

Binomial Distribution

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| 1 | A fair coin is tossed 12 times and X is the number of heads.
(a) State the distribution of X .
(b) Find $P(X = 7)$. | [1]
[2] |
| 2 | $X \sim B(8, 0.3)$. Find $P(X = 2)$. | [2] |
| 3 | $X \sim B(10, 0.5)$. Write down $E(X)$ and $\text{Var}(X)$. | [2] |
| 4 | $X \sim B(20, 0.4)$. Use your calculator to find $P(X \leq 6)$. | [1] |
| 5 | $X \sim B(6, \frac{1}{3})$. Find $P(X = 0)$, giving your answer as a fraction. | [2] |
| 6 | $X \sim B(40, 0.15)$. Find the mean and standard deviation of X , to 3 significant figures where appropriate. | [2] |
| 7 | A multiple-choice test has 10 questions, each with four options of which exactly one is correct. A student guesses every answer at random. Find the probability that the student gets at least 3 questions correct. | [3] |
| 8 | The random variable X has a binomial distribution with mean 6 and variance 2.4. Find the parameters n and p . | [3] |
| 9 | $X \sim B(12, 0.25)$. Find $P(3 \leq X < 6)$. | [3] |
| 10 | 5% of the bulbs made by a machine are defective. A box contains 20 bulbs, chosen independently. Find the probability that the box contains
(a) no defective bulbs,
(b) at most two defective bulbs.
(c) more than one defective bulb. | [1]
[2]
[2] |
| 11 | Two fair dice are rolled together 4 times. Let X be the number of times that the two dice show the same score. Find the probability distribution of X , giving each probability as a fraction with denominator 1296. | [4] |
| 12 | $X \sim B(15, 0.4)$. By comparing $P(X = r)$ with $P(X = r + 1)$, or otherwise, find the most likely value of X . | [3] |
| 13 | 30 students are chosen from a large school and X is the number who walk to school. State two assumptions needed for X to have a binomial distribution, and give a reason why one of them might not be true. | [3] |
| 14 | Each time Mia shoots at a target she hits it with probability 0.1, independently. Find the smallest number of shots n so that the probability of at least one hit is greater than 0.99. | [4] |
| 15 | $X \sim B(10, 0.3)$. Find $P(X = 2 \mid X \geq 1)$. | [4] |
| 16 | $X \sim B(8, p)$. It is given that $P(X = 2) = 3P(X = 1)$. Find the value of p . | [5] |
| 17 | $X \sim B(12, 0.2)$ and $Y \sim B(8, 0.2)$ are independent. Find $P(X + Y = 4)$, giving the distribution of $X + Y$ and your reasoning. | [4] |

- 18** $X \sim B(n, p)$ with $q = 1 - p$.
- (a) Show that $\frac{P(X = r + 1)}{P(X = r)} = \frac{(n - r)p}{(r + 1)q}$. [3]
- (b) Hence find the mode of $B(20, 0.35)$. [2]
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- 19** The discrete random variable X has a binomial distribution $B(15, p)$ and $\text{Var}(X) = 3.15$. The random variable Y has a binomial distribution $B(n, 0.3)$ and its mean is three times its standard deviation.
- (a) Find the two possible values of p . [3]
- (b) Find the value of n . [3]
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- 20** An examiner marks 25 scripts. Each script independently contains a marking error with probability 0.08. Each error costs the examiner £15 to correct. Find the expected total cost and its standard deviation. [5]
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- 21** A biased coin shows heads with probability $p = 0.3$ and is tossed 10 times. By considering the expansions of $(q + p)^{10}$ and $(q - p)^{10}$, find the probability that the number of heads is even. [6]
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- 22** $X \sim B(10, 0.4)$ and $Y \sim B(10, 0.6)$ are independent. Find $P(X = Y)$, correct to 3 significant figures. [6]
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- 23** $X \sim B(n, p)$. Writing $X = I_1 + I_2 + \dots + I_n$, where $I_j = 1$ if the j -th trial is a success and 0 otherwise, and using $E(I_j)$ and $\text{Var}(I_j)$, prove that $E(X) = np$ and $\text{Var}(X) = np(1 - p)$. [5]
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- 24** A batch is accepted if a random sample of 25 items contains at most 2 defective items. Each item is defective independently with probability p .
- (a) Find the probability that the batch is accepted when $p = 0.04$. [2]
- (b) Find the probability that the batch is accepted when $p = 0.10$. [1]
- (c) Batches are accepted independently. Out of 6 batches, each with $p = 0.04$, find the probability that at least 5 are accepted. [4]
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- 25** Eva calculates $P(X \geq 3)$ for $X \sim B(12, 0.2)$ as $1 - P(X \leq 3) = 1 - 0.7946 = 0.2054$. Explain her error and find the correct value. [5]
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- 26** A doctor can see at most 20 patients in a day. Each patient with an appointment turns up independently with probability 0.8.
- (a) The doctor books 22 appointments. Find the probability that more than 20 patients turn up. [2]
- (b) The doctor wants the probability in part (a) to be less than 0.1. Find the greatest number of appointments that can be booked. [4]
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- 27** $X \sim B(n, p)$ with $0 < p < 1$, and k is an integer with $0 < k < n$. By considering the derivative of $P(X = k)$ with respect to p , show that $P(X = k)$ is greatest when $p = \frac{k}{n}$. [8]
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- 28** $X \sim B(2n, \frac{1}{2})$ and the central probability is $c_n = P(X = n)$.
- Show that $\frac{c_{n+1}}{c_n} = \frac{2n + 1}{2n + 2}$, and hence find c_4 to 4 decimal places given that $c_1 = 0.5$. [8]
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- 29** Anil tosses a fair coin 5 times and Beth tosses a fair coin 7 times. Let X and Y be the numbers of heads obtained by Anil and Beth respectively. Find $P(X > Y)$, giving your answer as a fraction in its lowest terms. [9]
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- 30** A bag contains two coins. One is fair; the other is biased and shows heads with probability 0.8. A coin is chosen at random and tossed 10 times, giving exactly 7 heads.
- (a) Show that the probability that the fair coin was chosen is approximately 0.368. [5]
- (b) The same coin is tossed 5 more times. Find the probability that all five are heads. [4]

