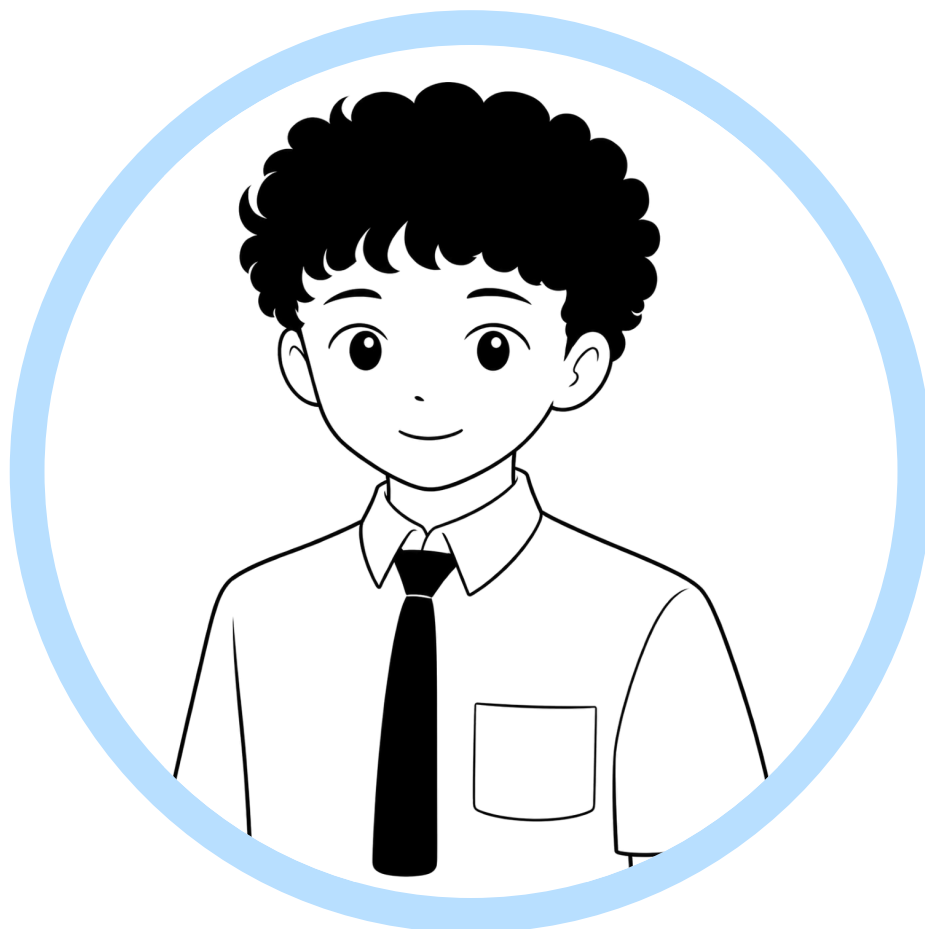


A LEVEL FURTHER MATHS • EXTREME PAPERS

# MATHS



**MODEL ANSWERS**

**EXTREME PAPER: FURTHER DECISION MATHEMATICS**

**SCORE**

**/80**

**PERCENTAGE**

**%**

**STEMUP**

## Paper: Further Decision Mathematics (80 marks)

### 1 (11 marks) Edexcel | OCR MEI

(a) Key idea: reduce rows, then columns, then keep adjusting until five lines are needed.

