

Binomial Distribution

1 (3 marks)

(a) $X \sim B(12, 0.5)$

B1

Answer: **B(12, 0.5)**

1 mark

(b) ${}^{12}C_7(0.5)^{12}$

M1

$= 0.1934$

A1

Answer: **0.1934**

2 marks

2 (2 marks)

$P(X = 2) = {}^8C_2(0.3)^2(0.7)^6$

M1

$= 0.2965$

A1

Answer: **0.2965**

2 marks

3 (2 marks)

$E(X) = np = 5$

B1

$\text{Var}(X) = npq = 10 \times 0.5 \times 0.5 = 2.5$

B1

Answer: **5 and 2.5**

2 marks

4 (1 mark)

$P(X \leq 6) = 0.2500$

B1

Answer: **0.2500**

1 mark

5 (2 marks)

$P(X = 0) = \left(\frac{2}{3}\right)^6$

M1

$= \frac{64}{729}$

A1

Answer: **$\frac{64}{729}$**

2 marks

6 (2 marks)

Mean $= 40 \times 0.15 = 6$

B1

$SD = \sqrt{40 \times 0.15 \times 0.85} = \sqrt{5.1} = 2.26$

B1

Answer: **Mean 6, SD 2.26**

2 marks

7 (3 marks)

$X \sim B(10, 0.25)$

B1

$P(X \geq 3) = 1 - P(X \leq 2)$

M1

$= 0.4744$

A1

Answer: **0.4744**

3 marks

8 (3 marks)

$np = 6$ and $np(1 - p) = 2.4$

B1

$1 - p = \frac{2.4}{6} = 0.4$, so $p = 0.6$

M1

$n = \frac{6}{0.6} = 10$

A1

Answer: **$n = 10$, $p = 0.6$**

3 marks

9 (3 marks)

$P(3 \leq X < 6) = P(X \leq 5) - P(X \leq 2)$

M1

$= 0.9456 - 0.3907$

M1

$= 0.5549$

A1

Answer: **0.5549**

3 marks

10 (5 marks)

(a) $(0.95)^{20} = 0.3585$

B1

Answer: 0.3585

1 mark

(b) $X \sim B(20, 0.05)$

M1

$P(X \leq 2) = 0.9245$

A1

Answer: 0.9245

2 marks

(c) $1 - P(X \leq 1)$

M1

$= 0.2642$

A1

Answer: 0.2642

2 marks

11 (4 marks)

On one roll, $P(\text{same score}) = \frac{6}{36} = \frac{1}{6}$

B1

$X \sim B(4, \frac{1}{6})$

M1

$P(X = r) = {}^4C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{4-r} = \frac{{}^4C_r 5^{4-r}}{1296}$

M1

$P(0) = \frac{625}{1296}, P(1) = \frac{500}{1296}, P(2) = \frac{150}{1296}, P(3) = \frac{20}{1296}, P(4) = \frac{1}{1296}$

A1

Answer: $\frac{625}{1296}, \frac{500}{1296}, \frac{150}{1296}, \frac{20}{1296}, \frac{1}{1296}$ for $X = 0, 1, 2, 3, 4$

4 marks

12 (3 marks)

Mean = 6, and $P(X = r + 1)/P(X = r) = \frac{(15 - r) \times 0.4}{(r + 1) \times 0.6}$

M1

This is greater than 1 when $2(15 - r) > 3(r + 1)$, that is $r < 5.4$

M1

The mode is $X = 6$

A1

Answer: 6

3 marks

13 (3 marks)

Each student walks with the same probability p

B1

Students' journeys are independent of each other

B1

e.g. students from the same family or street are likely to travel the same way, so independence may fail

E1

Answer: Constant p , independence; e.g. siblings travel together

3 marks

14 (4 marks)

$P(\text{at least one hit}) = 1 - (0.9)^n > 0.99$

M1

$(0.9)^n < 0.01$

A1

$n > \frac{\ln 0.01}{\ln 0.9} = 43.7$

M1

$n = 44$

A1

Answer: 44

4 marks

15 (4 marks)

$P(X \geq 1) = 1 - 0.7^{10} = 0.9718$

M1 A1

$P(X = 2) = 0.2335$

A1

$P(X = 2 | X \geq 1) = 0.2403$

A1

Answer: 0.2403

4 marks

16 (5 marks)

${}^8C_2 p^2 (1 - p)^6 = 3 \times {}^8C_1 p (1 - p)^7$

M1 A1

$28p^2 (1 - p)^6 = 24p (1 - p)^7$

A1

Divide by $4p(1 - p)^6$ ($p \neq 0, 1$): $7p = 6(1 - p)$

M1

$p = \frac{6}{13}$

A1

Answer: $\frac{6}{13}$

5 marks

17 (4 marks)

$X + Y$ is the number of successes in $12 + 8 = 20$ independent trials, each with probability 0.2

M1

So $X + Y \sim B(20, 0.2)$

A1

$$P(X + Y = 4) = {}^{20}C_4(0.2)^4(0.8)^{16}$$

M1

$$= 0.2182$$

A1

Answer: 0.2182

4 marks

18 (5 marks)

$$(a) P(X = r + 1) = \frac{n!}{(r + 1)!(n - r - 1)!} p^{r+1} q^{n-r-1}$$

M1

$$\text{Divide by } P(X = r) = \frac{n!}{r!(n - r)!} p^r q^{n-r}$$

M1

$$\text{Ratio} = \frac{(n - r)}{(r + 1)} \times \frac{p}{q}$$

A1

$$\text{Answer: } \frac{P(X = r + 1)}{P(X = r)} = \frac{(n - r)p}{(r + 1)q}$$

3 marks

(b) Ratio > 1 when $(20 - r) \times 0.35 > (r + 1) \times 0.65$, i.e. $7 - 0.35r > 0.65r + 0.65$, $r < 6.35$

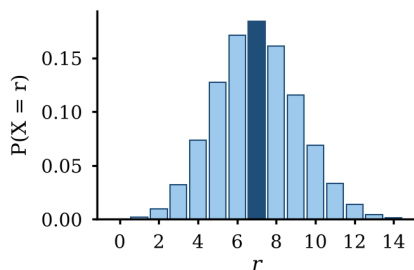
M1

P increases up to $r = 7$, so the mode is 7

A1

Answer: 7

2 marks



19 (6 marks)

$$(a) 15p(1 - p) = 3.15$$

M1

$$p^2 - p + 0.21 = 0, \text{ so } (p - 0.3)(p - 0.7) = 0$$

M1

$$p = 0.3 \text{ or } p = 0.7$$

A1

Answer: $p = 0.3$ or $p = 0.7$

3 marks

$$(b) 0.3n = 3\sqrt{n \times 0.3 \times 0.7}$$

M1

$$\text{Square: } 0.09n^2 = 9 \times 0.21n, \text{ so } n = \frac{1.89}{0.09}$$

M1

$$n = 21$$

A1

Answer: $n = 21$

3 marks

20 (5 marks)

Number of errors $X \sim B(25, 0.08)$

B1

$$E(X) = 2, \text{ Var}(X) = 25 \times 0.08 \times 0.92 = 1.84$$

M1 A1

$$\text{Cost} = 15X: E = £30$$

A1

$$SD = 15 \times \sqrt{1.84} = £20.35$$

A1

Answer: £30 and £20.35

5 marks

21 (6 marks)

$$(q + p)^n = \sum_{r=0}^n {}^nC_r p^r q^{n-r} = 1$$

M1

$$(q - p)^n = \sum_{r=0}^n (-1)^r \times {}^nC_r p^r q^{n-r}$$

M1 A1

$$\text{Adding: } 1 + (q - p)^n = 2 \times P(\text{even number of heads})$$

M1

$$q - p = 0.7 - 0.3 = 0.4$$

A1

$$P(\text{even}) = \frac{1 + 0.4^{10}}{2} = 0.500052$$

A1

Answer: 0.500052

6 marks

22 (6 marks)

$$P(X = Y) = \sum_{k=0}^{10} P(X = k)P(Y = k)$$

M1 A1

$Y \sim B(10, 0.6)$ has $P(Y = k) = P(X' = 10 - k)$ for $X' \sim B(10, 0.4)$

M1

$$\text{So } P(X = Y) = \sum_k P(X = k)P(X = 10 - k)$$

M1

Evaluate and add the eleven terms ($k = 0$ to 10)

M1

$$P(X = Y) = 0.117$$

A1

Answer: 0.117

6 marks

23 (5 marks)

$$E(I_j) = 1 \times p + 0 \times q = p$$

B1

$$E(X) = E(I_1) + \dots + E(I_n) = np$$

M1 A1

$$\text{Var}(I_j) = E(I_j^2) - p^2 = p - p^2 = p(1 - p), \text{ as } I_j^2 = I_j$$

M1

The I_j are independent, so $\text{Var}(X) = np(1 - p)$

A1

Answer: $E(X) = np$, $\text{Var}(X) = np(1 - p)$

5 marks

24 (7 marks)

$$(a) X \sim B(25, 0.04)$$

M1

$$P(X \leq 2) = 0.9235$$

A1

Answer: 0.9235

2 marks

$$(b) P(X \leq 2) \text{ for } B(25, 0.1) = 0.5371$$

B1

Answer: 0.5371

1 mark

$$(c) Y \sim B(6, 0.9235)$$

M1

$$P(Y \geq 5) = P(Y = 5) + P(Y = 6)$$

M1

$$= 6(0.9235)^5(0.0765) + (0.9235)^6$$

M1

$$= 0.9287$$

A1

Answer: 0.9287

4 marks

25 (5 marks)

The complement of $X \geq 3$ is $X \leq 2$, not $X \leq 3$

E1

She has also removed $P(X = 3)$, so her answer is too small by $P(X = 3)$

E1

$$\text{Correct: } P(X \geq 3) = 1 - P(X \leq 2) = 1 - 0.5583$$

M1 A1

$$= 0.4417$$

A1

Answer: 0.4417

5 marks

26 (6 marks)

$$(a) T \sim B(22, 0.8), P(T \geq 21) = P(T = 21) + P(T = 22)$$

M1

$$= 22(0.8)^{21}(0.2) + (0.8)^{22} = 0.0480$$

A1

Answer: 0.0480

2 marks

(b) For 23 appointments, $P(T \geq 21) = P(T = 21) + P(T = 22) + P(T = 23) = 0.1332$, which is more than 0.1

M1 A1

For 22 appointments the probability is $0.0480 < 0.1$, and it increases with the number of appointments

A1

Greatest number of appointments is 22

A1

Answer: 22 appointments

4 marks

27 (8 marks)

$$P(X = k) = {}^nC_k p^k (1 - p)^{n-k}$$

B1

$$\text{Differentiate (product rule): } {}^nC_k [k p^{k-1} (1 - p)^{n-k} - (n - k) p^k (1 - p)^{n-k-1}]$$

M1 A1

$$= {}^nC_k p^{k-1} (1 - p)^{n-k-1} [k(1 - p) - (n - k)p]$$

M1

$$\text{Stationary when } k(1 - p) - (n - k)p = 0, \text{ so } k - np = 0$$

M1 A1

$$p = \frac{k}{n}$$

A1

For $p < \frac{k}{n}$ the bracket $k - np$ is positive (probability increasing), and for $p > \frac{k}{n}$ it is negative, so this is a maximum

E1

$$\text{Answer: } p = \frac{k}{n}$$

8 marks

28 (8 marks)

$$c_n = {}^{2n}C_n \left(\frac{1}{2}\right)^{2n}$$

B1

$$c_{n+1} = {}^{2n+2}C_{n+1} \left(\frac{1}{2}\right)^{2n+2}$$

B1

$$\frac{c_{n+1}}{c_n} = \frac{(2n+2)!}{(n+1)!(n+1)!} \times \frac{n!n!}{(2n)!} \times \frac{1}{4}$$

M1 A1

$$= \frac{(2n+2)(2n+1)}{(n+1)^2} \times \frac{1}{4} = \frac{2n+1}{2n+2}$$

M1 A1

$$c_2 = 0.5 \times \frac{3}{4} = 0.375, c_3 = 0.375 \times \frac{5}{6} = 0.3125$$