Poisson Distribution

- 1 $X \sim \text{Po}(2)$. Find $P(X = 3)$. [2]
- 2 $X \sim \text{Po}(4.5)$. Use your calculator to find $P(X \leq 2)$. [1]
- 3 $X \sim \text{Po}(6)$. Write down $E(X)$, $\text{Var}(X)$ and the standard deviation of X , to 3 significant figures. [2]
- 4 $X \sim \text{Po}(3)$. Find the exact value of $P(X = 0)$, and $P(X = 1)$ in terms of e . [2]
- 5 $X \sim \text{Po}(0.8)$. Find $P(X \geq 1)$. [2]
- 6 A helpline receives calls at an average rate of 5 per hour. State the distribution of the number of calls received in a 30 minute period, and find $P(\text{no calls in 30 minutes})$. [2]
- 7 The number of typing errors on a page of a book has a Poisson distribution with mean 0.8, independently from page to page.
 - (a) Find the probability that a page has no errors. [1]
 - (b) Find the probability that a chapter of 5 pages has at most 3 errors. [3]
- 8 $X \sim \text{Po}(\lambda)$ and $P(X = 0) = 0.2$. Find the value of λ , correct to 3 significant figures. [3]
- 9 $X \sim \text{Po}(\lambda)$, where $\lambda > 0$, and $P(X = 1) = P(X = 2)$. Find λ and hence $P(X = 3)$ to 4 decimal places. [3]
- 10 Cars pass a checkpoint at an average rate of 3.2 per minute. Find the probability that more than 2 cars pass in a given minute, stating one assumption you have made. [4]
- 11 The number of typing errors on a page of a textbook has a Poisson distribution with mean 0.7, independently from page to page. The book is proof-read page by page, starting with page 1. Find the probability that page 4 is the first page to contain any errors. [3]
- 12 $X \sim \text{Po}(\lambda)$. Show that $\frac{P(X = r + 1)}{P(X = r)} = \frac{\lambda}{r + 1}$. Hence find the mode of X when $\lambda = 5.5$. [3]
- 13 The number of customers entering a shop in a minute is modelled as a Poisson variable. Give two conditions needed for this to be a good model, and explain why the model may not be appropriate for a shop at lunchtime. [3]
- 14 $X \sim \text{Po}(3)$. Find the smallest integer k such that $P(X > k) < 0.01$. [4]
- 15 $X \sim \text{Po}(2)$. Given that $X \geq 3$, show that $P(X = 5 | X \geq 3) = \frac{4}{15(e^2 - 5)}$, and evaluate this probability to 3 significant figures. [5]
- 16 $X \sim \text{Po}(2)$ and $Y \sim \text{Po}(3)$ are independent. Find $P(X = 1 | X + Y = 4)$, correct to 3 significant figures. [5]
- 17 Faults occur in a cable at random, at an average rate of 0.4 per 10 m. Find the probability that a 25 m length contains at least 2 faults. [4]
- 18 The number of houses sold by an estate agent follows a Poisson distribution with mean 2.5 per week.
 - (a) Find the probability that more than 6 houses are sold in a period of four weeks. [2]
 - (b) The agent looks at twelve consecutive four-week periods. Find the probability that more than 6 houses are sold in exactly 7 of these periods. [3]

- 19** $X \sim \text{Po}(\lambda)$. Prove that $E(X) = \lambda$. [5]
- 20** Patients arrive at a clinic at random at an average rate of 6 per hour.
 (a) Find the probability that more than 8 patients arrive in a given hour. [2]
 (b) Find the probability that at least 3 patients arrive in each of two consecutive half-hour periods. [3]
- 21** $X \sim \text{Po}(\lambda)$. By first finding $E[X(X - 1)]$, prove that $\text{Var}(X) = \lambda$. [6]
- 22** $X \sim \text{Po}(3)$. Aisha says, " $X_1 + X_2$ has mean 6 and variance 6, so it is $\text{Po}(6)$. Also $2X$ has mean 6 and variance 6, so $2X$ is $\text{Po}(6)$ as well, and they have the same distribution." Show that Aisha is wrong about $2X$, where X_1 and X_2 are independent copies of X . [5]
- 23** $X \sim \text{Po}(2)$. Show that $E(X | X \geq 1) = \frac{2}{1 - e^{-2}}$ and evaluate it to 3 significant figures. Explain why this is larger than $E(X)$. [6]
- 24** The number of cars N arriving at a toll booth in a minute is $\text{Po}(\lambda)$. Each car independently has a foreign registration with probability p . Let Y be the number of cars with foreign registration in that minute. Show that $Y \sim \text{Po}(\lambda p)$. [7]
- 25** $X \sim \text{Po}(\lambda)$ where λ is a positive integer. Show that X has exactly two modes, and state them. Use this to find the modes of $\text{Po}(4)$. [5]
- 26** Faults in a roll of fabric occur at random at average rate 2 per roll for machine A and 5 per roll for machine B. A roll is chosen at random from the output of the two machines (equally likely to be from either) and contains exactly 4 faults. Find the probability that it came from machine A. [6]
- 27** $X \sim \text{Po}(\lambda)$.
 (a) By considering $e^\lambda + e^{-\lambda}$, show that $P(X \text{ is even}) = \frac{1 + e^{-2\lambda}}{2}$. [4]
 (b) Find $P(X \text{ is even})$ when $\lambda = 1.5$. [1]
 (c) Show that $P(X \text{ is even}) > \frac{1}{2}$ for all $\lambda > 0$, and state what happens to this probability as λ becomes large. [3]
- 28** $X \sim \text{Po}(2)$ and $Y \sim \text{Po}(1)$ are independent. Find $P(X > Y)$, correct to 3 significant figures. [9]
- 29** Buses arrive at a stop according to a Poisson process with an average of 0.5 buses per minute. Let T be the time in minutes until the next bus arrives.
 (a) Show that $P(T > t) = e^{-0.5t}$. [2]
 (b) Find the median waiting time. [3]
 (c) Find the probability that the wait is between 1 minute and 3 minutes. [3]
- 30** The number N of calls per minute received by a switchboard is $\text{Po}(2)$ on half of all minutes and $\text{Po}(4)$ on the other half of all minutes (which kind of minute it is does not depend on N).
 (a) Find $E(N)$ and $\text{Var}(N)$. [5]
 (b) Explain why N does not have a Poisson distribution. [1]
 (c) Find $P(N = 0)$. [3]