Rows A to E, columns 1 to 5, with  $x = 11$ : 
$$\begin{pmatrix} 4 & 13 & 8 & 14 & 5 \\ 5 & 11 & 11 & 10 & 6 \\ 10 & 6 & 11 & 6 & 4 \\ 2 & 12 & 11 & 8 & 11 \\ 13 & 2 & 2 & 14 & 9 \end{pmatrix}$$
. Reduce each row (subtract

M1 M1

its minimum), then each column: 
$$\begin{pmatrix} 0 & 9 & 4 & 8 & 1 \\ 0 & 6 & 6 & 3 & 1 \\ 6 & 2 & 7 & 0 & 0 \\ 0 & 10 & 9 & 4 & 9 \\ 11 & 0 & 0 & 10 & 7 \end{pmatrix}$$

Zeros covered by rows C, E and column 1 (3 lines, fewer than 5). Smallest uncovered entry 1:

M1

subtract it from uncovered entries, add it at the crossings of lines: 
$$\begin{pmatrix} 0 & 8 & 3 & 7 & 0 \\ 0 & 5 & 5 & 2 & 0 \\ 7 & 2 & 7 & 0 & 0 \\ 0 & 9 & 8 & 3 & 8 \\ 12 & 0 & 0 & 10 & 7 \end{pmatrix}$$

Zeros covered by row E and columns 1, 4, 5 (4 lines). Smallest uncovered entry 2, giving

A1

$$\begin{pmatrix} 0 & 6 & 1 & 7 & 0 \\ 0 & 3 & 3 & 2 & 0 \\ 7 & 0 & 5 & 0 & 0 \\ 0 & 7 & 6 & 3 & 8 \\ 14 & 0 & 0 & 12 & 9 \end{pmatrix}$$
. Zeros covered by rows C, E and columns 1, 5 (4 lines), smallest uncovered

entry 1, giving 
$$\begin{pmatrix} 0 & 5 & 0 & 6 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 8 & 0 & 5 & 0 & 1 \\ 0 & 6 & 5 & 2 & 8 \\ 15 & 0 & 0 & 12 & 10 \end{pmatrix}$$

Five lines are now needed. D has only one zero (column 1), so D1; then B5 (B's other zero is column 1); then A3; then C4 and E2. Time  $8 + 6 + 6 + 2 + 2 = 24$  hours.

A1

**Answer: A3, B5, C4, D1, E2; 24 hours**

5 marks

(b) Key idea: if  $u_i$  and  $v_j$  are the total amounts subtracted from row  $i$  and column  $j$ , the total time of any allocation is  $24 +$  (the sum of its entries in the final matrix), and an entry that contains  $x$  changes by exactly the change in  $x$ .

M1

At  $x = 11$  the entries of B3 and D5 in the final matrix are 2 and 8, so for general  $x$  they are  $x - 9$  and  $x - 3$ . Every other entry is as in the final matrix and is at least 0.

M1 A1

An allocation uses none, one or both of B3 and D5. None: the least extra is 0 (A3, B5, C4, D1, E2). B3 only: extra  $x - 9 + 0$ , with A5, B3, C4, D1, E2. D5 only: extra  $x - 3 + 0$ , with A3, B1, C4, D5, E2. Both: extra  $2x - 12 + 0$ , with A1, B3, C4, D5, E2. Each has the extra shown because the other three entries used are zeros of the final matrix.

M1

Since  $x - 3 > x - 9$  the D5-only case is never best. So  $f(x) = 24 +$  the least of 0,  $x - 9$  and  $2x - 12$ :  $f(x) = 12 + 2x$  for  $0 < x \leq 3$ ;  $f(x) = 15 + x$  for  $3 \leq x \leq 9$ ;  $f(x) = 24$  for  $x \geq 9$ .

A1

Allocations:  $0 < x < 3$ : A1, B3, C4, D5, E2 (uses both).  $3 < x < 9$ : A5, B3, C4, D1, E2.  $x > 9$ : A3, B5, C4, D1, E2. At  $x = 3$  the first two are both optimal (18 hours) and at  $x = 9$  the last two are both optimal (24 hours); the allocation is unique for every other  $x$ .