M1

$$c_4 = 0.3125 \times \frac{7}{8} = 0.2734$$

A1

$$\text{Answer: } 0.2734$$

8 marks

29 (9 marks)

Let $X \sim B(5, 0.5)$, $Y \sim B(7, 0.5)$ independently

B1

$$P(X > Y) = \sum_{x=1}^5 P(X = x)P(Y < x)$$

M1 A1

$$P(X = x) = {}^5C_x / 32 \text{ and } P(Y \leq x - 1) = \left(\sum_{j=0}^{x-1} {}^7C_j\right) / 128$$

M1

Cumulative counts of ${}^7C_j$: 1, 8, 29, 64, 99

M1 A1

$$\text{Terms: } 5 \times 1 + 10 \times 8 + 10 \times 29 + 5 \times 64 + 1 \times 99 = 5 + 80 + 290 + 320 + 99 = 794$$

M1 A1

$$P(X > Y) = \frac{794}{4096} = \frac{397}{2048}$$

A1

$$\text{Answer: } \frac{397}{2048}$$

9 marks

30 (9 marks)

$$(a) P(7 \text{ heads} | \text{fair}) = {}^{10}C_7 (0.5)^{10} = 0.1172$$

M1 A1

$$P(7 \text{ heads} | \text{biased}) = {}^{10}C_7 (0.8)^7 (0.2)^3 = 0.2013$$

M1

$$P(\text{fair} | 7 \text{ heads}) = \frac{0.5 \times 0.1172}{0.5 \times 0.1172 + 0.5 \times 0.2013}$$

M1

$$= 0.3679 \approx 0.368$$

A1

$$\text{Answer: } \approx 0.368$$

5 marks

$$(b) P(5 \text{ heads} | \text{fair}) = \frac{1}{32}, P(5 \text{ heads} | \text{biased}) = 0.8^5 = 0.32768$$

B1

$$P(\text{fair}) = 0.3679, P(\text{biased}) = 0.6321$$

M1

$$P(5 \text{ heads}) = 0.3679 \times \frac{1}{32} + 0.6321 \times 0.32768$$

M1

$$= 0.2186$$

A1

$$\text{Answer: } 0.2186$$

4 marks

Poisson Distribution

1 (2 marks)

$$P(X = 3) = \frac{e^{-2} \times 2^3}{3!}$$

M1

$$= 0.1804$$

A1

Answer: 0.1804

2 marks

2 (1 mark)

$$P(X \leq 2) = 0.1736$$

B1

Answer: 0.1736

1 mark

3 (2 marks)

$$E(X) = \text{Var}(X) = 6$$

B1

$$SD = \sqrt{6} = 2.45$$

B1

Answer: 6, 6, 2.45

2 marks

4 (2 marks)

$$P(X = 0) = e^{-3}$$

B1

$$P(X = 1) = 3e^{-3}$$

B1

Answer: e^{-3} and $3e^{-3}$

2 marks

5 (2 marks)

$$P(X \geq 1) = 1 - P(X = 0)$$

M1

$$= 1 - e^{-0.8} = 0.5507$$

A1

Answer: 0.5507

2 marks

6 (2 marks)

Number of calls $\sim \text{Po}(2.5)$

B1

$$P(0) = e^{-2.5} = 0.0821$$

B1

Answer: Po(2.5); 0.0821

2 marks

7 (4 marks)

(a) $e^{-0.8} = 0.4493$

B1

Answer: 0.4493

1 mark

(b) Total errors $\sim \text{Po}(4)$

M1 A1

$$P(\leq 3) = 0.4335$$

A1

Answer: 0.4335

3 marks

8 (3 marks)

$$e^{-\lambda} = 0.2$$

M1

$$\lambda = -\ln 0.2 = \ln 5$$

M1

$$= 1.61$$

A1

Answer: 1.61

3 marks

9 (3 marks)

$$\lambda e^{-\lambda} = \frac{\lambda^2}{2} e^{-\lambda}$$

M1

$$\lambda = 2$$

A1

$$P(X = 3) = \frac{e^{-2} 2^3}{6} = 0.1804$$

A1

Answer: $\lambda = 2$; 0.1804

3 marks

10 (4 marks)

Number of cars $X \sim \text{Po}(3.2)$

B1

$$P(X > 2) = 1 - P(X \leq 2)$$

M1

$$= 1 - 0.3799 = 0.6201$$

A1

Assumption: cars pass independently and at a constant average rate

E1

Answer: 0.6201

4 marks

11 (3 marks)

$P(\text{a page has no errors}) = e^{-0.7}$, $P(\text{a page has errors}) = 1 - e^{-0.7}$

B1

Pages 1, 2 and 3 have no errors and page 4 has errors: $(e^{-0.7})^3(1 - e^{-0.7})$

M1

$$= e^{-2.1}(1 - e^{-0.7}) = 0.0616$$

A1

Answer: 0.0616

3 marks

12 (3 marks)

$$P(X = r + 1)/P(X = r) = \frac{e^{-\lambda}\lambda^{r+1}}{(r+1)!} \times \frac{r!}{e^{-\lambda}\lambda^r} = \frac{\lambda}{r+1}$$

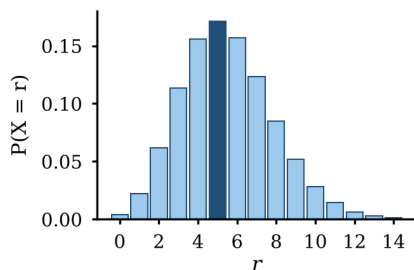
M1 A1

The ratio exceeds 1 while $r + 1 < 5.5$, i.e. $r \leq 4$, so the probabilities rise up to $r = 5$: mode = 5

A1

Answer: $\frac{\lambda}{r+1}$; mode 5

3 marks



13 (3 marks)

Customers arrive independently of each other

B1

Customers arrive at a constant average rate

B1

At lunchtime the rate is not constant (it rises and falls), or groups arrive together, so the conditions fail

E1

Answer: Independent arrivals at a constant rate

3 marks

14 (4 marks)

$$P(X > k) = 1 - P(X \leq k) < 0.01 \text{ so } P(X \leq k) > 0.99$$

M1

$$P(X \leq 7) = 0.9881 < 0.99$$

A1

$$P(X \leq 8) = 0.9962 > 0.99$$

M1

$$k = 8$$

A1

Answer: 8

4 marks

15 (5 marks)

$$P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2) = 1 - e^{-2}(1 + 2 + 2) = 1 - 5e^{-2}$$

M1 A1

$$P(X = 5) = \frac{e^{-2} \times 2^5}{5!} = \frac{4}{15}e^{-2}$$

M1

$$P(X = 5 | X \geq 3) = \frac{\frac{4}{15}e^{-2}}{1 - 5e^{-2}} = \frac{4}{15(e^2 - 5)} \text{ (multiply top and bottom by } e^2)$$

A1

$$= 0.112$$

A1

Answer: 0.112

5 marks

16 (5 marks)

$$P(X = 1 | X + Y = 4) = \frac{P(X = 1)P(Y = 3)}{P(X + Y = 4)}$$

M1 A1

$$X + Y \sim \text{Po}(5)$$

B1

$$= \frac{2e^{-2} \times \frac{27}{6}e^{-3}}{\frac{625}{24}e^{-5}}$$

M1

$$= 0.346$$

A1

Answer: 0.346

5 marks

17 (4 marks)

$$\text{Faults in 25 m} \sim \text{Po}(1)$$

M1 A1

$$P(\geq 2) = 1 - P(0) - P(1)$$

M1

$$= 1 - e^{-1} - e^{-1} = 0.2642$$

A1

Answer: 0.2642

4 marks

18 (5 marks)

$$\text{(a) Four weeks: } \text{Po}(10)$$

B1

$$P(X > 6) = 1 - P(X \leq 6) = 0.8699$$

A1

Answer: 0.8699

2 marks

$$\text{(b) Number of such periods } Y \sim B(12, 0.8699)$$

M1 A1

$$P(Y = 7) = {}^{12}C_7(0.8699)^7(0.1301)^5 = 0.0111$$

A1

Answer: 0.0111

3 marks

19 (5 marks)

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$

M1

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

M1 A1

$$= \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

M1

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

A1

Answer: E(X) = λ

5 marks

20 (5 marks)

$$\text{(a) Po}(6): P(X > 8) = 1 - P(X \leq 8)$$

M1

$$= 0.1528$$

A1

Answer: 0.1528

2 marks

$$\text{(b) Half hour: } Y \sim \text{Po}(3), P(Y \geq 3) = 1 - 0.4232 = 0.5768$$

M1 A1

$$\text{Independent periods: } (0.5768)^2 = 0.3327$$

A1

Answer: 0.3327

3 marks

21 (6 marks)

$$E[X(X-1)] = \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!}$$

M1

$$= e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!}$$

M1 A1

$$= \lambda^2 e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = \lambda^2$$

M1 A1

$$\text{Var}(X) = E[X(X-1)] + E(X) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

A1

Answer: E[X(X-1)] = λ², so Var(X) = λ

6 marks

22 (5 marks)

$X_1 + X_2 \sim \text{Po}(6)$ is correct (sum of independent Poissons)

B1

$E(2X) = 6$

B1

$\text{Var}(2X) = 4 \times 3 = 12$, not 6

M1 A1

A Poisson variable has equal mean and variance, so $2X$ is not Poisson

E1

Answer: $\text{Var}(2X) = 12 \neq 6$, so $2X$ is not Poisson

5 marks

23 (6 marks)

$$E(X | X \geq 1) = \frac{\sum_{x=1}^{\infty} xP(X=x)}{P(X \geq 1)}$$

M1

The sum equals $E(X)$ because the $x = 0$ term contributes 0, so the numerator is 2

M1 A1

$$P(X \geq 1) = 1 - e^{-2}$$

B1

$$E(X | X \geq 1) = \frac{2}{1 - e^{-2}} = 2.31$$

A1

Conditioning removes the value 0, so the mean is higher

E1

Answer: 2.31

6 marks

24 (7 marks)

$$P(Y=y) = \sum_{n=y}^{\infty} P(N=n)P(Y=y | N=n)$$

M1

$$= \sum_{n=y}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \times \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}$$

M1 A1

$$= \frac{e^{-\lambda} (\lambda p)^y}{y!} \sum_{n=y}^{\infty} \frac{(\lambda(1-p))^{n-y}}{(n-y)!}$$

M1 A1

The sum is $e^{\lambda(1-p)}$

M1

$$P(Y=y) = \frac{e^{-\lambda p} (\lambda p)^y}{y!}, \text{ so } Y \sim \text{Po}(\lambda p)$$

A1

Answer: $Y \sim \text{Po}(\lambda p)$

7 marks

25 (5 marks)

$$\text{Ratio } P(X=r+1)/P(X=r) = \frac{\lambda}{r+1}$$

M1

For $r = \lambda - 1$ the ratio is exactly 1, so $P(X = \lambda - 1) = P(X = \lambda)$

M1 A1

For $r < \lambda - 1$ the ratio exceeds 1 and for $r > \lambda$ it is below 1, so these are the maxima

E1

Po(4): the modes are 3 and 4

A1

Answer: $\lambda - 1$ and λ ; modes 3 and 4

5 marks

26 (6 marks)

$$P(4 | A) = \frac{e^{-2} 2^4}{4!} = 0.0902$$

M1 A1

$$P(4 | B) = \frac{e^{-5} 5^4}{4!} = 0.1755$$

M1 A1

$$P(A | 4) = \frac{0.5 \times 0.0902}{0.5 \times 0.0902 + 0.5 \times 0.1755}$$

M1

$$= 0.3396$$

A1

Answer: 0.3396

6 marks

27 (8 marks)

$$(a) e^{\lambda} + e^{-\lambda} = 2 \sum_{x \text{ even}} \frac{\lambda^x}{x!} \quad \text{M1 A1}$$

$$P(X \text{ even}) = e^{-\lambda} \sum_{x \text{ even}} \frac{\lambda^x}{x!} \quad \text{M1}$$

$$= \frac{e^{-\lambda}(e^{\lambda} + e^{-\lambda})}{2} = \frac{1 + e^{-2\lambda}}{2} \quad \text{A1}$$

$$\text{Answer: } P(X \text{ is even}) = \frac{1 + e^{-2\lambda}}{2} \quad \text{4 marks}$$

$$(b) \frac{1 + e^{-3}}{2} = 0.5249 \quad \text{B1}$$

Answer: 0.5249 1 mark

$$(c) e^{-2\lambda} > 0, \text{ so } P(X \text{ even}) > \frac{1}{2} \quad \text{M1 A1}$$

$$\text{As } \lambda \rightarrow \infty, e^{-2\lambda} \rightarrow 0 \text{ and } P(X \text{ even}) \rightarrow \frac{1}{2} \quad \text{A1}$$