Geometric Distribution

- 1** $X \sim \text{Geo}(0.3)$. Find $P(X = 4)$. [2]

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| 2 | $X \sim \text{Geo}(0.25)$. Write down $E(X)$ and find $\text{Var}(X)$. | [2] |
| 3 | $X \sim \text{Geo}(0.2)$. Find $P(X > 3)$. | [2] |
| 4 | A fair die is rolled repeatedly until a six is obtained. Find the probability that the first six occurs on the fourth roll, giving your answer as a fraction. | [2] |
| 5 | The random variable X has a geometric distribution with $p = \frac{1}{8}$. Write down $E(X)$. | [1] |
| 6 | $X \sim \text{Geo}(0.5)$. Find $P(X \leq 3)$. | [2] |
| 7 | $X \sim \text{Geo}(0.15)$. Find $P(4 \leq X < 10)$. | [3] |
| 8 | A basketball player scores with each shot with probability 0.6, independently. Find the probability that
(a) the first score occurs on the third shot,
(b) the player needs at most 2 shots to score for the first time. | [1]
[2] |
| 9 | X has a geometric distribution with mean 5. Find p and $P(X = 3)$. | [3] |
| 10 | $X \sim \text{Geo}(p)$, where $q = 1 - p$. Show that $P(X > r) = q^r$. | [3] |
| 11 | The random variable X has a geometric distribution with mean 8. Find $\text{Var}(X)$. | [3] |
| 12 | A machine produces items and each is independently defective with probability 0.04. Items are inspected one by one until a defective item is found. Find the expected number of items inspected, and the probability that more than 30 items must be inspected. | [4] |
| 13 | A learner driver passes each attempt at a driving test with probability 0.45. Let X be the number of attempts up to and including the first pass.
State two assumptions required for X to be modelled by a geometric distribution, and say why one of them may be unrealistic. | [3] |
| 14 | $X \sim \text{Geo}(0.15)$. Show that $P(X > 7 X > 4) = P(X > 3)$, and find its value to 3 significant figures. | [4] |
| 15 | $X \sim \text{Geo}(0.2)$. Find the smallest n such that $P(X \leq n) > 0.95$. | [4] |
| 16 | $X \sim \text{Geo}(0.1)$. Find the median of X , that is the smallest m for which $P(X \leq m) \geq 0.5$. | [4] |
| 17 | Amir and Beth take turns shooting at a target, Amir first. Amir hits with probability 0.3 and Beth hits with probability 0.5 on each shot, independently. The first person to hit the target wins. Find the probability that Amir wins. | [5] |
| 18 | A game player pays £4 each time they play. Each play wins a prize independently with probability 0.2. The player keeps playing until the first win. Find the expected total amount paid, and the probability that the player pays more than £40 in total before the first win. | [5] |
| 19 | X and Y are independent and each has the distribution $\text{Geo}(0.25)$. Find $P(X + Y = 4)$. | [5] |
| 20 | $X \sim \text{Geo}(p)$. Given that $X > 2$, show that $E(X X > 2) = 2 + \frac{1}{p}$, and find its value when $p = 0.4$. | [5] |
| 21 | $X \sim \text{Geo}(p)$, where $q = 1 - p$. Starting from $E(X) = \sum_{r=1}^{\infty} r q^{r-1} p$, prove that $E(X) = \frac{1}{p}$. | [6] |