A1

**Answer:  $f(x) = 12 + 2x$  ( $x \leq 3$ );  $15 + x$  ( $3 \leq x \leq 9$ );  $24$  ( $x \geq 9$ ); not unique only at  $x = 3$  and  $x = 9$**  6 marks

**2 (9 marks) Edexcel | OCR MEI**

(a) Let  $f(v)$  be the least time from  $v$  to  $T$ . Stage D:  $f(D1) = 7$ ,  $f(D2) = 4$ . Stage C:  $f(C1) = 10$ ,  $f(C2) = 13$ ,  $f(C3) = 10$ . Stage B:  $f(B1) = 15$ ,  $f(B2) = 17$ ,  $f(B3) = 14$ . Stage A:  $f(A1) = 20$ ,  $f(A2) = 22$ ,  $f(A3) = 23$ .  $f(S) = 23$ .

M1 A1

The route is S A1 B2 C3 D1 T (23 hours), with modes b, r, r, b. Legs 2 and 3 both use rail, so the rule is broken.

A1

**Answer: S A1 B2 C3 D1 T, 23 hours; the rule is broken**

3 marks

(b) Key idea: the best way to continue from a town depends on the mode used to arrive there, so the state must be the pair (town, mode of the leg just used).

B1

Let  $g(v, m)$  be the least time from  $v$  to  $T$  when the leg into  $v$  used mode  $m$ , so the next leg must use the other mode (write  $-$  for impossible). Stage D: D1: r 7, b  $-$ ; D2: r  $-$ , b 4.

M1

Stage C: C1: r 10, b  $-$ , C2: r  $-$ , b  $-$ , C3: r 13, b 10. Stage B: B1: r  $-$ , b  $-$ , B2: r  $-$ , b 19, B3: r  $-$ , b 17.

A1

Stage A: A1: r  $-$ , b  $-$ , A2: r 26, b  $-$ , A3: r 25, b  $-$ . At S there is no previous leg, so compare the three first legs:  $3 + g(A1, b) = -$ ,  $2 + g(A2, r) = 28$ ,  $9 + g(A3, r) = 34$ , so the least is 28. Following the choices: S A2 B2 C1 D2 T (modes r, b, r, b, r), 28 hours.

A1

**Answer: S A2 B2 C1 D2 T, 28 hours**

4 marks

(c) Let  $N(v, m)$  be the number of obeying routes from  $v$  to  $T$  when the leg into  $v$  used mode  $m$ .

M1

$N(T, r) = N(T, b) = 1$ . Stage D: D1: r 1, b 0, D2: r 0, b 1. Stage C: C1: r 1, b 0, C2: r 0, b 0, C3: r 1, b 1.

Stage B: B1: r 0, b 0, B2: r 0, b 2, B3: r 0, b 1. Stage A: A1: r 0, b 0, A2: r 2, b 0, A3: r 3, b 0.

A1

Adding the three first legs gives 5 routes in total.

**Answer: 6 routes**

2 marks

**3 (10 marks) Edexcel**

(a) Initial distance table (rows and columns A to F,  $\infty$  for no direct road):

B1

|          |          |          |          |          |          |
|----------|----------|----------|----------|----------|----------|
| 0        | 12       | 2        | 10       | $\infty$ | $\infty$ |
| 12       | 0        | 2        | $\infty$ | 6        | $\infty$ |
| 2        | 2        | 0        | $\infty$ | $\infty$ | 4        |
| 10       | $\infty$ | $\infty$ | 0        | 5        | 3        |
| $\infty$ | 6        | $\infty$ | 5        | 0        | 10       |
| $\infty$ | $\infty$ | 4        | 3        | 10       | 0        |

Pivot A: BD:  $\infty \rightarrow 22$ , CD:  $\infty \rightarrow 12$ ; Pivot B: AE:  $\infty \rightarrow 18$ , CE:  $\infty \rightarrow 8$ ; Pivot C: AB:  $12 \rightarrow 4$ , AE:  $18 \rightarrow 10$ , AF:  $\infty \rightarrow 6$ , BD:  $22 \rightarrow 14$ , BF:  $\infty \rightarrow 6$  (each change is also made in the symmetric entry)