Answer: Greater than $\frac{1}{2}$; tends to $\frac{1}{2}$ 3 marks

28 (9 marks)

$$P(X > Y) = \sum_{y=0}^{\infty} P(Y = y)P(X > y) \quad \text{M1 A1}$$

$$P(Y = y) \text{ for } y = 0, 1, 2, 3, 4, \dots: 0.3679, 0.3679, 0.1839, 0.0613, 0.0153 \quad \text{B1}$$

$$P(X > y) \text{ for } y = 0, 1, 2, 3, 4: 0.8647, 0.5940, 0.3233, 0.1429, 0.0527 \quad \text{M1 A1}$$

$$\text{Products: } 0.3181, 0.2185, 0.0595, 0.0088, 0.0008 \quad \text{M1}$$

$$\text{Remaining terms } (y \geq 5) \text{ are negligible: they total about } 0.0001 \quad \text{M1 A1}$$

$$\text{Sum} = 0.6057 \quad \text{A1}$$

Answer: 0.606 9 marks

29 (8 marks)

(a) $T > t$ means no buses in the first t minutes; the number of buses in t minutes $\sim \text{Po}(0.5t)$ M1

$$P(0 \text{ buses}) = e^{-0.5t} \quad \text{A1}$$

$$\text{Answer: } P(T > t) = e^{-0.5t} \quad \text{2 marks}$$

$$(b) e^{-0.5t} = 0.5 \quad \text{M1}$$

$$-0.5t = \ln 0.5 \quad \text{M1}$$

$$t = 2 \ln 2 = 1.39 \text{ minutes} \quad \text{A1}$$

Answer: 1.39 minutes 3 marks

$$(c) P(1 < T < 3) = P(T > 1) - P(T > 3) \quad \text{M1}$$

$$= e^{-0.5} - e^{-1.5} \quad \text{M1}$$

$$= 0.3834 \quad \text{A1}$$

Answer: 0.3834 3 marks

30 (9 marks)

$$(a) E(N) = \frac{1}{2}(2) + \frac{1}{2}(4) = 3 \quad \text{M1 A1}$$

$$E(N^2) = \frac{1}{2}(\text{Var} + \text{mean}^2 \text{ for Po}(2)) + \frac{1}{2}(\text{same for Po}(4)) = \frac{1}{2}(2 + 4) + \frac{1}{2}(4 + 16) \quad \text{M1}$$

$$= 13 \quad \text{A1}$$

$$\text{Var}(N) = 13 - 9 = 4 \quad \text{A1}$$

Answer: $E = 3$, $\text{Var} = 4$ 5 marks

(b) For a Poisson variable $\text{Var} = E$, but $\text{Var}(N) = 4 \neq 3$ E1

Answer: Variance is not equal to the mean 1 mark

$$(c) \frac{1}{2}e^{-2} + \frac{1}{2}e^{-4} \quad \text{M1 A1}$$

$$= 0.0768 \quad \text{A1}$$

Answer: 0.0768 3 marks

Geometric Distribution

1 (2 marks)

$$P(X = 4) = (0.7)^3(0.3)$$

$$= 0.1029$$

M1

A1

Answer: 0.1029

2 marks

2 (2 marks)

$$E(X) = \frac{1}{0.25} = 4$$

B1

$$\text{Var}(X) = \frac{1 - 0.25}{0.25^2} = 12$$

B1

Answer: 4 and 12

2 marks

3 (2 marks)

$$P(X > 3) = (1 - 0.2)^3$$

$$= 0.512$$

M1

A1

Answer: 0.512

2 marks

4 (2 marks)

$$\left(\frac{5}{6}\right)^3 \times \frac{1}{6}$$

M1

$$= \frac{125}{1296}$$

A1

Answer: $\frac{125}{1296}$

2 marks

5 (1 mark)

$$E(X) = \frac{1}{p} = 8$$

B1

Answer: 8

1 mark

6 (2 marks)

$$P(X \leq 3) = 0.5 + 0.25 + 0.125$$

$$= 0.875$$

M1

A1

Answer: 0.875

2 marks

7 (3 marks)

$$P(4 \leq X < 10) = P(X \geq 4) - P(X \geq 10)$$

M1

$$P(X \geq 4) = P(X > 3) = (0.85)^3 \text{ and } P(X \geq 10) = P(X > 9) = (0.85)^9$$

$$= 0.6141 - 0.2316 = 0.3825$$

M1

A1

Answer: 0.3825

3 marks

8 (3 marks)

$$(a) (0.4)^2(0.6) = 0.096$$

B1

Answer: 0.096

1 mark

$$(b) P(X \leq 2) = 1 - (0.4)^2$$

$$= 0.84$$

M1

A1

Answer: 0.84

2 marks

9 (3 marks)

$$E(X) = \frac{1}{p} = 5, \text{ so } p = 0.2$$

M1 A1

$$P(X = 3) = (0.8)^2(0.2) = 0.128$$

A1

Answer: p = 0.2; 0.128

3 marks

10 (3 marks)

$$P(X > r) = \sum_{k=r+1}^{\infty} q^{k-1}p \quad \text{M1}$$

$$= pq^r(1 + q + q^2 + \dots) = \frac{pq^r}{1 - q} \quad \text{M1}$$

$$= q^r \quad \text{A1}$$

Answer: $P(X > r) = q^r$ 3 marks

11 (3 marks)

$$p = \frac{1}{8} \quad \text{B1}$$

$$\text{Var}(X) = \frac{q}{p^2} = \frac{7}{8} \times 64 \quad \text{M1}$$

$$= 56 \quad \text{A1}$$

Answer: 56 3 marks

12 (4 marks)

$$X \sim \text{Geo}(0.04) \quad \text{B1}$$

$$E(X) = \frac{1}{0.04} = 25 \quad \text{B1}$$

$$P(X > 30) = (0.96)^{30} \quad \text{M1}$$

$$= 0.2939 \quad \text{A1}$$

Answer: 25; 0.2939 4 marks

13 (3 marks)

The probability of passing is the same, 0.45, at every attempt B1

Attempts are independent of one another B1

Learners usually improve with practice, so the pass probability probably rises with each attempt E1

Answer: Constant p and independence; p probably increases 3 marks

14 (4 marks)

$$P(X > 7 | X > 4) = \frac{P(X > 7)}{P(X > 4)} \quad \text{M1}$$

$$= \frac{0.85^7}{0.85^4} = 0.85^3 \quad \text{M1 A1}$$

$$= P(X > 3) = 0.614 \quad \text{A1}$$

Answer: 0.614 4 marks

15 (4 marks)

$$P(X \leq n) = 1 - (0.8)^n > 0.95 \quad \text{M1}$$

$$(0.8)^n < 0.05 \quad \text{A1}$$

$$n > \frac{\ln 0.05}{\ln 0.8} = 13.4 \quad \text{M1}$$

$$n = 14 \quad \text{A1}$$

Answer: 14 4 marks

16 (4 marks)

$$P(X \leq m) = 1 - (0.9)^m \geq 0.5, \text{ so } (0.9)^m \leq 0.5 \quad \text{M1 A1}$$

$$m \geq \frac{\ln 0.5}{\ln 0.9} = 6.58 \quad \text{M1}$$

$$m = 7 \quad \text{A1}$$

Answer: 7 4 marks

17 (5 marks)

Amir wins on shot 1, 3, 5, ... (his k-th shot)

M1

$$P(\text{Amir wins}) = 0.3 + (0.7)(0.5)(0.3) + [(0.7)(0.5)]^2(0.3) + \dots$$

M1 A1

Geometric series with first term 0.3 and common ratio 0.35

A1

$$\text{Sum} = \frac{0.3}{1 - 0.35} = \frac{6}{13}$$

A1

Answer: $\frac{6}{13}$

5 marks

18 (5 marks)

$X \sim \text{Geo}(0.2)$ is the number of plays

B1

$$E(\text{cost}) = 4E(X) = 4 \times 5 = \text{£}20$$

M1 A1

Cost > 40 means $X > 10$

M1

$$P(X > 10) = (0.8)^{10} = 0.1074$$

A1

Answer: £20; 0.1074

5 marks

19 (5 marks)

$X + Y = 4$ for $(X, Y) = (1, 3), (2, 2), (3, 1)$

M1 A1

$$P(1, 3) = (0.25)(0.75)^2(0.25)$$

M1

Each pair has probability $(0.25)^2(0.75)^2$

A1

$$P = 3 \times 0.0625 \times 0.5625 = 0.1055$$

A1

Answer: 0.1055

5 marks

20 (5 marks)

Given $X > 2$, the number of further trials needed is again geometric with parameter p (memoryless)

M1 A1

So $X - 2 \mid X > 2 \sim \text{Geo}(p)$

A1

$$E(X \mid X > 2) = 2 + \frac{1}{p}$$

A1

$$p = 0.4: 2 + 2.5 = 4.5$$

A1

Answer: 4.5

5 marks

21 (6 marks)

$$E(X) = p \sum_{r=1}^{\infty} r q^{r-1}$$

M1

$\sum_{r=1}^{\infty} r q^{r-1}$ is the derivative of $\sum_{r=0}^{\infty} q^r = \frac{1}{1-q}$ with respect to q

M1 A1

$$\frac{d}{dq} \left(\frac{1}{1-q} \right) = \frac{1}{(1-q)^2}$$

M1 A1

$$E(X) = \frac{p}{p^2} = \frac{1}{p}$$

A1

Answer: $E(X) = \frac{1}{p}$

6 marks

22 (4 marks)

$$P(X > m + n \mid X > m) = \frac{P(X > m + n)}{P(X > m)}$$

M1 A1

$$= \frac{q^{m+n}}{q^m} = q^n$$

M1

$$= P(X > n)$$

A1

Answer: $P(X > m + n \mid X > m) = q^n = P(X > n)$

4 marks

23 (6 marks)

$$E[X(X-1)] = \sum_{r=2}^{\infty} r(r-1)q^{r-1}p = pq \sum_{r=2}^{\infty} r(r-1)q^{r-2} \quad \text{M1 A1}$$

$$\sum_{r=2}^{\infty} r(r-1)q^{r-2} = \frac{d^2}{dq^2} \left(\frac{1}{1-q} \right) = \frac{2}{(1-q)^3} \quad \text{M1 A1}$$

$$E[X(X-1)] = \frac{2pq}{p^3} = \frac{2q}{p^2} \quad \text{A1}$$

$$\text{Var}(X) = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q + p - 1}{p^2} = \frac{q}{p^2} \quad \text{A1}$$

Answer: $\text{Var}(X) = \frac{q}{p^2}$ 6 marks

24 (5 marks)

The complement of $X \geq 4$ is $X \leq 3$, not $X \leq 4$ E1

$$P(X \geq 4) = 1 - P(X \leq 3) \quad \text{M1}$$

$$P(X \leq 3) = 1 - (0.8)^3 \quad \text{M1}$$

$$\text{So } P(X \geq 4) = (0.8)^3 \quad \text{A1}$$

$$= 0.512 \quad \text{A1}$$

Answer: 0.512 5 marks

25 (6 marks)

$$P(X \text{ even}) = P(X=2) + P(X=4) + \dots = qp + q^3p + q^5p + \dots \quad \text{M1 A1}$$

Geometric series with first term pq and common ratio q^2 M1

$$= \frac{pq}{1-q^2} = \frac{pq}{(1-q)(1+q)} = \frac{q}{1+q} \quad \text{A1}$$

$$q = \frac{5}{6} : \frac{\frac{5}{6}}{\frac{11}{6}} \quad \text{M1}$$

$$= \frac{5}{11} \quad \text{A1}$$

Answer: $\frac{5}{11}$ 6 marks

26 (6 marks)