- 22** $X \sim \text{Geo}(p)$. Prove the memoryless property: $P(X > m + n \mid X > m) = P(X > n)$ for positive integers m and n . [4]
- 23** $X \sim \text{Geo}(p)$, $q = 1 - p$. By finding $E[X(X - 1)]$ first, show that $\text{Var}(X) = \frac{q}{p^2}$. [6]
- 24** Sofia says, "For $X \sim \text{Geo}(0.2)$, $P(X \geq 4) = 1 - P(X \leq 4) = 1 - (1 - 0.8^4) = 0.8^4 = 0.4096$." Identify her error and find the correct value of $P(X \geq 4)$. [5]
- 25** $X \sim \text{Geo}(p)$, $q = 1 - p$. Show that the probability that X is even is $\frac{q}{1 + q}$. A fair die is rolled until a six appears; find the probability that the number of rolls needed is even. [6]
- 26** Maya and Noor take turns to shoot at a target, Maya first. Maya hits the target with probability 0.25 on each shot and Noor hits it with probability p on each shot, independently. The first to hit the target wins. The probability that Maya wins is $\frac{5}{11}$.
 (a) Show that Maya's probability of winning is $\frac{0.25}{1 - 0.75(1 - p)}$. [4]
 (b) Hence find the value of p . [2]
- 27** $X \sim \text{Geo}(p)$ and $Y \sim \text{Geo}(r)$ are independent, and $M = \min(X, Y)$.
 (a) Show that $P(M > m) = [(1 - p)(1 - r)]^m$, and hence that M has a geometric distribution, stating its parameter. [4]
 (b) Two players each toss a fair coin once in every round, independently. Find the expected number of rounds until at least one of them has tossed a head. [4]
- 28** A casino game: a biased coin shows heads with probability 0.6. The coin is tossed until the first head, which occurs on toss X , and the player then wins $\text{£}2^X$.
 (a) Show that the expected winnings are finite for this coin and find them, by evaluating $E(2^X)$. [5]
 (b) Explain what happens to $E(2^X)$ if the coin is fair, and why. [2]
- 29** $X \sim \text{Geo}(0.3)$ and $Y \sim \text{Geo}(0.4)$ are independent. Find $P(X > Y)$, giving your answer as a fraction in its lowest terms. [9]
- 30** X and Y are independent random variables, each with the distribution $\text{Geo}(p)$.
 (a) Show that $P(X = k \mid X + Y = n) = \frac{1}{n - 1}$ for $k = 1, 2, \dots, n - 1$. [6]
 (b) Hence, given that $X + Y = 6$, find $P(X \geq 4)$. [3]

Negative Binomial Distribution

- 1** $X \sim \text{NB}(2, 0.5)$. Find $P(X = 4)$. [2]
- 2** $X \sim \text{NB}(3, 0.25)$. Find $E(X)$ and $\text{Var}(X)$. [2]
- 3** $X \sim \text{NB}(4, p)$. Write down the smallest value that X can take and the expression for $P(X = 4)$ in terms of p . [2]
- 4** $X \sim \text{NB}(3, 0.4)$. Find $P(X = 5)$. [2]
- 5** Show that $\text{NB}(1, p)$ is the same as the geometric distribution $\text{Geo}(p)$, and find $P(X = 3)$ when $p = 0.2$. [2]

- 6** A fair coin is tossed until 3 heads have been obtained. Write down the expected number of tosses. [2]
- 7** A fair die is rolled until two sixes have been obtained. Find the probability that the second six is obtained on the fifth roll. [3]
- 8** The random variable X has a negative binomial distribution $NB(r, p)$ with mean 12 and variance 36. Find r and p . [3]
- 9** $X \sim NB(4, p)$ and $P(X = 4) = 0.0256$. Find the value of p and hence find $P(X = 6)$. [3]
- 10** For $X \sim NB(r, p)$, explain why $P(X = x) = {}^{x-1}C_{r-1}p^r(1-p)^{x-r}$ for $x = r, r+1, \dots$ [3]
- 11** A fair coin is tossed repeatedly until 3 heads have appeared. Find the probability that more than 5 tosses are needed. [4]
- 12** Each item inspected on a production line is independently faulty with probability 0.1. Inspection costs £2 per item and continues until 5 faulty items have been found. Find the expected total cost. [3]
- 13** Candidates for a post are interviewed one at a time until 2 suitable candidates have been found. Each candidate is suitable independently with probability 0.3. Find the probability that exactly 6 candidates are interviewed. [3]
- 14** $X \sim NB(3, 0.5)$. Find $P(X \leq 6)$ by considering the number of heads in 6 tosses of a fair coin. [4]
- 15** $X \sim NB(3, 0.4)$, the number of trials needed for the third success. Given that the first trial is a success, find the probability that $X = 5$. [5]
- 16** X and Y are independent and each is $\text{Geo}(0.3)$, and $T = X + Y$. By considering all the ways in which $T = 4$ can occur, find $P(T = 4)$ and verify that it equals $P(Z = 4)$ for $Z \sim NB(2, 0.3)$. [5]
- 17** $X \sim NB(4, 0.4)$ is the number of trials needed for 4 successes. Let $Y = X - 4$ be the number of failures. Find $E(Y)$ and $\text{Var}(Y)$. [4]
- 18** $X \sim NB(2, 0.4)$ and $Y \sim NB(3, 0.4)$ are independent. By considering the trials in which successes occur, explain why $X + Y \sim NB(5, 0.4)$, and find $P(X + Y = 7)$. [5]
- 19** $X \sim NB(3, 0.5)$. Find the smallest n such that $P(X \leq n) \geq 0.9$. [5]
- 20** $X \sim NB(3, 0.3)$. Show that $\frac{P(X = x+1)}{P(X = x)} = \frac{0.7x}{x-2}$, and hence find the mode of X . [5]
- 21** Let $X_1, \dots, X_r$ be independent random variables, each $\sim \text{Geo}(p)$. The random variable $X = X_1 + \dots + X_r$ has a negative binomial distribution $NB(r, p)$. Use this to derive $E(X)$ and $\text{Var}(X)$. [4]
- 22** Use the series $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ ($|q| < 1$) to prove that the probabilities of $NB(2, p)$ sum to 1. [5]
- 23** Two players play a match in which the first to win 3 games wins the match (so at most 5 games are played). Aisha wins each game independently with probability 0.6. Find the probability that Aisha wins the match. [6]
- 24** Dan says, " $X \sim NB(2, 0.3)$ is the number of trials to get two successes, so $P(X = 5) = {}^5C_2(0.3)^2(0.7)^3$." Explain what is wrong with Dan's calculation, and find the correct value. [5]