M1

Pivot D: EF:  $10 \rightarrow 8$ ; Pivot E: BD:  $14 \rightarrow 11$ ; Pivot F: AD:  $10 \rightarrow 9$ , BD:  $11 \rightarrow 9$ , CD:  $12 \rightarrow 7$

A1

Final table (rows and columns A to F):

|    |   |   |   |    |   |
|----|---|---|---|----|---|
| 0  | 4 | 2 | 9 | 10 | 6 |
| 4  | 0 | 2 | 9 | 6  | 6 |
| 2  | 2 | 0 | 7 | 8  | 4 |
| 9  | 9 | 7 | 0 | 5  | 3 |
| 10 | 6 | 8 | 5 | 0  | 8 |
| 6  | 6 | 4 | 3 | 8  | 0 |

A1

**Answer: Final table above**

4 marks

(b) Key idea: all six junctions are odd, so they must be paired up and the least distances between paired junctions are repeated. The total of the roads is 54 km.

B1

There are 15 ways to pair six junctions. Using the table, the least pairing is AC BE DF with  $2 + 6 + 3 = 11$ . The next best are AB CF DE with 13 and AC BF DE with 13, and every other pairing is larger still.

M1

Shortest closed route:  $54 + 11 = 65$  km.

A1

**Answer: 65 km; pairing AC, BE, DF**

3 marks

(c) Key idea: the start and finish are the two odd vertices that are allowed to stay odd; the other four are paired.

B1

Finishing at B: the other four are C, D, E, F, pairing cost 9, route 63. Finishing at C: 9, route 63.

M1

Finishing at D: 10, route 64. Finishing at F: 7, route 61. Finishing at E: the other four B, C, D, F are paired BC 2 and DF 3, cost 5, route 59.

The shortest is to finish at E, with length  $54 + 5 = 59$  km.

A1

**Answer: Finish at E; 59 km****3 marks****4 (14 marks) Edexcel | AQA | OCR MEI****(a)** Arcs SP [0, 10], SQ [0, 7], XT [2, 11], YT [2, 8].**B1**

The given flow has  $SP = 2 + 5 + 0 = 7$  and  $SQ = 3 + 0 = 3$  (equivalently  $XT = 2$  and  $YT = 8$ ), so its value is 10.

**B1****Answer: SP [0,10], SQ [0,7], XT [2,11], YT [2,8]; value 10****2 marks**

**(b)** Key idea: a backward step along an arc can only cancel flow down to that arc's lower capacity, so on PB (flow 5, lower capacity 3) at most 2 can be cancelled.

Route S P A X T: residuals SP 3, PA 7, AX 6, XT 9, so increase 3. Flow is now 13 (SP 10).

**M1**

Route S Q B P A X T, using PB backwards: residuals SQ 4, QB 3, PB backwards  $5 - 3 = 2$ , PA 4, AX 3, XT 6, so increase 2. PB becomes 3, PA 7, QB 5, SQ 5.

**M1**

Flow value 15, with SP 10, SQ 5, XT 7, YT 8.

**A1****Answer: Maximum flow 15 (SP 10, SQ 5, XT 7, YT 8)****3 marks**

**(c)** Cut with S, Q, B, C, Y on one side and P, A, X, T on the other (the S side is shaded on the diagram of the final flow).

**B1**

Arcs from the S side to the T side: SP (upper 10) and YT (upper 8). Arcs from the T side to the S side: PB (lower 3) and PQ (lower 0). The net flow across any cut is (flow forward) – (flow back)  $\leq$  (sum of forward upper capacities) – (sum of backward lower capacities).

**M1**

Value of the cut =  $10 + 8 - 3 - 0 = 15$ , which equals the flow in (b), so 15 is the maximum.

**A1****Answer: Cut value 15 = 10 + 8 – 3****3 marks**

**(d)** Key idea: look at the set  $U = (B, C, Y)$ . Its only inputs are PB, QB and QC, and its only output is YT.

**B1**

Conservation gives  $PB + QB + QC = YT \leq 8$ . Since  $QB \geq 3$  and  $QC \geq 0$ ,  $PB \leq 8 - 3 = 5$ , so  $m \leq 5$ .