(a) Maya wins on shot 1, 3, 5, ...: $P = 0.25 + (0.75)(1-p)(0.25) + [(0.75)(1-p)]^2(0.25) + \dots$ M1 A1

Geometric series with first term 0.25 and common ratio $0.75(1-p)$ M1

$$\text{Sum to infinity gives } \frac{0.25}{1-0.75(1-p)} \quad \text{A1}$$

Answer: $\frac{0.25}{1-0.75(1-p)}$ 4 marks

(b) $\frac{0.25}{1-0.75(1-p)} = \frac{5}{11}$, so $1-0.75(1-p) = 0.55$ M1

$$1-p = 0.6, \text{ so } p = 0.4 \quad \text{A1}$$

Answer: $p = 0.4$ 2 marks

27 (8 marks)

(a) $M > m$ means $X > m$ and $Y > m$ M1

$$P(M > m) = P(X > m)P(Y > m) = (1-p)^m(1-r)^m \quad \text{M1 A1}$$

This is q^m with $q = (1-p)(1-r)$, so $M \sim \text{Geo}(1-(1-p)(1-r))$ A1

Answer: $\text{Geo}(1-(1-p)(1-r))$ 4 marks

(b) Each player's rounds to first head is $\text{Geo}(0.5)$ B1

M is the min, parameter $1-0.5 \times 0.5 = 0.75$ M1 A1

$$E(M) = \frac{1}{0.75} = \frac{4}{3} \quad \text{A1}$$

Answer: $\frac{4}{3}$ 4 marks

28 (7 marks)

$$(a) E(2^X) = \sum_{r=1}^{\infty} 2^r (0.4)^{r-1} (0.6) \quad \text{M1}$$

$$= 1.2 \sum_{r=1}^{\infty} (0.8)^{r-1} \quad \text{M1 A1}$$

The series is geometric with ratio $0.8 < 1$, so it converges E1

$$= \frac{1.2}{1 - 0.8} = £6 \quad \text{A1}$$

Answer: £6 5 marks

(b) The terms are $2^r (0.5)^{r-1} (0.5) = 1$ for every r M1

The series $1 + 1 + 1 + \dots$ diverges, so the expected winnings are infinite E1

Answer: Infinite (the series diverges) 2 marks

29 (9 marks)

$$P(X > Y) = \sum_{y=1}^{\infty} P(Y = y) P(X > y) \quad \text{M1 A1}$$

$$P(Y = y) = (0.6)^{y-1} (0.4) \text{ and } P(X > y) = (0.7)^y \quad \text{B1 B1}$$

$$= 0.4 \sum_{y=1}^{\infty} (0.6)^{y-1} (0.7)^y \quad \text{M1}$$

$$= 0.4 \times 0.7 \sum_{y=1}^{\infty} (0.42)^{y-1} \quad \text{M1 A1}$$

$$= \frac{0.28}{1 - 0.42} = \frac{0.28}{0.58} \quad \text{M1}$$

$$= \frac{14}{29} \quad \text{A1}$$

Answer: $\frac{14}{29}$ 9 marks

30 (9 marks)

$$(a) P(X = k, Y = n - k) = q^{k-1} p \times q^{n-k-1} p = p^2 q^{n-2} \quad \text{M1 A1}$$

This does not depend on k E1

$$P(X + Y = n) = (n - 1) p^2 q^{n-2} \text{ (} n - 1 \text{ pairs)} \quad \text{M1 A1}$$

$$\text{Ratio} = \frac{1}{n - 1} \quad \text{A1}$$

Answer: $P(X = k | X + Y = n) = \frac{1}{n - 1}$ 6 marks

(b) Given $X + Y = 6$, X is uniform on 1, ..., 5 M1

$X \geq 4$: two values (4 and 5) M1

Probability $\frac{2}{5}$ A1

Answer: $\frac{2}{5}$ 3 marks

Negative Binomial Distribution**1 (2 marks)**

$$P(X = 4) = {}^3C_1 (0.5)^2 (0.5)^2 \quad \text{M1}$$

$$= \frac{3}{16} = 0.1875 \quad \text{A1}$$

Answer: $\frac{3}{16}$ 2 marks

2 (2 marks)

$$E(X) = \frac{r}{p} = \frac{3}{0.25} = 12$$

B1

$$\text{Var}(X) = \frac{rq}{p^2} = \frac{3 \times 0.75}{0.0625} = 36$$

B1

Answer: 12 and 36

2 marks

3 (2 marks)

The fourth success needs at least 4 trials: smallest value 4

B1

$$P(X = 4) = p^4$$

B1

Answer: 4; p^4

2 marks

4 (2 marks)

$$P(X = 5) = {}^4C_2(0.4)^3(0.6)^2$$

M1

$$= 0.1382$$

A1

Answer: 0.1382

2 marks

5 (2 marks)NB(1, p): $P(X = x) = {}^{x-1}C_0p(1 - p)^{x-1} = p(1 - p)^{x-1}$, the geometric pmf

B1

$$P(X = 3) = (0.8)^2(0.2) = 0.128$$

B1

Answer: 0.128

2 marks

6 (2 marks)

$$X \sim \text{NB}(3, 0.5)$$

M1

$$E(X) = \frac{3}{0.5} = 6$$

A1

Answer: 6

2 marks

7 (3 marks)

$$X \sim \text{NB}(2, \frac{1}{6})$$

B1

$$P(X = 5) = {}^4C_1(\frac{1}{6})^2(\frac{5}{6})^3$$

M1

$$= \frac{500}{7776} = 0.0643$$

A1

Answer: 0.0643

3 marks

8 (3 marks)

$$\frac{r}{p} = 12 \text{ and } \frac{rq}{p^2} = 36, \text{ so } \frac{q}{p} = 3$$

M1

$$1 - p = 3p, p = \frac{1}{4}$$

A1

$$r = 12 \times \frac{1}{4} = 3$$

A1

Answer: $r = 3, p = \frac{1}{4}$

3 marks

9 (3 marks)

$$P(X = 4) = p^4 = 0.0256$$

M1

$$p = 0.4$$

A1

$$P(X = 6) = {}^5C_3(0.4)^4(0.6)^2 = 0.0922$$

A1

Answer: $p = 0.4; 0.0922$

3 marks

10 (3 marks)The r -th success is on trial x , so trial x is a success with probability p

E1

The first $x - 1$ trials contain exactly $r - 1$ successes: ${}^{x-1}C_{r-1}p^{r-1}(1 - p)^{x-r}$

E1

Multiplying the two gives the pmf

E1

Answer: $P(X = x) = {}^{x-1}C_{r-1}p^r(1 - p)^{x-r}$

3 marks

11 (4 marks)

$$P(X > 5) = 1 - P(X \leq 5) \quad \text{M1}$$

$$P(X = 3) = \frac{1}{8}, P(X = 4) = \frac{3}{16}, P(X = 5) = \frac{6}{32} \quad \text{M1 A1}$$

$$= 1 - \frac{1}{2} = \frac{1}{2} \quad \text{A1}$$

$$\text{Answer: } \frac{1}{2} \quad \text{4 marks}$$

12 (3 marks)

Number inspected $X \sim \text{NB}(5, 0.1)$ B1

$$E(X) = \frac{5}{0.1} = 50 \quad \text{M1}$$

$$\text{Expected cost} = 2 \times 50 = \text{£}100 \quad \text{A1}$$

$$\text{Answer: £100} \quad \text{3 marks}$$

13 (3 marks)

$X \sim \text{NB}(2, 0.3)$ B1

$$P(X = 6) = {}^5C_1(0.3)^2(0.7)^4 \quad \text{M1}$$

$$= 0.1080 \quad \text{A1}$$

$$\text{Answer: 0.1080} \quad \text{3 marks}$$

14 (4 marks)

$X \leq 6$ means the third head occurs within the first 6 tosses M1

This happens if and only if at least 3 heads occur in 6 tosses: $Y \sim \text{B}(6, 0.5)$, $Y \geq 3$ A1

$$P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - \frac{1 + 6 + 15}{64} \quad \text{M1}$$

$$= \frac{42}{64} = \frac{21}{32} \quad \text{A1}$$

$$\text{Answer: } \frac{21}{32} \quad \text{4 marks}$$

15 (5 marks)

Given a success on trial 1, two more successes are needed in the later trials M1

The number of further trials $Y \sim \text{NB}(2, 0.4)$, and $X = 1 + Y$ A1

$X = 5$ means $Y = 4$ M1

$$P(Y = 4) = {}^3C_1(0.4)^2(0.6)^2 \quad \text{M1}$$

$$= 0.1728 \quad \text{A1}$$

$$\text{Answer: 0.1728} \quad \text{5 marks}$$

16 (5 marks)

$T = 4$ for $(X, Y) = (1, 3), (2, 2), (3, 1)$ M1 A1

Each has probability $(0.3)^2(0.7)^2$ M1

$$P(T = 4) = 3 \times 0.09 \times 0.49 = 0.1323 \quad \text{A1}$$

$$P(Z = 4) = {}^3C_1(0.3)^2(0.7)^2 = 0.1323 \quad \text{A1}$$

$$\text{Answer: 0.1323} \quad \text{5 marks}$$

17 (4 marks)

$$E(X) = \frac{4}{0.4} = 10, \text{ so } E(Y) = 10 - 4 = 6 \quad \text{M1 A1}$$

$$\text{Var}(X) = \frac{4 \times 0.6}{0.16} = 15 \quad \text{M1}$$

$$\text{Var}(Y) = \text{Var}(X) = 15 \quad \text{A1}$$

$$\text{Answer: } E(Y) = 6, \text{ Var}(Y) = 15 \quad \text{4 marks}$$

18 (5 marks)

X is the number of trials to get 2 successes; then Y is the number of further trials for 3 more

M1

Together $X + Y$ is the number of trials to obtain 5 successes in total

A1

$X + Y \sim \text{NB}(5, 0.4)$

A1

$$P(X + Y = 7) = {}^6C_4(0.4)^5(0.6)^2$$

M1

$$= 0.0553$$

A1

Answer: 0.0553

5 marks

19 (5 marks)

$P(X \leq n) = P(\text{at least 3 heads in } n \text{ tosses of a fair coin})$

M1

$$n = 6: 0.6562$$

A1

$$n = 7: 0.7734$$

M1 A1

$n = 8: 0.8555 \geq 0.9$, so the smallest is $n = 9$

A1

Answer: 9

5 marks

20 (5 marks)

$$P(X = x + 1) = {}^xC_2(0.3)^3(0.7)^{x-2}, P(X = x) = {}^{x-1}C_2(0.3)^3(0.7)^{x-3}$$

M1 A1

$$\text{Ratio} = \frac{x}{x-2} \times 0.7 \text{ (since } {}^xC_2 / {}^{x-1}C_2 = \frac{x}{x-2} \text{)}$$

M1

Ratio > 1 when $0.7x > x - 2$, i.e. $x < 6.67$

M1

Probabilities increase up to $x = 7$, so the mode is 7

A1

Answer: 7

5 marks

21 (4 marks)

$$E(X_i) = \frac{1}{p} \text{ and } \text{Var}(X_i) = \frac{q}{p^2}$$

B1

$$E(X) = r \times \frac{1}{p} = \frac{r}{p}$$

M1 A1

$$\text{Var}(X) = r \times \frac{q}{p^2} \text{ (independent)} = \frac{rq}{p^2}$$

A1

Answer: $\frac{r}{p}$ and $\frac{rq}{p^2}$

4 marks

22 (5 marks)

$$P(X = x) = (x - 1)p^2q^{x-2} \text{ for } x = 2, 3, \dots$$

B1

$$\sum_{x=2}^{\infty} (x - 1)q^{x-2} = \sum_{k=1}^{\infty} kq^{k-1}$$

M1

$$= \frac{d}{dq} \left(\frac{1}{1 - q} \right) = \frac{1}{(1 - q)^2}$$

M1 A1

$$\text{Total} = p^2 \times \frac{1}{p^2} = 1$$

A1

Answer: The probabilities of NB(2, p) sum to 1

5 marks

23 (6 marks)

Aisha wins the match if her 3rd game win occurs on game 3, 4 or 5, i.e. $X \sim \text{NB}(3, 0.6)$ with $X \leq 5$