- 25** The third success in a sequence of independent trials, each with success probability p , occurs on trial 7. Find the probability that the first trial was a success, giving your answer as a fraction. [6]
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- 26** A company must collect 6 faulty components for testing. Components arrive and are independently faulty with probability 0.15. Handling each component costs £3 and there is a fixed set-up cost of £20. Find the expected total cost and the standard deviation of the total cost. [6]
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- 27** Alan needs 2 more game wins to win a match and Beth needs 3 more. Each game is won independently by Alan or Beth with probability 0.5.
- (a) Explain why Alan wins the match if and only if he wins at least 2 of the next 4 games (even if all 4 are not played). [3]
- (b) Hence find the probability that Alan wins the match. [3]
- (c) Check your answer using the negative binomial distribution $X \sim \text{NB}(2, 0.5)$ for the game on which Alan wins his second game. [3]
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- 28** $X \sim \text{NB}(r, p)$, where $q = 1 - p$.
- (a) Show that the probability generating function of X is $G(t) = \left(\frac{pt}{1 - qt}\right)^r$. [3]
- (b) Use $G(t)$ to show that $E(X) = \frac{r}{p}$. [3]
- (c) For $r = 2$, find $P(X = 5)$ from the expansion of $G(t)$ when $p = 0.5$. [2]
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- 29** $X \sim \text{NB}(4, 0.3)$ is the number of trials needed for 4 successes.
- (a) Explain why, for any n , $P(X \leq n) = P(Y \geq 4)$, where $Y \sim B(n, 0.3)$. [3]
- (b) Hence find $P(X \leq 12)$ using binomial tables or a calculator. [2]
- (c) Find the smallest n such that $P(X \leq n) > 0.95$. [3]
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- 30** $X \sim \text{NB}(2, p)$, where $q = 1 - p$.
- (a) Show that $P(X \text{ is even}) = \frac{1 + q^2}{(1 + q)^2}$. [6]
- (b) Evaluate $P(X \text{ is even})$ when $p = 0.5$, and comment on the answer. [3]

Hypergeometric Distribution

- 1** A bag contains 4 red and 6 blue counters. Three counters are taken at random without replacement. Find the probability that exactly 2 are red. [2]
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- 2** X has a hypergeometric distribution: a sample of 8 items is taken without replacement from 20 items of which 5 are defective, and X is the number of defective items in the sample. Find $E(X)$. [2]
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- 3** A box holds 10 balls of which 6 are red. Seven balls are taken at random without replacement. State the smallest and largest possible numbers of red balls in the sample. [2]
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- 4** A committee of 2 is chosen at random from 5 men and 3 women. Find the probability that both members are women. [2]
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- 5** Five cards are dealt at random from a standard pack of 52 cards. Find the probability that exactly 2 are hearts. [2]
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- 6** A population of N objects contains K of a certain type. A random sample of n objects is taken without replacement and X is the number of this type in the sample. Write down $P(X = x)$. [1]

- 7** In a lottery 6 numbers are drawn at random without replacement from 1, 2, ..., 49. A player has chosen 6 numbers. Find the probability that exactly 3 of the player's numbers are drawn. [3]
- 8** A bag contains 12 beads, of which 5 are blue. Three beads are taken at random without replacement and X is the number of blue beads taken. Find the probability distribution of X . [4]
- 9** X has a hypergeometric distribution: a sample of 5 is taken without replacement from 20 items of which 8 are of a certain type. Using $E(X) = n\frac{K}{N}$ and $\text{Var}(X) = n\frac{K}{N}(1 - \frac{K}{N})\frac{N-n}{N-1}$, find $E(X)$ and $\text{Var}(X)$. [3]
- 10** A box contains 25 items of which 4 are defective. A sample of 5 items is taken at random without replacement. Find the probability that the sample contains at least one defective item. [3]
- 11** A box contains 25 items of which 4 are defective. A sample of 5 is taken without replacement. Find the exact probability of exactly one defective item, and compare it with the value obtained using a binomial model $B(5, \frac{4}{25})$. [4]
- 12** A teacher selects 4 students at random, without replacement, from a class of 12. Explain why the number of girls selected does not have a binomial distribution. [3]
- 13** Find the probability that a 5-card hand dealt from a standard pack contains no aces, as a decimal to 4 significant figures. [3]
- 14** An urn contains N balls, exactly 2 of which are red. Two balls are taken at random without replacement. The probability that both are red is $\frac{1}{45}$. Find N . [4]
- 15** A sample of 3 is taken at random without replacement from 9 items, of which 4 are faulty. X is the number of faulty items in the sample. Find the distribution of X , and $E(X)$ and $\text{Var}(X)$. [5]
- 16** A lake contains 100 fish, of which 20 have been tagged. A sample of 15 fish is caught at random without replacement. Find the probability that exactly 2 tagged fish are caught, to 3 significant figures. [4]
- 17** Three counters are taken at random without replacement from 4 red and 6 blue counters. Given that at least one is red, find the probability that exactly two are red. [5]
- 18** A fair coin is tossed. If it shows heads, 3 balls are taken without replacement from Box A, which has 4 red and 6 white balls. If it shows tails, 3 balls are taken without replacement from Box B, which has 7 red and 3 white balls. Find the probability that exactly 2 of the 3 balls taken are red. [5]
- 19** X is the number of successes when n items are sampled without replacement from N items containing K successes. Prove that, for $n = 2$, $E(X) = \frac{2K}{N}$. [5]
- 20** Five cards are dealt from a standard pack. Given that at least 2 are spades, find the probability that exactly 3 are spades, correct to 3 significant figures. [5]
- 21** For a sample of size 2 taken without replacement from N items containing K of a given type, with X the number of that type in the sample, prove that $\text{Var}(X) = \frac{2K(N-K)(N-2)}{N^2(N-1)}$. [6]

