. A1

$$V'' = m g a [\frac{3}{2} \sin(\frac{\theta}{2}) + 2 \cos \theta].$$
M1

At θ_0 : $\cos \theta_0 = 1 - 2 \times \frac{9}{16} = -\frac{1}{8}$, so $V'' = m g a (\frac{9}{8} - \frac{1}{4}) = \frac{7}{8} m g a > 0$: stable. At $\theta = \pi$: $V'' = m g a (\frac{3}{2} - 2) = -\frac{1}{2} m g a < 0$: unstable. A1

Answer: $\sin(\frac{\theta_0}{2}) = \frac{3}{4}$; θ_0 stable, $\theta = \pi$ unstable 5 marks

(c) Energy: $\frac{1}{2} m a^2 (\frac{d\theta}{dt})^2 + V(\theta) = \text{constant}$. Differentiating with respect to t : $m a^2 \frac{d^2\theta}{dt^2} = -V'(\theta)$. M1

Put $\theta = \theta_0 + \varepsilon$ with ε small and $V'(\theta_0) = 0$: $V'(\theta) \approx V''(\theta_0) \varepsilon = \frac{7}{8} m g a \varepsilon$. M1

$$\frac{d^2\varepsilon}{dt^2} = -\frac{7g}{(8a)} \varepsilon: \text{ SHM with } \omega^2 = \frac{7g}{(8a)}.$$
A1

$$\text{Period } T = \frac{2\pi}{\omega} = 2\pi \sqrt{8a/(7g)}.$$
A1

Answer: $T = 2\pi \sqrt{8a/(7g)}$ 4 marks

8 (12 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Energy: $\frac{1}{2} m v_A^2 + \frac{1}{2} M v_B^2 = \frac{1}{2} (X^2 + Y^2)$ is conserved (elastic). M1

Momentum: $m v_A + M v_B = \sqrt{m} X + \sqrt{M} Y$ is conserved, i.e. $(X, Y) \cdot n$ is unchanged, where $n = (\sqrt{m}, \sqrt{M})/\sqrt{m+M}$. M1

So $|(X, Y)|$ and the component along n are unchanged; the component perpendicular to n can only change sign (it must change because the velocities change): reflection in the line along n . A1

Answer: shown 3 marks

(b) After an A-B impact the particles separate, so $v_A \leq v_B$; the only possible next impact is A-W, needing $v_A < 0$. After an A-W impact $v_A > 0$, so the next impact can only be A-B, needing $v_A > v_B$. So impacts alternate, the first being A-B. B1

Hence no further impact occurs exactly when $0 \leq v_A \leq v_B$: after an A-B impact $v_A \leq v_B$ already holds, and after an A-W impact $v_A > 0$ already holds. M1

$$v_A = \frac{X}{\sqrt{m}} \text{ and } v_B = \frac{Y}{\sqrt{M}}, \text{ giving } X \geq 0 \text{ and } \sqrt{M} X \leq \sqrt{m} Y. \text{ The line through O with direction}$$
A1

$(\sqrt{m}, \sqrt{M})$ has $\tan \psi = \sqrt{M/m} = \cot \theta$, that is $\psi = 90^\circ - \theta$, so the region is the sector $90^\circ - \theta \leq \psi \leq 90^\circ$.

Answer: shown 3 marks

(c) Key idea: each impact is a reflection, so ψ moves by a fixed pattern. Reflection in a line through O at polar angle λ sends ψ to $2\lambda - \psi$. An A-W impact ($X \rightarrow -X$) is reflection in the Y-axis: $\psi \rightarrow 180^\circ - \psi$. An A-B impact is reflection in the line $\psi = 90^\circ - \theta$: $\psi \rightarrow 180^\circ - 2\theta - \psi$. M1

ψ .

Initially $(X, Y) = (0, -\sqrt{M}u)$, so $\psi_0 = -90^\circ$. The impacts go A-B, A-W, A-B, ... so $\psi_1 = 270^\circ - 2\theta$, $\psi_2 = -90^\circ + 2\theta$, $\psi_3 = 270^\circ - 4\theta$, $\psi_4 = -90^\circ + 4\theta$. By induction (each step is one of the two reflections above): $\psi_k = -90^\circ + k\theta$ for even k and $\psi_k = 270^\circ - (k+1)\theta$ for odd k . A1

By (b) the process stops at the first k with ψ_k in the sector $90^\circ - \theta \leq \psi \leq 90^\circ$. For even k this is $k\theta \in [180^\circ - \theta, 180^\circ]$; for odd k it is $(k+1)\theta \in [180^\circ, 180^\circ + \theta]$, which is again $k\theta \in [180^\circ - \theta, 180^\circ]$. While $k\theta < 180^\circ - \theta$, ψ_k lies in $[-90^\circ, 90^\circ - \theta]$ for even k and in $(90^\circ, 270^\circ)$ for odd k , outside the sector. M1

$k\theta$ increases in steps of θ , so the first k with $k\theta \geq 180^\circ - \theta$ has $k\theta < 180^\circ$. So N is the least integer with $(N+1)\theta \geq \pi$, which is the greatest integer with $N\theta < \pi$. A1

Answer: $N = \text{greatest integer with } N\theta < \pi$ 4 marks

(d) $N \geq 10 \iff 10\theta < \pi \iff \theta < 18^\circ$. Since $M/m = \cot^2\theta$ is decreasing for $0 < \theta < 90^\circ$, this is $M/m > \cot^2 18^\circ$. M1

$$\cot^2 18^\circ = \frac{(1 + \cos 36^\circ)}{(1 - \cos 36^\circ)} = \frac{(5 + \sqrt{5})}{(3 - \sqrt{5})} = \frac{(5 + \sqrt{5})(3 + \sqrt{5})}{4} = 5 + 2\sqrt{5}. \text{ So } M/m > 5 + 2\sqrt{5} (\approx 9.47). \quad \text{A1}$$

Answer: $M/m > 5 + 2\sqrt{5}$ 2 marks