**M1**

For  $m = 5$  a feasible flow exists: PQ 0, PA 2, PB 5, QB 3, QC 0, BC 1, CB 0, BY 7, CY 1, AX 2 (SP 7, SQ 3, XT 2, YT 8), so the greatest  $m$  is 5.

**A1****Answer: m = 5****3 marks**

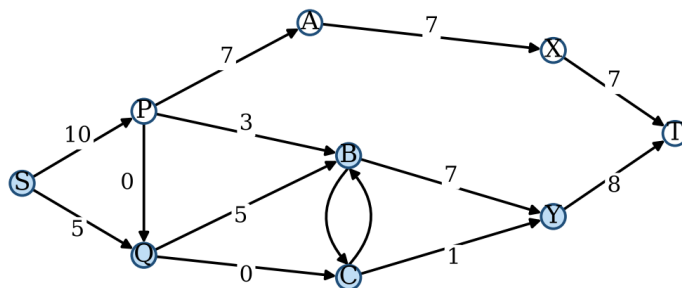
**(e)** Key idea: the pipes PB and QB force at least  $3 + 3 = 6$  units into B, and everything that enters B, C or Y can only leave through YT.

**B1**

For  $U = (B, C, Y)$ : inflow =  $PB + QB + QC \geq 3 + 3 + 0 = 6$ , and the only outflow is YT, so  $YT \geq 6$ . Also  $XT \geq 2$ , and the flow through X is separate from the flow through Y (A leads only to X). So the flow value is at least  $6 + 2 = 8$ .

**M1**

A feasible flow of value 8 exists: PQ 0, PA 2, PB 3, QB 3, QC 0, BC 0, CB 0, BY 6, CY 0, AX 2 (SP 5, SQ 3, XT 2, YT 6), which satisfies every lower and upper capacity. So the smallest value is 8, not 4.

**A1****Answer: 8****3 marks**

Final flow (value 15; BC 1, CB 0). Shaded: S side of the minimum cut

**5 (8 marks) Edexcel | AQA | OCR A | OCR MEI**

**(a)** Forward pass, earliest start times: A 0, B 0, C 2, D 1, E 1, F 0, G 2, H 3, I 4; the completion time is 6 days. Backward pass, latest finish times: A 2, B 2, C 6, D 5, E 6, F 6, G 3, H 6, I 6.

**M1**

Critical activities A, C, G, H (zero float); E has earliest start 1 and latest start 4, so its float is 3 days.

**A1****Answer: 6 days; critical A, C, G, H; float on E is 3 days****2 marks**

**(b)** Total worker-days =  $2 + 2 + 12 + 9 + 4 + 6 + 3 + 9 + 1 = 48$ , and  $48 \div 5 = 9.6$ , which only

**M1**

gives 10 days.

Activities C, D, G, H each need 3 or more workers, and  $3 + 3 > 5$ , so no two of them can run together. They take  $4 + 3 + 1 + 3 = 11$  days one after another, so at least 11 days.

**Answer: At least 11 days**

2 marks

(c) Key idea: the four large activities occupy 11 days, but they cannot all start at the very beginning.

B1

Each of C, D, G, H has a predecessor, which takes at least 1 day (B needs 1 day, A needs 2 days), so none of them can start before day 1. The 11 days of large activities all lie after day 1, so the project takes at least  $1 + 11 = 12$  days.

A1

**Answer: At least 12 days**

2 marks

(d) Starts: A 0, B 0, D 1, E 2, C 4, F 4, I 7, G 8, H 9. Precedences: D starts at 1 after B (ends 1); E at 2 after B; C at 4 after A (ends 2); I at 7 after B and D (ends 4); G at 8 after A and B; H at 9 after G (ends 9). Finish: H ends at 12.

B1

Workers in use on days 1 to 12: 3, 4, 5, 5, 5, 5, 5, 4, 3, 3, 3, 3, never more than 5.