M1 A1

$$P(X = 3) = 0.6^3 = 0.216$$

B1

$$P(X = 4) = {}^3C_2(0.6)^3(0.4) = 0.2592$$

M1

$$P(X = 5) = {}^4C_2(0.6)^3(0.4)^2 = 0.20736$$

M1

$$\text{Total} = 0.68256$$

A1

Answer: 0.68256

6 marks

24 (5 marks)

${}^5C_2 p^2 q^3$ is the binomial probability of exactly 2 successes in 5 trials

E1

It does not insist that the 5th trial is a success

E1

Correct: the 5th trial is a success and exactly 1 success in the first 4

M1

$$P = {}^4C_1 (0.3)^2 (0.7)^3$$

M1

$$= 0.1235$$

A1

Answer: 0.1235

5 marks

25 (6 marks)

Successes 1 and 2 occur in the first 6 trials and the 7th trial is a success

M1

Number of equally likely arrangements of two successes in trials 1-6: ${}^6C_2 = 15$

M1 A1

Arrangements with trial 1 a success: the other success is in trials 2-6: ${}^5C_1 = 5$

M1 A1

$$\text{Probability} = \frac{5}{15} = \frac{1}{3}$$

A1

Answer: $\frac{1}{3}$

6 marks

26 (6 marks)

Components handled $X \sim \text{NB}(6, 0.15)$

B1

$$E(X) = \frac{6}{0.15} = 40$$

M1

$$\text{Var}(X) = \frac{6 \times 0.85}{0.0225} = 226.67$$

M1 A1

$$\text{Cost} = 20 + 3X, E = £140$$

A1

$$\text{SD} = 3 \times \sqrt{226.67} = £45.17$$

A1

Answer: £140 and £45.17

6 marks

27 (9 marks)

(a) Imagine 4 games are played in any case

E1

If Alan wins at least 2 of them, he reaches 2 wins before Beth gets 3

E1

If he wins at most 1, Beth wins at least 3 first, so Alan loses

E1

Answer: Alan wins if and only if he wins at least 2 of 4 games

3 marks

(b) $Y \sim B(4, 0.5)$, $P(Y \geq 2) = 1 - P(Y \leq 1)$

M1

$$P(Y \leq 1) = \frac{1+4}{16} = \frac{5}{16}$$

M1

$$P = \frac{11}{16}$$

A1

Answer: $\frac{11}{16}$

3 marks

(c) Alan wins if $X \leq 4$

M1

$$P(X=2) + P(X=3) + P(X=4) = \frac{1}{4} + \frac{2}{8} + \frac{3}{16}$$

M1

$$= \frac{11}{16}$$

A1

Answer: $\frac{11}{16}$

3 marks

28 (8 marks)

(a) $X = X_1 + \dots + X_r$, with X_i independent geometric

M1

Each geometric has PGF $\frac{pt}{1-qt}$

B1

The PGF of the sum is the product, giving $(\frac{pt}{1-qt})^r$

A1

Answer: $G(t) = (\frac{pt}{1-qt})^r$

3 marks

$$(b) G'(t) = r(\frac{pt}{1-qt})^{r-1} \times \frac{p}{(1-qt)^2}$$

M1 A1

$$G'(1) = r \times 1 \times \frac{p}{p^2} = \frac{r}{p}$$

A1

Answer: $\frac{r}{p}$

3 marks

(c) $G(t) = \frac{t^2}{4}(1 - 0.5t)^{-2} = \frac{t^2}{4}(1 + t + \frac{3}{4}t^2 + \frac{1}{2}t^3 + \dots)$

M1

Coefficient of t^5 : $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$ (check: ${}^4C_1(0.5)^5 = \frac{1}{8}$)

A1

Answer: $\frac{1}{8}$

2 marks

29 (8 marks)(a) $X \leq n$ means the 4th success occurs within the first n trials

E1

That is the same as at least 4 successes in n trials

E1

The number of successes in n trials is $B(n, 0.3)$

E1

Answer: $P(X \leq n) = P(Y \geq 4)$ for $Y \sim B(n, 0.3)$

3 marks

(b) $P(X \leq 12) = P(Y \geq 4) = 1 - P(Y \leq 3)$

M1

$= 0.5075$

A1

Answer: 0.5075

2 marks

(c) Need $P(B(n, 0.3) \geq 4) > 0.95$

M1

$n = 23: 0.9462 ; n = 24: 0.9576$

M1

$n = 24$

A1

Answer: 24

3 marks

30 (9 marks)

(a) $P(X = x) = (x - 1)p^2q^{x-2}$

B1

$P(X \text{ even}) = p^2 \sum_{k=1}^{\infty} (2k - 1)q^{2k-2}$

M1

With $s = q^2$: $\sum_{k=1}^{\infty} (2k - 1)s^{k-1} = \frac{2}{(1 - s)^2} - \frac{1}{1 - s} = \frac{1 + s}{(1 - s)^2}$

M1 A1

$P = \frac{p^2(1 + q^2)}{(1 - q^2)^2}$, and $1 - q^2 = (1 - q)(1 + q) = p(1 + q)$

M1

$P(X \text{ even}) = \frac{1 + q^2}{(1 + q)^2}$

A1

Answer: $P(X \text{ is even}) = \frac{1 + q^2}{(1 + q)^2}$

6 marks

(b) $\frac{1 + 0.25}{2.25} = \frac{5}{9}$

M1 A1

Greater than $\frac{1}{2}$: the smallest value is 2 (even) and the early, more likely values favour even X

E1

Answer: $\frac{5}{9}$

3 marks

Hypergeometric Distribution**1 (2 marks)**

$\frac{{}^4C_2 \times {}^6C_1}{{}^{10}C_3}$

M1

$= \frac{36}{120} = \frac{3}{10}$

A1

Answer: $\frac{3}{10}$

2 marks

2 (2 marks)

$E(X) = n \times \frac{K}{N} = 8 \times \frac{5}{20}$

M1

$= 2$

A1

Answer: 2

2 marks

3 (2 marks)

There are only 4 non-red balls, so at least 3 red balls are taken

B1

At most 6 red balls (there are only 6)

B1

Answer: 3 and 6

2 marks

4 (2 marks)

$$\frac{{}^3C_2}{{}^8C_2}$$

M1

$$= \frac{3}{28}$$

A1

Answer: $\frac{3}{28}$

2 marks

5 (2 marks)

$$\frac{{}^{13}C_2 \times {}^{39}C_3}{{}^{52}C_5}$$

M1

$$= 0.2743$$

A1

Answer: 0.2743

2 marks

6 (1 mark)

$$P(X = x) = \frac{{}^K C_x \times {}^{N-K} C_{n-x}}{{}^N C_n}$$

B1

Answer: $\frac{{}^K C_x \times {}^{N-K} C_{n-x}}{{}^N C_n}$

1 mark

7 (3 marks)

Of the 49 numbers, 6 are the player's

B1

$$\frac{{}^6 C_3 \times {}^{43} C_3}{{}^{49} C_6}$$

M1

$$= 0.0177$$

A1

Answer: 0.0177

3 marks

8 (4 marks)

$$P(X = x) = \frac{{}^5 C_x \times {}^7 C_{3-x}}{{}^{12} C_3}, {}^{12} C_3 = 220$$

M1 A1

$$P(0) = \frac{35}{220} = \frac{7}{44}, P(1) = \frac{105}{220} = \frac{21}{44}$$

A1

$$P(2) = \frac{70}{220} = \frac{7}{22}, P(3) = \frac{10}{220} = \frac{1}{22}$$

A1

Answer: $\frac{7}{44}, \frac{21}{44}, \frac{7}{22}, \frac{1}{22}$

4 marks

9 (3 marks)

$$E(X) = 5 \times \frac{8}{20} = 2$$

B1

$$\text{Var}(X) = 5 \times \frac{2}{5} \times \frac{3}{5} \times \frac{15}{19}$$

M1

$$= \frac{18}{19}$$

A1

Answer: $E(X) = 2, \text{Var}(X) = \frac{18}{19}$

3 marks

10 (3 marks)

$$P(\text{none defective}) = \frac{{}^{21}C_5}{{}^{25}C_5} \quad \text{M1}$$

$$= \frac{969}{2530} \quad \text{A1}$$

$$P(\text{at least one}) = 1 - \frac{969}{2530} = \frac{1561}{2530} \quad \text{A1}$$

$$\text{Answer: } \frac{1561}{2530} \quad \text{3 marks}$$

11 (4 marks)

$$P(X = 1) = \frac{{}^4C_1 \times {}^{21}C_4}{{}^{25}C_5} \quad \text{M1}$$

$$= \frac{4 \times 5985}{53130} = 0.4506 \quad \text{A1}$$

$$\text{Binomial: } 5 \times 0.16 \times 0.84^4 = 0.3983 \quad \text{M1}$$

The values differ by 0.0523: the binomial model ignores that sampling is without replacement A1

Answer: 0.4506 (binomial 0.3983) 4 marks

12 (3 marks)

Binomial requires independent trials with constant probability E1

Each selection changes the class composition, so the probability that the next student is a girl changes E1

Selections are therefore not independent: the distribution is hypergeometric E1

Answer: Selection without replacement: not independent 3 marks

13 (3 marks)

$$\frac{{}^4C_0 \times {}^{48}C_5}{{}^{52}C_5} \quad \text{M1}$$

$$= \frac{1712304}{2598960} \quad \text{A1}$$

$$= 0.6588 \quad \text{A1}$$

Answer: 0.6588 3 marks

14 (4 marks)

$$P(\text{both red}) = \frac{{}^2C_2}{{}^N C_2} = \frac{2}{N(N-1)} \quad \text{M1 A1}$$

$$N(N-1) = 90 \quad \text{M1}$$

$$N = 10 \quad \text{A1}$$

Answer: 10 4 marks

15 (5 marks)

$${}^9C_3 = 84; P(0) = \frac{10}{84}, P(1) = \frac{40}{84}, P(2) = \frac{30}{84}, P(3) = \frac{4}{84} \quad \text{M1 A1}$$

$$E(X) = \frac{40 + 60 + 12}{84} = \frac{4}{3} \quad \text{M1}$$

$$E(X^2) = \frac{40 + 120 + 36}{84} = \frac{196}{84} \quad \text{M1}$$

$$\text{Var}(X) = \frac{7}{3} - \frac{16}{9} = \frac{5}{9} \quad \text{A1}$$

Answer: $E(X) = \frac{4}{3}, \text{Var}(X) = \frac{5}{9}$ 5 marks

16 (4 marks)

X is hypergeometric with $N = 100$, $K = 20$, $n = 15$

B1

$$P(X = 2) = \frac{{}^{20}C_2 \times {}^{80}C_{13}}{{}^{100}C_{15}}$$

M1 A1

$$= 0.236$$

A1

Answer: 0.236

4 marks

17 (5 marks)

$$P(\text{no red}) = \frac{{}^6C_3}{{}^{10}C_3} = \frac{20}{120} = \frac{1}{6}$$

M1 A1

$$P(\text{at least one red}) = \frac{5}{6}$$

A1

$$P(\text{exactly 2 red}) = \frac{3}{10}$$

B1

$$P(2 \text{ red} \mid \text{at least one}) = \frac{3}{10} \div \frac{5}{6} = \frac{9}{25}$$

A1

$$\text{Answer: } \frac{9}{25}$$

5 marks

18 (5 marks)

$$\text{Box A: } P(2 \text{ red}) = \frac{{}^4C_2 \times {}^6C_1}{{}^{10}C_3} = \frac{3}{10}$$

M1 A1

$$\text{Box B: } P(2 \text{ red}) = \frac{{}^7C_2 \times {}^3C_1}{{}^{10}C_3} = \frac{63}{120} = \frac{21}{40}$$

M1 A1

$$P = \frac{1}{2} \times \frac{3}{10} + \frac{1}{2} \times \frac{21}{40} = \frac{33}{80}$$