- 22** To estimate the number N of fish in a lake, 20 fish are caught, tagged and returned. Later 30 fish are caught at random (without replacement) and 6 of them are found to be tagged. The value of N that maximises the probability of this outcome is to be found.
- (a) Write down the probability $L(N)$ that exactly 6 of the 30 are tagged. [2]
- (b) Show that $\frac{L(N)}{L(N-1)} = \frac{(N-20)(N-30)}{N(N-44)}$ (for $N > 44$). [3]
- (c) Hence find the value or values of N that maximise $L(N)$. [3]
-
- 23** Maya says, "I take 2 cards at random from a standard pack without replacement. The chance both are aces is $(\frac{4}{52})^2 = 0.0059$." Explain her error and find the correct probability. [5]
-
- 24** From a group of N people, K are women. A committee of n people is chosen at random, and X is the number of women on it. Another way to choose the committee is first to choose, at random, K people to be called "special", and then count how many of the committee are special. Prove that $P(X = x) = \frac{{}^K C_x \times {}^{N-K} C_{n-x}}{{}^N C_n} = \frac{{}^n C_x \times {}^{N-n} C_{K-x}}{{}^N C_K}$, which shows that the hypergeometric distributions with parameters (N, K, n) and (N, n, K) are the same. [5]
-
- 25** A team of 4 is chosen at random from 5 forwards, 4 midfielders and 3 defenders. Find the probability that the team contains exactly 2 forwards, 1 midfielder and 1 defender. [6]
-
- 26** A sample of 10 is taken without replacement from 50 items, of which 20 are defective. Let X be the number of defective items in the sample and let $Y \sim B(10, 0.4)$.
- (a) Find $E(X)$ and $\text{Var}(X)$. [3]
- (b) Compare $\text{Var}(X)$ with $\text{Var}(Y)$, and explain why $\text{Var}(X)$ is smaller. [3]
-
- 27** By equating the coefficients of x^n in $(1+x)^K(1+x)^{N-K} = (1+x)^N$, prove that the probabilities of the hypergeometric distribution $P(X = x) = \frac{{}^K C_x \times {}^{N-K} C_{n-x}}{{}^N C_n}$ sum to 1 over all possible x . [8]
-
- 28** X is the number of defective items in a sample of 5 taken without replacement from N items, of which K are defective. $E(X) = 2$ and $\text{Var}(X) = 1$. Find N and K . [8]
-
- 29** A batch of 50 components contains 5 that are faulty. A random sample of n components is taken without replacement. Find the smallest n for which the probability that the sample contains at least one faulty component is at least 0.9. [8]
-
- 30** X is hypergeometric: n items are sampled without replacement from N items of which K are special, where $K = pN$ for a fixed p with $0 < p < 1$ (N large).
- (a) Show that $P(X = x) = {}^n C_x \times \frac{K(K-1)\dots(K-x+1) \times (N-K)(N-K-1)\dots(N-K-n+x+1)}{N(N-1)\dots(N-n+1)}$. [3]
- (b) Hence show that, as $N \rightarrow \infty$ with n , x and p fixed, $P(X = x) \rightarrow {}^n C_x p^x (1-p)^{n-x}$, so that the hypergeometric distribution tends to $B(n, p)$. [5]
- (c) For $N = 1000$, $K = 300$ and $n = 5$, verify numerically that $P(X = 2)$ is close to the binomial value $B(5, 0.3)$. [2]

Discrete Uniform Distribution

- 1** X is the score on a fair six-sided die. State the distribution of X and find $P(X \geq 5)$. [2]
- 2** X has a discrete uniform distribution on 1, 2, ..., 10. Find $E(X)$ and $\text{Var}(X)$. [2]

- 3** X is chosen at random from 1, 2, 3, 4, 5, all values being equally likely. Find the probability that X is prime. [2]
- 4** A fair spinner has the numbers 1 to 8 on it, each equally likely. Find the probability that it shows a number greater than 5. [1]
- 5** A random digit D is chosen from 0, 1, 2, ..., 9. Write down $P(D = 7)$ and $E(D)$. [2]
- 6** X is the score on a fair die. Use $\text{Var}(X) = \frac{n^2 - 1}{12}$ to find the variance of X. [2]
- 7** X has a discrete uniform distribution on 1, 2, ..., n with $E(X) = 10.5$. Find n and $\text{Var}(X)$. [3]
- 8** X is the score on a fair die and $Y = 2X + 1$. Find $E(Y)$ and $\text{Var}(Y)$. [3]
- 9** X is equally likely to take each of the values 5, 6, 7, ..., 12. Write $X = 4 + W$, where W has a discrete uniform distribution on 1, 2, ..., 8, and hence find $E(X)$ and $\text{Var}(X)$. [3]
- 10** A number is chosen at random from the integers 1 to 100 inclusive. Find the probability that it is
 (a) a multiple of 7, [1]
 (b) a perfect square, [1]
 (c) both a multiple of 7 and a perfect square. [2]
- 11** X has a discrete uniform distribution on 0, 1, 2, ..., 9. Show that $E(X) = 4.5$ and find $\text{Var}(X)$. [3]
- 12** Explain why the score on one fair die has a discrete uniform distribution, but the total score on two fair dice does not. [3]
- 13** A contestant opens one of ten envelopes chosen at random and wins the amount inside. The envelopes contain £10, £20, ..., £100 (one envelope for each amount). Find the contestant's expected winnings and the standard deviation of the winnings, to 3 significant figures. [4]
- 14** X has a discrete uniform distribution on 1, 2, ..., n and $\text{Var}(X) = 33.25$. Find the value of n. [4]
- 15** X is uniform on 1, 2, ..., n and $P(X > 12) = 0.4$. Find n. [4]
- 16** Two fair dice are rolled, with scores X and Y. Find $P(X = Y)$, $P(X < Y)$ and $P(|X - Y| \geq 4)$. [4]
- 17** Two fair four-sided dice, with faces numbered 1 to 4, are rolled and M is the larger of the two scores. Find the distribution of M and $E(M)$. [5]
- 18** X is uniform on 1, 2, ..., 20. Find the median and the lower and upper quartiles of X, defined as the values below which at least 25%, 50% and 75% of the distribution lie, taking the smallest such integer value for the quartiles and the usual convention (the mean of the two middle values) for the median. [4]
- 19** X is uniform on 1, 2, 3, 4, 5. The random variable $Y = aX + b$, where $a > 0$, has mean 30 and variance 8. Find a and b. [5]
- 20** X is uniform on 1, 2, ..., 20. Find $P(X > 12 \mid X \text{ is even})$. [4]
- 21** X has a discrete uniform distribution on 1, 2, ..., n. Using $\sum_{x=1}^n x = \frac{n(n+1)}{2}$ and $\sum_{x=1}^n x^2 = \frac{n(n+1)(2n+1)}{6}$, prove that $E(X) = \frac{n+1}{2}$ and $\text{Var}(X) = \frac{n^2 - 1}{12}$. [6]