A1

**Answer: A0 B0 D1 E2 C4 F4 I7 G8 H9; 12 days**

2 marks

## 6 (12 marks) Edexcel | OCR MEI

(a) Key idea: after each pivot every entry of the right-hand column must stay non-negative; use the least ratio to choose the row.

Initial tableau (rows r, s, t, P; columns x, y, z, r, s, t, value):

$$\begin{pmatrix} 6 & 3 & 2 & 1 & 0 & 0 & 26 \\ 2 & 6 & 5 & 0 & 1 & 0 & 26 \\ 6 & 4 & 4 & 0 & 0 & 1 & 30 \\ -7 & -4 & -6 & 0 & 0 & 0 & 0 \end{pmatrix}$$

B1

Pivot on x (entry 6 in row r, ratios  $\frac{13}{3}$ , 13, 5):

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & \frac{13}{3} \\ 0 & 5 & \frac{13}{3} & -\frac{1}{3} & 1 & 0 & \frac{52}{3} \\ 0 & 1 & 2 & -1 & 0 & 1 & 4 \\ 0 & -\frac{1}{2} & -\frac{11}{3} & \frac{7}{6} & 0 & 0 & \frac{91}{3} \end{pmatrix}$$

M1

Pivot on z (most negative entry  $-\frac{11}{3}$ ; ratios of rows s and t give least ratio in row t):

A1

$$\begin{pmatrix} 1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & -\frac{1}{6} & \frac{11}{3} \\ 0 & \frac{17}{6} & 0 & \frac{11}{6} & 1 & -\frac{13}{6} & \frac{26}{3} \\ 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 2 \\ 0 & \frac{4}{3} & 0 & -\frac{2}{3} & 0 & \frac{11}{6} & \frac{113}{3} \end{pmatrix}$$

Pivot on the r column (entry  $-\frac{2}{3}$  is most negative; least ratio in row s):

A1

$$\begin{pmatrix} 1 & -\frac{2}{11} & 0 & 0 & -\frac{2}{11} & \frac{5}{22} & \frac{23}{11} \\ 0 & \frac{17}{11} & 0 & 1 & \frac{6}{11} & -\frac{13}{11} & \frac{52}{11} \\ 0 & \frac{14}{11} & 1 & 0 & \frac{3}{11} & -\frac{1}{11} & \frac{48}{11} \\ 0 & \frac{26}{11} & 0 & 0 & \frac{4}{11} & \frac{23}{22} & \frac{449}{11} \end{pmatrix}$$

No negative entries in the profit row, so optimal:  $P = \frac{449}{11}$  ( $\approx 40.8$ ),  $x = \frac{23}{11}$ ,  $y = 0$ ,  $z = \frac{48}{11}$ ,  $r =$

A1

$\frac{52}{11}$ ,  $s = 0$ ,  $t = 0$ .

**Answer:  $P = 449/11$ ,  $x = 23/11$ ,  $y = 0$ ,  $z = 48/11$ ,  $r = 52/11$ ,  $s = t = 0$**

5 marks

(b) Key idea: the column under t in the final tableau tells how the basic variables respond to extra of the third resource.

New values:  $x = \frac{23}{11} + \frac{5}{22}\delta$ ,  $r = \frac{52}{11} - \frac{13}{11}\delta$ ,  $z = \frac{48}{11} - \frac{1}{11}\delta$ , and the profit row entry under t is

M1

$$\frac{23}{22}, \text{ so } P = \frac{449}{11} + \frac{23}{22}\delta.$$

Need  $x \geq 0$ :  $\delta \geq -\frac{46}{5}$ ;  $r \geq 0$ :  $\delta \leq 4$ ;  $z \geq 0$ :  $\delta \leq 48$ . So  $-\frac{46}{5} \leq \delta \leq 4$  (the resource can rise only to 34).