A1

$$\text{Answer: } \frac{33}{80}$$

5 marks

19 (5 marks)

$$P(X = 1) = \frac{K(N - K)}{{}^N C_2}, P(X = 2) = \frac{{}^K C_2}{{}^N C_2}$$

M1 A1

$$E(X) = \frac{K(N - K) + 2 \times \frac{K(K - 1)}{2}}{\frac{N(N - 1)}{2}}$$

M1

$$= \frac{2[K(N - K) + K(K - 1)]}{N(N - 1)} = \frac{2K(N - 1)}{N(N - 1)}$$

M1

$$= \frac{2K}{N}$$

A1

$$\text{Answer: } E(X) = \frac{2K}{N}$$

5 marks

20 (5 marks)

$$P(X = x) = \frac{{}^{13}C_x \times {}^{39}C_{5-x}}{{}^{52}C_5}$$

B1

$$P(X = 2) = 0.2743, P(X = 3) = 0.0815$$

M1 A1

$$P(X \geq 2) = 0.3670$$

M1

$$P(X = 3 \mid X \geq 2) = 0.222$$

A1

Answer: 0.222

5 marks

21 (6 marks)

Each of the two items chosen is of the given type with probability $\frac{K}{N}$, so $E(X) = \frac{2K}{N}$

B1

$$E(X^2) = P(X = 1) + 4P(X = 2) = \frac{2K(N - K) + 4K(K - 1)}{N(N - 1)}$$

M1 A1

$$\text{Var}(X) = \frac{2K(N - K) + 4K(K - 1)}{N(N - 1)} - \frac{4K^2}{N^2}$$

M1

Common denominator $N^2(N - 1)$: numerator $2KN(N - K) + 4KN(K - 1) - 4K^2(N - 1)$

M1

$$= 2K[N^2 - NK + 2NK - 2N - 2KN + 2K] = 2K(N - K)(N - 2)$$

A1

$$\text{Answer: } \text{Var}(X) = \frac{2K(N - K)(N - 2)}{N^2(N - 1)}$$

6 marks

22 (8 marks)

(a) N is hypergeometric: K = 20 tagged, sample 30

M1

$$L(N) = \frac{{}^{20}C_6 \times {}^{N-20}C_{24}}{{}^NC_{30}}$$

A1

$$\text{Answer: } \frac{{}^{20}C_6 \times {}^{N-20}C_{24}}{{}^NC_{30}}$$

2 marks

$$(b) L(N)/L(N - 1) = \frac{{}^{N-20}C_{24}}{{}^{N-21}C_{24}} \times \frac{{}^{N-1}C_{30}}{{}^NC_{30}}$$

M1

$$\frac{{}^{N-20}C_{24}}{{}^{N-21}C_{24}} = \frac{N - 20}{N - 44}$$

A1

$$\frac{{}^{N-1}C_{30}}{{}^NC_{30}} = \frac{N - 30}{N}, \text{ giving the result}$$

A1

$$\text{Answer: } \frac{L(N)}{L(N - 1)} = \frac{(N - 20)(N - 30)}{N(N - 44)}$$

3 marks

$$(c) \text{Ratio} \geq 1 \Leftrightarrow (N - 20)(N - 30) \geq N(N - 44) \Leftrightarrow N^2 - 50N + 600 \geq N^2 - 44N$$

M1

$$\Leftrightarrow 600 \geq 6N \Leftrightarrow N \leq 100$$

M1

L increases up to N = 100 and L(100) = L(99) (ratio exactly 1), so the maximum is at N = 99 and N = 100

A1

Answer: N = 99 or N = 100

3 marks

23 (5 marks)

$(\frac{4}{52})^2$ treats the draws as independent, as if the first card were replaced

E1

After one ace is removed there are 3 aces among 51 cards

E1

$$P = \frac{4}{52} \times \frac{3}{51}$$

M1

$$= \frac{1}{221}$$

A1

= 0.00452, noticeably less than Maya's 0.0059

A1

$$\text{Answer: } \frac{1}{221}$$

5 marks

24 (5 marks)

$$\frac{{}^KC_x {}^{N-K}C_{n-x}}{{}^NC_n} = \frac{K!(N - K)!n!(N - n)!}{x!(K - x)!(n - x)!(N - K - n + x)!N!}$$

M1 A1

The expression is unchanged when K and n are exchanged

M1 A1

Hence the two forms are equal

E1

Answer: The two expressions for P(X = x) are equal, so the distributions are the same

5 marks

25 (6 marks)

$${}^{12}C_4 = 495 \text{ equally likely teams}$$

B1

$$\text{Forwards: } {}^5C_2 = 10, \text{ midfielders: } {}^4C_1 = 4, \text{ defenders: } {}^3C_1 = 3$$

M1 A1

$$\text{Favourable} = 10 \times 4 \times 3 = 120$$

M1

$$P = \frac{120}{495}$$

M1

$$= \frac{8}{33}$$

A1

$$\text{Answer: } \frac{8}{33}$$

6 marks

26 (6 marks)

$$(a) E(X) = 10 \times \frac{20}{50} = 4$$

B1

$$\text{Var}(X) = 10 \times 0.4 \times 0.6 \times \frac{40}{49}$$

M1

$$= 1.9592$$

A1

$$\text{Answer: } E(X) = 4, \text{ Var}(X) = 1.9592$$

3 marks

$$(b) \text{Var}(Y) = 2.4, \text{ and } \text{Var}(X)/\text{Var}(Y) = \frac{40}{49}$$

M1 A1

Sampling without replacement removes some variability: the sample cannot stray as far from its expected composition

E1

$$\text{Answer: } \text{Var}(X) = \frac{40}{49} \text{Var}(Y)$$

3 marks

27 (8 marks)

$$(1+x)^K = \sum_j {}^KC_j x^j \text{ and } (1+x)^{N-K} = \sum_k {}^{N-K}C_k x^k$$

M1

$$\text{Coefficient of } x^n \text{ in the product: } \sum_j {}^KC_j \times {}^{N-K}C_{n-j}$$

M1 A1

$$\text{Coefficient of } x^n \text{ in } (1+x)^N: {}^NC_n$$

M1 A1

$$\text{Equating: } \sum_j {}^KC_j {}^{N-K}C_{n-j} = {}^NC_n$$

M1

$$\text{Divide by } {}^NC_n$$

M1

So the probabilities sum to 1

A1

$$\text{Answer: The probabilities sum to 1}$$

8 marks

28 (8 marks)

$$E(X) = \frac{5K}{N} = 2, \text{ so } \frac{K}{N} = 0.4$$

M1 A1

$$\text{Var}(X) = 5 \times 0.4 \times 0.6 \times \frac{N-5}{N-1} = 1$$

M1 A1

$$1.2(N-5) = N-1$$

M1

$$0.2N = 5, N = 25$$

A1

$$K = 0.4 \times 25 = 10$$

A1

$$\text{Check: } 5 \times 0.4 \times 0.6 \times \frac{20}{24} = 1$$

B1

$$\text{Answer: } N = 25, K = 10$$

8 marks

29 (8 marks)

$$P(\text{none faulty}) = \frac{{}^{45}C_n}{{}^{50}C_n} = \frac{45!(50-n)!}{(45-n)!50!}$$

M1 A1

$$= \frac{(50-n)(49-n)(48-n)(47-n)(46-n)}{50 \times 49 \times 48 \times 47 \times 46}$$

M1 A1

$$\text{Need this } \leq 0.1$$

B1

$$n = 17: 0.1120 > 0.1$$

M1 A1

$$n = 18: 0.0950 \leq 0.1, \text{ so } n = 18$$

A1

$$\text{Answer: } 18$$

8 marks

30 (10 marks)

$$(a) {}^K C_x = \frac{K(K-1)\dots(K-x+1)}{x!} \text{ and } {}^{N-K} C_{n-x} = \frac{(N-K)\dots(N-K-n+x+1)}{(n-x)!} \quad \text{M1}$$

$${}^N C_n = \frac{N(N-1)\dots(N-n+1)}{n!} \quad \text{B1}$$

$$\text{Combining, } \frac{n!}{x!(n-x)!} = {}^n C_x \quad \text{A1}$$

$$\text{Answer: } P(X=x) = {}^n C_x \times \frac{K(K-1)\dots(K-x+1) \times (N-K)(N-K-1)\dots(N-K-n+x+1)}{N(N-1)\dots(N-n+1)} \quad \text{3 marks}$$

$$(b) \text{ Numerator has } x \text{ factors of the form } (K-i) = (pN-i) \approx pN \quad \text{M1}$$

$$\text{and } n-x \text{ factors of the form } (N-K-j) \approx (1-p)N \quad \text{M1}$$

$$\text{Denominator has } n \text{ factors each } \approx N \quad \text{M1}$$

$$\text{So } P(X=x) \rightarrow {}^n C_x \times \frac{(pN)^x ((1-p)N)^{n-x}}{N^n} \quad \text{M1}$$

$$= {}^n C_x p^x (1-p)^{n-x} \quad \text{A1}$$

$$\text{Answer: Tends to } B(n, p) \quad \text{5 marks}$$

$$(c) \text{ Hypergeometric: } \frac{{}^{300} C_2 \times {}^{700} C_3}{{}^{1000} C_5} = 0.30944 \quad \text{M1}$$

$$\text{Binomial: } 10 \times 0.09 \times 0.343 = 0.30870, \text{ very close} \quad \text{A1}$$

$$\text{Answer: } 0.30944 \text{ vs } 0.30870 \quad \text{2 marks}$$

Discrete Uniform Distribution

1 (2 marks)

$$X \text{ has a discrete uniform distribution on } 1, 2, 3, 4, 5, 6, P(X=x) = \frac{1}{6} \quad \text{B1}$$

$$P(X \geq 5) = \frac{2}{6} = \frac{1}{3} \quad \text{B1}$$

$$\text{Answer: } \frac{1}{3} \quad \text{2 marks}$$

2 (2 marks)

$$E(X) = \frac{n+1}{2} = 5.5 \quad \text{B1}$$

$$\text{Var}(X) = \frac{n^2-1}{12} = \frac{99}{12} = 8.25 \quad \text{B1}$$

$$\text{Answer: } 5.5 \text{ and } 8.25 \quad \text{2 marks}$$

3 (2 marks)

$$\text{Primes: } 2, 3, 5 \quad \text{M1}$$

$$P = \frac{3}{5} \quad \text{A1}$$

$$\text{Answer: } \frac{3}{5} \quad \text{2 marks}$$

4 (1 mark)

$$\frac{3}{8} \quad \text{B1}$$

$$\text{Answer: } \frac{3}{8} \quad \text{1 mark}$$

5 (2 marks)

$$P(D=7) = 0.1 \quad \text{B1}$$

$$E(D) = \frac{0+9}{2} = 4.5 \quad \text{B1}$$

$$\text{Answer: } 0.1 \text{ and } 4.5 \quad \text{2 marks}$$

6 (2 marks)

$$n = 6$$

B1

$$\text{Var}(X) = \frac{35}{12}$$

B1

$$\text{Answer: } \frac{35}{12}$$

2 marks

7 (3 marks)

$$\frac{n+1}{2} = 10.5, \text{ so } n = 20$$

M1 A1

$$\text{Var}(X) = \frac{400-1}{12} = \frac{399}{12} = 33.25$$

A1

$$\text{Answer: } n = 20, \text{ Var}(X) = 33.25$$

3 marks

8 (3 marks)

$$E(X) = 3.5 \text{ and } \text{Var}(X) = \frac{35}{12}$$

B1

$$E(Y) = 2 \times 3.5 + 1 = 8$$

M1

$$\text{Var}(Y) = 4 \times \frac{35}{12} = \frac{35}{3}$$

A1

$$\text{Answer: } E(Y) = 8, \text{ Var}(Y) = \frac{35}{3}$$

3 marks

9 (3 marks)

$$E(W) = 4.5, \text{ so } E(X) = 8.5$$

M1 A1

$$\text{Var}(X) = \text{Var}(W) = \frac{63}{12} = \frac{21}{4}$$

A1

$$\text{Answer: } E(X) = 8.5, \text{ Var}(X) = \frac{21}{4}$$

3 marks

10 (4 marks)

$$\text{(a) 14 multiples (7 to 98): } \frac{14}{100} = \frac{7}{50}$$

B1

$$\text{Answer: } \frac{7}{50}$$

1 mark

$$\text{(b) 1, 4, ..., 100: 10 squares, } \frac{10}{100} = \frac{1}{10}$$

B1

$$\text{Answer: } \frac{1}{10}$$

1 mark

(c) Must be a multiple of 49: only 49 (and 196 is too large)