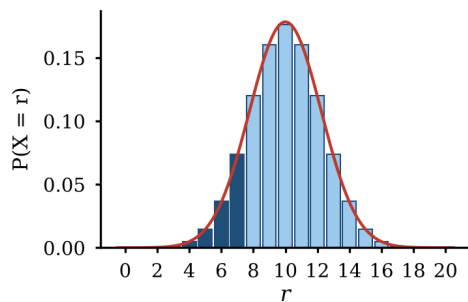
- 22** X and Y are independent and each is uniform on 1, 2, ..., n. Let $T = X + Y$.
- (a) Show that $P(T = n + 1) = \frac{1}{n}$ and that $P(T = t) = \frac{t-1}{n^2}$ for $2 \leq t \leq n + 1$. [3]
- (b) Find $E(T)$ and $\text{Var}(T)$ in terms of n. [3]
-
- 23** Sam says: "X is the score on a fair die, with variance $\frac{35}{12}$. So the variance of $2X$ is $2 \times \frac{35}{12} = \frac{35}{6}$, which is also the variance of the total of two dice."
- Show that Sam has made an error, and find the correct variances of $2X$ and of the total $X_1 + X_2$ of two independent dice. [5]
-
- 24** A three-digit code is formed by choosing each of its three digits independently and uniformly from 0, 1, ..., 9. Find the mean and variance of the sum of the three digits, and the probability that the sum of the three digits is exactly 27 or exactly 0. [6]
-
- 25** Three numbers are chosen independently and uniformly from 1, 2, ..., 10 and M is the largest of the three. Show that $P(M \leq m) = \left(\frac{m}{10}\right)^3$ and find $E(M)$ as a decimal. [6]
-
- 26** An integer is chosen at random from 1, 2, ..., 210. Use the inclusion-exclusion principle to find the probability that it is not divisible by 2, 3, 5 or 7. Give your answer as a fraction in its lowest terms. [6]
-
- 27** Three fair dice are rolled. Find the probability that the total score is 10, and the probability that the total is 10 given that the total is even. [8]
-
- 28** X and Y are independent and each is uniform on 1, 2, ..., n.
- (a) Show that, for $d = 1, 2, \dots, n - 1$, $P(|X - Y| = d) = \frac{2(n-d)}{n^2}$. [3]
- (b) Hence show that $E(|X - Y|) = \frac{n^2 - 1}{3n}$. [4]
- (c) Check the formula for $n = 6$ by direct calculation of the expected absolute difference between the scores on two dice. [2]
-
- 29** X is uniform on 1, 2, ..., n. Given $X = x$, the random variable Y is uniform on 1, 2, ..., x.
- (a) Show that $E(Y) = \frac{n+3}{4}$. [4]
- (b) For $n = 6$, find $P(Y = 1)$ and $P(Y = 6)$. [3]
- (c) For $n = 6$, find $P(X = 6 | Y = 6)$. [2]
-
- 30** Two numbers X_1 and X_2 are chosen at random, without replacement, from 1, 2, ..., 6, all pairs being equally likely.
- (a) Find $E(X_1 + X_2)$ and explain why $P(X_1 = 3) = \frac{1}{6}$. [3]
- (b) Show that $\text{Var}(X_1 + X_2) = \frac{14}{3}$ by listing the possible values of the sum and their probabilities. [6]
- (c) Compare this with the variance $2 \times \frac{35}{12} = \frac{35}{6}$ for two independent throws of a die, and explain the difference. [2]

Approximations Between Distributions

- 1** $X \sim B(100, 0.02)$. State a suitable Poisson approximation for X, and use it to estimate $P(X = 1)$. [2]