**Answer:  $-46/5 \leq \delta \leq 4$**

2 marks

(c) Key idea: the z row of the final tableau is  $\frac{14}{11}y + z + \frac{3}{11}s - \frac{1}{11}t = \frac{48}{11}$ . Changing the z

M1

coefficient adds k times this row to the profit row, so the new profit row entries are  $y: \frac{26}{11} + \frac{14}{11}k$ ,

$$s: \frac{4}{11} + \frac{3}{11}k, t: \frac{23}{22} - \frac{1}{11}k.$$

All must stay  $\geq 0$ :  $k \geq -\frac{13}{7}$ ,  $k \geq -\frac{4}{3}$ ,  $k \leq \frac{23}{2}$ . So  $-\frac{4}{3} \leq k \leq \frac{23}{2}$ .

A1

**Answer:  $-4/3 \leq k \leq 23/2$**

2 marks

(d) Key idea: the values  $\frac{4}{11}$  and  $\frac{23}{22}$  in the profit row under s and t are the profit values of one unit of the second and third resources.

M1

One hundred units of W use resources worth  $4 \times \frac{4}{11} + 1 \times \frac{23}{22} = \frac{5}{2}$ , so making W adds profit  $q - \frac{5}{2}$  per hundred units until a basic variable reaches 0. So W is worth making if and only if  $q > \frac{5}{2}$ .

A1

For  $q = 4$ , the column of W in the final tableau is  $-\frac{1}{2}$  (row x), 1 (row z), 1 (row r). The ratios are

A1

$\frac{48}{11}$  for z and  $\frac{52}{11}$  for r (row x is ignored since its entry is negative), so z leaves. Then  $w = \frac{48}{11}$ ,  $x =$

$$\frac{23}{11} + \frac{1}{2} \times \frac{48}{11} = \frac{47}{11}, r = \frac{52}{11} - \frac{48}{11} = \frac{4}{11}, y = z = s = t = 0, \text{ and } P = \frac{449}{11} + \left(\frac{3}{2}\right) \times \frac{48}{11} = \frac{521}{11}.$$

**Answer:  $q > 5/2$ ;  $w = 48/11$ ,  $x = 47/11$ ,  $r = 4/11$ ,  $y = z = 0$ ;  $P = 521/11$**

3 marks

## 7 (16 marks) Edexcel | OCR MEI

(a) Row minima:  $-2, 0, -1$ , so Rowan's play-safe choice is R2 with maximin 0. Column maxima:  $2, 2, 4$ , so Colin's play-safe choice is C1 (or C2) with minimax 2.

M1

Maximin 0  $\neq$  minimax 2, so there is no stable solution (no saddle point).

A1

**Answer: Maximin 0, minimax 2: not stable**

2 marks

(b) Adding 3 to every pay-off makes all entries positive (smallest was  $-2$ ) and so the value  $v'$  of

B1

the new game is positive; the new matrix is  $\begin{pmatrix} 5 & 1 & 5 \\ 3 & 5 & 7 \\ 4 & 5 & 2 \end{pmatrix}$  and  $v' = v + 3$ .

Let Colin play C1, C2, C3 with probabilities  $q_1, q_2, q_3$ , and let  $y_j = q_j/v'$ . Each row of the game must give Rowan at most  $v'$ , so dividing by  $v'$ :  $5y_1 + y_2 + 5y_3 \leq 1$ ,  $3y_1 + 5y_2 + 7y_3 \leq 1$ ,  $4y_1 + 5y_2 + 2y_3 \leq 1$ .

M1

Since  $q_1 + q_2 + q_3 = 1$ ,  $y_1 + y_2 + y_3 = 1/v'$ . Colin wants  $v'$  as small as possible, i.e. P maximised. Then  $v' = 1/P$ ,  $v = 1/P - 3$  and  $q_j = y_j/P$ .

A1

**Answer: LP as above;  $v = 1/P - 3$ ,  $q_j = y_j/P$**

3 marks

(c) Key idea: if there is an optimal mixed strategy using all choices, it makes the opponent indifferent between all their choices.