M1

$$\frac{1}{100}$$

A1

$$\text{Answer: } \frac{1}{100}$$

2 marks

11 (3 marks)

Write $X = W - 1$ with W uniform on 1, ..., 10

M1

$$E(X) = 5.5 - 1 = 4.5$$

A1

$$\text{Var}(X) = \text{Var}(W) = \frac{99}{12} = 8.25$$

A1

$$\text{Answer: } 8.25$$

3 marks

12 (3 marks)

One die: six equally likely outcomes

E1

Two dice: totals 2 to 12 can occur in different numbers of ways

E1

e.g. a total of 7 can occur in 6 ways but a total of 2 in only 1 way, so the probabilities are not equal

E1

Answer: Totals are not equally likely

3 marks

13 (4 marks)

Winnings $W = 10X$ where X is uniform on 1, ..., 10

M1

$$E(W) = 10 \times 5.5 = \text{£}55$$

A1

$$\text{Var}(W) = 100 \times 8.25 = 825$$

M1

$$SD = \sqrt{825} = \text{£}28.7$$

A1

Answer: £55 and £28.7

4 marks

14 (4 marks)

$$\frac{n^2 - 1}{12} = 33.25$$

M1

$$n^2 - 1 = 399$$

A1

$$n^2 = 400$$

M1

$$n = 20$$

A1

Answer: 20

4 marks

15 (4 marks)

$$P(X > 12) = \frac{n - 12}{n}$$

M1

$$\frac{n - 12}{n} = 0.4$$

A1

$$0.6n = 12$$

M1

$$n = 20$$

A1

Answer: 20

4 marks

16 (4 marks)

$$P(X = Y) = \frac{6}{36} = \frac{1}{6}$$

B1

$$\text{By symmetry } P(X < Y) = \frac{1}{2} \left(1 - \frac{1}{6}\right) = \frac{5}{12}$$

M1 A1

$$|X - Y| \geq 4: \text{differences 4 and 5 occur } 2(2 + 1) = 6 \text{ times, } \frac{6}{36} = \frac{1}{6}$$

A1

$$\text{Answer: } \frac{1}{6}, \frac{5}{12}, \frac{1}{6}$$

4 marks

17 (5 marks)

16 equally likely outcomes

B1

$$P(M = m) = \frac{2m - 1}{16} \text{ for } m = 1, 2, 3, 4$$

M1 A1

$$\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}$$

A1

$$E(M) = \frac{1 + 6 + 15 + 28}{16} = \frac{25}{8}$$

A1

$$\text{Answer: } E(M) = \frac{25}{8}$$

5 marks

max	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

18 (4 marks)

Median: the 10th and 11th values are 10 and 11

M1

Median = 10.5

A1

Lower quartile: $P(X \leq 5) = 0.25$, so $Q_1 = 5$

M1

Upper quartile: $P(X \leq 15) = 0.75$, so $Q_3 = 15$

A1

Answer: $Q_1 = 5$, median 10.5, $Q_3 = 15$

4 marks

19 (5 marks)

$$E(X) = 3 \text{ and } \text{Var}(X) = \frac{25 - 1}{12} = 2$$

B1 B1

$$\text{Var}(Y) = 2a^2 = 8, \text{ so } a = 2$$

M1 A1

$$E(Y) = 6 + b = 30, \text{ so } b = 24$$

A1

Answer: a = 2, b = 24

5 marks

20 (4 marks)

$$P(X \text{ even}) = \frac{10}{20}$$

B1

Even values above 12: 14, 16, 18, 20

M1

$$P(X > 12 \text{ and even}) = \frac{4}{20}$$

A1

$$\text{Conditional probability} = \frac{4}{10} = \frac{2}{5}$$

A1

Answer: $\frac{2}{5}$

4 marks

21 (6 marks)

$$E(X) = \frac{1}{n} \sum_{x=1}^n x = \frac{n+1}{2}$$

M1 A1

$$E(X^2) = \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

M1 A1

$$\begin{aligned} \text{Var}(X) &= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)[2(2n+1) - 3(n+1)]}{12} \\ &= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12} \end{aligned}$$

M1

A1

Answer: $E(X) = \frac{n+1}{2}$, $\text{Var}(X) = \frac{n^2-1}{12}$

6 marks

22 (6 marks)**(a)** For $T = t$, X can be 1, 2, ..., $t-1$ ($Y = t - X$ must be between 1 and n , true for $t \leq n+1$)

M1 A1

$$P(T = t) = \frac{t-1}{n^2}; \text{ at } t = n+1 \text{ this is } \frac{n}{n^2} = \frac{1}{n}$$

A1

Answer: $P(T = n+1) = \frac{1}{n}$, $P(T = t) = \frac{t-1}{n^2}$ for $2 \leq t \leq n+1$

3 marks

$$\text{(b) } E(T) = 2 \times \frac{n+1}{2} = n+1$$

M1 A1

$$\text{Var}(T) = 2 \times \frac{n^2-1}{12} = \frac{n^2-1}{6}$$

A1

Answer: $n+1$ and $\frac{n^2-1}{6}$

3 marks

23 (5 marks)

$$\text{Var}(2X) = 2^2 \text{Var}(X)$$

M1

$$= 4 \times \frac{35}{12} = \frac{35}{3}$$

A1

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = \frac{35}{6}$$

M1 A1

Sam has treated $2X$ (one die doubled) as if it were two independent dice

E1

Answer: $\text{Var}(2X) = \frac{35}{3}$, $\text{Var}(X_1 + X_2) = \frac{35}{6}$

5 marks

24 (6 marks)

Each digit has mean 4.5 and variance 8.25

B1

Mean of the sum = 13.5

M1 A1

Variance of the sum = $3 \times 8.25 = 24.75$

A1

Sum 0 needs 0, 0, 0 and sum 27 needs 9, 9, 9: each has probability $(\frac{1}{10})^3$

M1

$$P = \frac{2}{1000} = 0.002$$

A1

Answer: 13.5, 24.75; 0.002

6 marks

25 (6 marks) $M \leq m$ requires all three numbers to be at most m

M1

$$P(M \leq m) = (\frac{m}{10})^3$$

A1

$$P(M = m) = \frac{m^3 - (m-1)^3}{1000} = \frac{3m^2 - 3m + 1}{1000}$$

M1

$$E(M) = \frac{1}{1000} \sum_{m=1}^{10} (3m^3 - 3m^2 + m)$$

M1

$$= \frac{1}{1000} (3 \times 3025 - 3 \times 385 + 55) = \frac{7975}{1000}$$

M1

$$= 7.975$$

A1

Answer: 7.975

6 marks

26 (6 marks)Multiples of 2, 3, 5, 7: $105 + 70 + 42 + 30 = 247$

M1 A1

Multiples of two of the primes (6, 10, 14, 15, 21, 35): $35 + 21 + 15 + 14 + 10 + 6 = 101$

M1 A1

Multiples of three (30, 42, 70, 105): $7 + 5 + 3 + 2 = 17$; multiples of all four (210): 1

M1

Not divisible by any: $210 - 247 + 101 - 17 + 1 = 48$, so $P = \frac{48}{210} = \frac{8}{35}$

A1

Answer: $\frac{8}{35}$

6 marks

27 (8 marks)

216 equally likely outcomes

B1

Total 10: count ordered triples: (1,3,6) type has 6 arrangements, (1,4,5) 6, (2,2,6) 3, (2,3,5) 6,

M1 A1

(2,4,4) 3, (3,3,4) 3

Total number of triples with total 10 = 27, so $P = \frac{27}{216} = \frac{1}{8}$

A1

Whatever the first two dice show, the third die makes the total even with probability $\frac{3}{6}$

M1 A1

$$P(\text{even}) = \frac{1}{2}$$

A1

$$P(10 | \text{even}) = \frac{1}{8} \div \frac{1}{2} = \frac{1}{4}$$

A1

Answer: $\frac{1}{8}; \frac{1}{4}$

8 marks

28 (9 marks)(a) For a given d there are $n - d$ pairs with $Y = X + d$

M1 A1

and $n - d$ pairs with $Y = X - d$, giving $2(n - d)$ pairs out of n^2

A1

$$\text{Answer: } P(|X - Y| = d) = \frac{2(n - d)}{n^2}$$

3 marks

$$(b) E|X - Y| = \sum_{d=1}^{n-1} d \times \frac{2(n - d)}{n^2}$$

M1

$$= \frac{2}{n^2} (n \sum_{d=1}^{n-1} d - \sum_{d=1}^{n-1} d^2)$$

M1

$$= \frac{2}{n^2} \left(\frac{n^2(n-1)}{2} - \frac{(n-1)n(2n-1)}{6} \right)$$

A1

$$= \frac{2(n-1)n}{6n^2} [3n - (2n-1)] = \frac{(n-1)(n+1)}{3n} = \frac{n^2-1}{3n} \quad \text{A1}$$

Answer: $E(|X - Y|) = \frac{n^2-1}{3n}$ 4 marks

(c) $\frac{n^2-1}{3n} = \frac{35}{18}$ B1

Direct: $\frac{1}{36}(2 \times 1 \times 5 + 2 \times 2 \times 4 + 2 \times 3 \times 3 + 2 \times 4 \times 2 + 2 \times 5 \times 1) = \frac{70}{36} = \frac{35}{18}$ B1

Answer: $\frac{35}{18}$ 2 marks

29 (9 marks)

(a) $E(Y | X = x) = \frac{x+1}{2}$ B1

$E(Y) = \sum_{x=1}^n \frac{1}{n} \times \frac{x+1}{2}$ M1

$= \frac{1}{2}(E(X) + 1) = \frac{1}{2}\left(\frac{n+1}{2} + 1\right)$ M1

$= \frac{n+3}{4}$ A1

Answer: $\frac{n+3}{4}$ 4 marks

(b) $P(Y = y) = \sum_{x=y}^n \frac{1}{n} \times \frac{1}{x}$ M1

$P(Y = 1) = \frac{1}{6}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) = \frac{1}{6} \times \frac{49}{20} = \frac{49}{120}$ A1

$P(Y = 6) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ A1

Answer: $\frac{49}{120}$ and $\frac{1}{36}$ 3 marks

(c) If $Y = 6$ then $X = 6$ necessarily M1

$P = 1$ A1

Answer: 1 2 marks

30 (11 marks)

(a) Each number is equally likely to be first, $E(X_1) = E(X_2) = 3.5$, so $E(\text{sum}) = 7$ M1 A1

By symmetry each of the six values is equally likely for the first pick, so $P(X_1 = 3) = \frac{1}{6}$ E1

Answer: 7 3 marks

(b) 15 equally likely pairs B1

Sums 3 to 11 with frequencies 1, 1, 2, 2, 3, 2, 2, 1, 1 M1 A1

$E(S^2) = \frac{1}{15}(9 + 16 + 50 + 72 + 147 + 128 + 162 + 100 + 121) = \frac{805}{15}$ M1 A1

$\text{Var}(S) = \frac{161}{3} - 49 = \frac{14}{3}$ A1

Answer: $\frac{14}{3}$ 6 marks

(c) $\frac{14}{3} < \frac{35}{6}$ B1

Without replacement the two numbers are negatively correlated (a high first value makes a high second value less likely), reducing the variance of the total E1

Answer: $\frac{14}{3}$ is smaller 2 marks

Approximations Between Distributions

1 (2 marks)

$np = 2$, so $X \approx \text{Po}(2)$

B1

$P(X = 1) \approx 2e^{-2} = 0.2707$

B1

Answer: 0.2707

2 marks

2 (2 marks)

$np = 24$ and $nq = 36$ are both large (greater than 10)

B1

$X \approx N(24, 14.4)$

B1

Answer: N(24, 14.4)

2 marks

3 (3 marks)

(a) $P(19.5 < Y < 20.5)$

B1

Answer: $P(19.5 < Y < 20.5)$

1 mark

(b) $X > 20$ means $X \geq 21$: $P(Y > 20.5)$

B1

Answer: $P(Y > 20.5)$

1 mark

(c) $P(Y < 20.5)$

B1

Answer: $P(Y < 20.5)$

1 mark

4 (1 mark)

$N(15, 15)$

B1

Answer: N(15, 15)

1 mark

5 (2 marks)