- | | | |
|----|--|-----|
| 2 | $X \sim B(60, 0.4)$. Show that a normal approximation is suitable and state its parameters. | [2] |
| 3 | X is a discrete random variable that is approximated by a continuous normal variable Y . Write down, with continuity corrections, the probability for Y that approximates | |
| | (a) $P(X = 20)$, | [1] |
| | (b) $P(X > 20)$, | [1] |
| | (c) $P(X \leq 20)$. | [1] |
| 4 | $X \sim \text{Po}(15)$. State the normal distribution that approximates X . | [1] |
| 5 | $X \sim B(200, 0.01)$. Use a Poisson approximation to estimate $P(X = 0)$. | [2] |
| 6 | State two conditions under which the binomial distribution $B(n, p)$ can be approximated by a Poisson distribution. | [2] |
| 7 | $X \sim B(80, 0.5)$. Use a normal approximation to estimate $P(X \leq 35)$, and compare it with the exact value 0.1572. | [3] |
| 8 | $X \sim \text{Po}(20)$. Use a normal approximation to estimate $P(X > 25)$. | [4] |
| 9 | Flaws occur at random in sheet glass at an average rate of 3 per square metre. Use a normal approximation to estimate the probability that a pane measuring 3.2 m by 2.5 m has at most 20 flaws. | [4] |
| 10 | For each of the following binomial distributions, say whether a Poisson approximation, a normal approximation, or neither is appropriate, giving a reason. | |
| | (a) $B(300, 0.5)$, | [1] |
| | (b) $B(200, 0.01)$, | [1] |
| | (c) $B(12, 0.4)$. | [1] |
| 11 | $X \sim B(50, 0.3)$. Use a normal approximation to estimate $P(12 \leq X \leq 18)$. | [4] |
| 12 | $X \sim \text{Po}(12)$. Use a normal approximation to estimate $P(X = 10)$, and compare it with the exact value 0.1048. | [4] |
| 13 | $X \sim B(n, p)$. Show that the variance npq is approximately equal to the mean np when p is small, and explain why this makes the Poisson approximation reasonable. | [3] |
| 14 | 0.8% of the items made by a machine are defective. A random sample of 250 items is taken. Use a Poisson approximation to estimate the probability that at least 4 items are defective, and compare it with the exact binomial value. | [5] |
| 15 | $X \sim \text{Po}(10)$ and $Y \sim \text{Po}(8)$ are independent. Use a normal approximation to estimate $P(X + Y < 14)$. | [5] |
| 16 | An airline sells 210 tickets for a flight with 200 seats. Each ticket-holder independently turns up with probability 0.95. Use a normal approximation to estimate the probability that more than 200 ticket-holders turn up. | [5] |

- 17** The diagram shows the probability distribution of $X \sim B(20, 0.5)$, with the normal curve used to approximate it. Find $P(X \leq 7)$ exactly and by a normal approximation with a continuity correction, and find the percentage error of the approximation, to 2 decimal places.



[5]

- 18** A waiter receives a tip from each customer independently with probability 0.4. In a week there are 225 customers and X is the number of tips received. Use a normal approximation to find the largest integer a such that $P(X > a) > \frac{1}{6}$.

[5]

- 19** A box of 1000 components contains 300 that are faulty. A sample of 10 components is taken without replacement. Find the exact probability that exactly 3 are faulty and compare with the value from a binomial model $B(10, 0.3)$.

[5]

- 20** $X \sim B(200, 0.4)$. Use a normal approximation to find the smallest integer k such that $P(X \geq k) < 0.05$.

[5]

- 21** $X \sim \text{Po}(10)$. Find $P(X \leq 8)$ exactly, and by a normal approximation with continuity correction. Explain why the normal approximation to a Poisson distribution is less accurate when the mean is small, and say what happens to the error as the mean increases.

[5]

- 22** $X \sim B(n, \frac{\lambda}{n})$, where λ is a constant. Prove that, for fixed x , $P(X = x) \rightarrow \frac{e^{-\lambda} \lambda^x}{x!}$ as $n \rightarrow \infty$.

[6]

- 23** Jack tosses a fair coin 100 times and wants to estimate $P(X \geq 55)$, where X is the number of heads. He says, " $X \approx N(50, 25)$, so $P(X \geq 55) = P(Z \geq 1) = 0.1587$." Explain what Jack has omitted, and find the better estimate.

[5]

- 24** A fair coin is tossed n times. Ignoring continuity corrections, use a normal approximation to find the smallest n for which the probability that the proportion of heads lies between 0.45 and 0.55 is at least 0.95.

[7]

- 25** $X \sim B(100, 0.5)$ and $Y \sim \text{Po}(30)$ are independent. Use normal approximations to estimate $P(X + Y > 85)$.

[6]

- 26** A garden centre sells two types of seed bag. In bags of type A, 80% of the seeds produce red flowers, and in bags of type B, 60% of the seeds produce red flowers. An unlabelled bag is found. A gardener plants 100 seeds from the bag and decides to label the bag A if more than 70 of them produce red flowers, and B otherwise. Use normal approximations to find the probability that the gardener

- (a) labels the bag A when it is in fact type B,
- (b) labels the bag B when it is in fact type A.

[3]

[3]

- 27** $X \sim B(n, \frac{\lambda}{n})$. Show that the probability generating function of X is $G_n(t) = (1 + \frac{\lambda(t-1)}{n})^n$, and hence show that $G_n(t) \rightarrow e^{\lambda(t-1)}$ as $n \rightarrow \infty$. State the conclusion for the distribution of X .

[8]

- 28** A lot of 2000 components contains 60 defective ones. A random sample of 100 components is taken without replacement and X is the number of defective components in it.
- (a) Explain why X is hypergeometric and why a binomial approximation $B(100, 0.03)$ is reasonable. [2]
 - (b) Use a Poisson approximation to the binomial to estimate $P(X \geq 5)$. [3]
 - (c) The exact hypergeometric value of $P(X \geq 5)$ is 0.1776 and the binomial value is 0.1821. Calculate the percentage errors of the two approximations, to 1 decimal place, and comment. [3]
-
- 29** $X \sim B(n, p)$, where n and p are such that X can be approximated by a normal distribution. Using this normal approximation (with continuity corrections), $P(X \leq 44) = 0.1357$ and $P(X < 57) = 0.9032$. Find the values of n and p . [8]
-
- 30** Forty fair dice are rolled and T is the total score.
- (a) Find $E(T)$ and $\text{Var}(T)$, and justify the use of a normal approximation. [3]
 - (b) Estimate $P(T > 150)$. [3]
 - (c) Estimate $P(130 \leq T \leq 150)$. [3]
-