Let Rowan play R1, R2, R3 with probabilities  $p_1, p_2, p_3$ . Equate expected winnings against C1, C2, C3 to  $v$ :  $2p_1 + p_3 = v$ ,  $-2p_1 + 2p_2 + 2p_3 = v$ ,  $2p_1 + 4p_2 - p_3 = v$ , with  $p_1 + p_2 + p_3 = 1$ .

M1 A1

Solving gives  $p = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$  and  $v = 1$ .

Colin:  $2q_1 - 2q_2 + 2q_3 = 1$ ,  $2q_2 + 4q_3 = 1$ ,  $q_1 + 2q_2 - q_3 = 1$  with  $q_1 + q_2 + q_3 = 1$  gives  $q = (\frac{5}{8},$

A1

$\frac{1}{4}, \frac{1}{8})$ . Check: Rowan's  $p$  gives exactly 1 against every column, so Rowan can guarantee at least 1; Colin's  $q$  gives exactly 1 against every row, so Colin can hold Rowan to at most 1. Hence the value is 1.

**Answer: Rowan  $(1/4, 1/4, 1/2)$ ; Colin  $(5/8, 1/4, 1/8)$ ; value 1**

3 marks

**(d)** Key idea: Rowan's optimal strategy in the original game is unique, so a new column helps Colin only if it beats that strategy.

Adding a choice can only help Colin, so the new value is at most 1. If  $k \geq 5$ : Rowan's mix  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$  against C4 gives  $\frac{3}{4} + \frac{k}{4} - 1 = \frac{(k-1)}{4} \geq 1$ , so Rowan still guarantees 1, and the value is 1.

M1 A1

Suppose the new value is 1. Then some mix  $p'$  of Rowan guarantees at least 1 against all four columns, so also against C1, C2, C3. Since  $q = (\frac{5}{8}, \frac{1}{4}, \frac{1}{8})$  has every entry positive and holds Rowan to at most 1,  $p'$  must give exactly 1 against each of C1, C2, C3.

M1

The matrix is invertible (determinant  $\neq 0$ ), so  $p' = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$  is the only possibility, and it must give at least 1 against C4, i.e.  $\frac{(k-1)}{4} \geq 1$ . So the value is 1 if and only if  $k \geq 5$ .

A1

**Answer:  $k \geq 5$**

4 marks

**(e)** Key idea: by part (d) the value is below 1, so Rowan's old strategy no longer works; guess that Rowan makes Colin indifferent between C1, C2 and C4, and Colin mixes C1, C2, C4 only.

Rowan:  $2p_1 + p_3 = v$ ,  $-2p_1 + 2p_2 + 2p_3 = v$ ,  $3p_1 + kp_2 - 2p_3 = v$  and  $p_1 + p_2 + p_3 = 1$ . Solving:  $p = (k+6, 11, 4k+2) \div (5k+19)$ , and  $v = \frac{6k+14}{5k+19}$ .

M1 A1

Colin plays C1, C2, C4 with probabilities  $q_1, q_2, q_4$ :  $2q_1 - 2q_2 + 3q_4 = v$ ,  $2q_2 + kq_4 = v$ ,  $q_1 + 2q_2 - 2q_4 = v$ , giving  $q = (4k+8, k+7, 0, 4) \div (5k+19)$  for (C1, C2, C3, C4). All entries are at least 0 for the given  $k$ .

M1

Check:  $p \geq 0$  requires  $k \geq -\frac{1}{2}$ . Rowan against C3 gets  $2p_1 + 4p_2 - p_3 = v + \frac{8(5-k)}{(5k+19)} > v$ , so

A1

Rowan does better than  $v$  against C3 and C3 is never worth playing. Rowan guarantees at least  $v$ , and Colin holds Rowan to exactly  $v$  against every row, so the value is  $v = \frac{6k+14}{5k+19}$ .

**Answer:  $v = (6k+14)/(5k+19)$ , with  $p$  and  $q$  as found**

4 marks