$np = 2$, so $X \approx \text{Po}(2)$

M1

$P(X = 0) \approx e^{-2} = 0.1353$

A1

Answer: 0.1353

2 marks

6 (2 marks)

n is large

B1

p is small (so that np is moderate)

B1

Answer: n large, p small

2 marks

7 (3 marks)

$X \approx N(40, 20)$

B1

$P(X \leq 35) \approx P(Y < 35.5) = P(Z < \frac{35.5 - 40}{\sqrt{20}}) = P(Z < -1.006)$

M1

$= 0.1572$ (close to the exact 0.1572)

A1

Answer: 0.1572

3 marks

8 (4 marks)

$X \approx N(20, 20)$

B1

$P(X > 25) = P(X \geq 26) \approx P(Y > 25.5)$

M1

$= P(Z > \frac{5.5}{\sqrt{20}}) = P(Z > 1.2298)$

M1

$= 0.1094$

A1

Answer: 0.1094

4 marks

9 (4 marks)

Area 8 m^2 , so the number of flaws $X \sim \text{Po}(24)$

B1

$X \approx N(24, 24)$

M1

$P(X \leq 20) \approx P(Y < 20.5) = P(Z < \frac{20.5 - 24}{\sqrt{24}}) = P(Z < -0.714)$

M1

$= 0.2375$

A1

Answer: 0.2375

4 marks

10 (3 marks)

(a) Normal: $np = 150$, $nq = 150$ both large, p not small

B1

Answer: Normal

1 mark

(b) Poisson: n large, p small, $np = 2$

B1

Answer: Poisson

1 mark

(c) Neither: n is small ($np = 4.8$, $nq = 7.2$ are too small for a normal approximation and p is not small)

B1

Answer: Neither

1 mark

11 (4 marks)

$X \approx N(15, 10.5)$

B1 B1

$P(12 \leq X \leq 18) \approx P(11.5 < Y < 18.5)$

M1

$= 0.7199$ (exact value 0.7204)

A1

Answer: 0.7199

4 marks

12 (4 marks)

$X \approx N(12, 12)$

B1

$P(X = 10) \approx P(9.5 < Y < 10.5)$

M1

$= \Phi(-0.433) - \Phi(-0.722)$

M1

$= 0.0973$

A1

Answer: 0.0973

4 marks

13 (3 marks)

$\text{Var}(X) = np(1 - p)$

B1

When p is small, $1 - p \approx 1$, so $\text{Var}(X) \approx np = E(X)$

M1

A Poisson distribution has equal mean and variance, so it can have the same mean and (approximately) the same variance as the binomial

E1

Answer: $\text{Var} \approx \text{mean}$

3 marks

14 (5 marks)

$X \sim B(250, 0.008)$, $np = 2$, so $X \approx \text{Po}(2)$

B1

$P(X \geq 4) = 1 - P(X \leq 3)$

M1

$P(X \leq 3) \approx 0.8571$

A1

$P(X \geq 4) \approx 0.1429$

A1

Exact binomial value 0.1421, so the approximation is close

A1

Answer: 0.1429

5 marks

15 (5 marks)

$X + Y \sim \text{Po}(18)$

M1 A1

Approximate by $N(18, 18)$

A1

$P(X + Y < 14) = P(X + Y \leq 13) \approx P(W < 13.5)$

M1

$= P(Z < \frac{13.5 - 18}{\sqrt{18}}) = \Phi(-1.0607) = 0.1444$

A1

Answer: 0.1444

5 marks

16 (5 marks)

$X \sim B(210, 0.95)$: mean 199.5, variance $210 \times 0.95 \times 0.05 = 9.975$

B1 B1

$X \approx N(199.5, 9.975)$

M1

$P(X > 200) = P(X \geq 201) \approx P(Y > 200.5) = P(Z > \frac{1}{3.158})$

M1

$= 0.3758$

A1

Answer: 0.3758

5 marks

17 (5 marks)

Exact $P(X \leq 7) = 0.1316$

B1

$X \approx N(10, 5)$

M1

$P(Y < 7.5) = \Phi\left(\frac{-2.5}{\sqrt{5}}\right) = \Phi(-1.118)$

M1

$= 0.1318$

A1

Percentage error $\frac{|0.1318 - 0.1316|}{0.1316} \times 100 \approx 0.14\%$

A1

Answer: 0.1316 exact, 0.1318 approx, error 0.14%

5 marks

18 (5 marks)

$X \sim B(225, 0.4) \approx N(90, 54)$

B1 B1

$P(X > a) = P(X \geq a + 1) \approx P(Y > a + 0.5) > \frac{1}{6}$

M1

$\frac{a + 0.5 - 90}{\sqrt{54}} < 0.9674 \text{ (the value with } P(Z > z) = \frac{1}{6})$

M1

$a + 0.5 < 90 + 0.9674 \times 7.348 = 97.11$, so the largest integer a is 96

A1

Answer: 96

5 marks

19 (5 marks)

Exact: $\frac{{}^{300}C_3 \times {}^{700}C_7}{{}^{1000}C_{10}}$

M1 A1

$= 0.2682$

A1

Binomial: ${}^{10}C_3(0.3)^3(0.7)^7 = 0.2668$

M1

The two agree closely because the sample is a small fraction (1%) of the population

A1

Answer: 0.2682 vs 0.2668

5 marks

20 (5 marks)

$X \approx N(80, 48)$, standard deviation $\sqrt{48} = 6.928$

B1 B1

$P(X \geq k) \approx P(Y > k - 0.5) < 0.05$

M1

$\frac{k - 0.5 - 80}{6.928} > 1.645$

M1

$k > 91.9$, so $k = 92$

A1

Answer: 92

5 marks

21 (5 marks)

Exact $P(X \leq 8) = 0.3328$

B1

$X \approx N(10, 10)$; $P(Y < 8.5) = \Phi(-0.474)$

M1

$= 0.3176$

A1

A Poisson distribution with a small mean is skewed (not symmetric), unlike the normal

E1

As the mean increases the Poisson becomes more symmetric and the error falls

E1

Answer: 0.3328 exact; 0.3176 approx

5 marks

22 (6 marks)

$P(X = x) = \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$

M1

$= \frac{\lambda^x}{x!} \times \frac{n(n-1)\dots(n-x+1)}{n^x} \times \left(1 - \frac{\lambda}{n}\right)^n \times \left(1 - \frac{\lambda}{n}\right)^{-x}$

M1 A1

$\frac{n(n-1)\dots(n-x+1)}{n^x} \rightarrow 1$

B1

$\left(1 - \frac{\lambda}{n}\right)^n \rightarrow e^{-\lambda}$ and $\left(1 - \frac{\lambda}{n}\right)^{-x} \rightarrow 1$

M1

So $P(X = x) \rightarrow \frac{\lambda^x e^{-\lambda}}{x!}$

A1

Answer: $P(X = x) \rightarrow \frac{e^{-\lambda} \lambda^x}{x!}$

6 marks

23 (5 marks)

Jack has omitted the continuity correction

E1

$$P(X \geq 55) \approx P(Y > 54.5)$$

M1

$$= P(Z > \frac{4.5}{5}) = P(Z > 0.9)$$

M1 A1

$$= 0.1841 \text{ (exact value 0.1841)}$$

A1

Answer: 0.1841

5 marks

24 (7 marks)

$$\text{Number of heads } X \approx N(\frac{n}{2}, \frac{n}{4})$$

B1

$$0.45n < X < 0.55n: |X - \frac{n}{2}| < 0.05n$$

M1

$$P(|Z| < \frac{0.05n}{\frac{\sqrt{n}}{2}}) = P(|Z| < 0.1\sqrt{n}) \geq 0.95$$

M1 A1

$$0.1\sqrt{n} \geq 1.96$$

M1

$$\sqrt{n} \geq 19.6, n \geq 384.16$$

A1

$$n = 385$$

A1

Answer: 385

7 marks

25 (6 marks)

$$X \approx N(50, 25) \text{ and } Y \approx N(30, 30)$$

M1 A1

$$X + Y \approx N(80, 55)$$

M1 A1

$$P(X + Y > 85) = P(X + Y \geq 86) \approx P(W > 85.5)$$

M1

$$= P(Z > \frac{5.5}{\sqrt{55}}) = 0.2292$$

A1

Answer: 0.2292

6 marks

26 (6 marks)

$$(a) X \sim B(100, 0.6) \approx N(60, 24)$$

B1

$$\text{Label A when } X > 70, \text{ i.e. } X \geq 71: P(Y > 70.5)$$

M1

$$= P(Z > \frac{10.5}{\sqrt{24}}) = P(Z > 2.143) = 0.0160$$

A1

Answer: 0.0160

3 marks

$$(b) X \sim B(100, 0.8) \approx N(80, 16)$$

B1

$$\text{Label B when } X \leq 70: P(Y < 70.5)$$

M1

$$= P(Z < \frac{-9.5}{4}) = P(Z < -2.375) = 0.0088$$

A1

Answer: 0.0088

3 marks

27 (8 marks)

$$\text{PGF of } B(n, p) \text{ is } (q + pt)^n$$

B1

$$\text{With } p = \frac{\lambda}{n}: q + pt = 1 - \frac{\lambda}{n} + \frac{\lambda t}{n} = 1 + \frac{\lambda(t-1)}{n}$$

M1 A1

$$G_n(t) = (1 + \frac{\lambda(t-1)}{n})^n$$

A1

$$\text{Use the standard limit } (1 + \frac{x}{n})^n \rightarrow e^x \text{ with } x = \lambda(t-1)$$

M1 A1

$$\text{The limit } e^{\lambda(t-1)} \text{ is the PGF of } \text{Po}(\lambda)$$

A1

$$\text{So } B(n, \lambda/n) \text{ tends to } \text{Po}(\lambda) \text{ as } n \rightarrow \infty$$

E1

Answer: Tends to Po(λ)

8 marks

28 (8 marks)

$$(a) \text{ Sampling is without replacement from a finite population with } K = 60, N = 2000, n = 100$$

B1

$$\text{The sample is only 5\% of the lot, so removing items hardly changes the proportion defective}$$

E1

Answer: Hypergeometric; binomial reasonable

2 marks

$$(b) B(100, 0.03) \approx \text{Po}(3)$$

M1

$$P(X \geq 5) = 1 - P(X \leq 4)$$

M1

$$= 0.1847$$

A1

Answer: 0.1847

3 marks

(c) Binomial error: $\frac{|0.1821 - 0.1776|}{0.1776} \times 100 = 2.5\%$

M1 A1

Poisson error: 4.0%; each approximation step adds some error, with Poisson the least accurate

A1

Answer: 2.5% and 4.0%

3 marks

29 (8 marks)

Let $X \approx N(\mu, \sigma^2)$. $P(X \leq 44) \approx P(Y < 44.5) = 0.1357$ and $P(X < 57) = P(X \leq 56) \approx P(Y < 56.5) = 0.9032$

M1 A1

$$\frac{44.5 - \mu}{\sigma} = -1.10 \text{ and } \frac{56.5 - \mu}{\sigma} = 1.30$$

M1 A1

Subtract: $\frac{12}{\sigma} = 2.4$, so $\sigma = 5$

M1

$$\mu = 44.5 + 1.1 \times 5 = 50$$

A1

$$np = 50 \text{ and } np(1 - p) = 25, \text{ so } 1 - p = 0.5, p = 0.5$$

M1

$$n = 100$$

A1

Answer: n = 100, p = 0.5

8 marks

30 (9 marks)

(a) $E(T) = 40 \times 3.5 = 140$

B1

$$\text{Var}(T) = 40 \times \frac{35}{12} = \frac{350}{3} = 116.67$$

M1

T is a sum of 40 independent identically distributed variables, so it is approximately normal

E1

Answer: 140 and 116.67

3 marks

(b) $T \approx N(140, 116.67)$

B1

$$P(T \geq 151) \approx P(Y > 150.5) = P(Z > \frac{10.5}{10.80})$$

M1

$$= 0.1655$$

A1

Answer: 0.1655

3 marks

(c) $P(129.5 < Y < 150.5)$

M1

$$= \Phi(0.972) - \Phi(-0.972)$$

M1

$$= 0.6690$$

A1

Answer: 0.6690

3 marks