

First-Order Differential Equations

1 (3 marks)

$$y = \int (6x^2 - 4x + 3) \, dx = 2x^3 - 2x^2 + 3x + C$$

M1 A1

$$\text{At } (1, 5): 5 = 2 - 2 + 3 + C, \text{ so } C = 2$$

A1

$$\text{Answer: } y = 2x^3 - 2x^2 + 3x + 2$$

3 marks

2 (2 marks)

$$y = x^2 + 3x^{-1}, \text{ so } \frac{dy}{dx} = 2x - 3x^{-2}$$

M1

$$x \frac{dy}{dx} + y = 2x^2 - 3x^{-1} + x^2 + 3x^{-1} = 3x^2$$

A1

$$\text{Answer: } x \frac{dy}{dx} + y = 3x^2$$

2 marks

3 (3 marks)

$$\int \frac{1}{y} \, dy = \int 5 \, dx, \text{ so } \ln|y| = 5x + c$$

M1 A1

$$y = Ae^{5x}; y = 2 \text{ when } x = 0 \text{ gives } A = 2, \text{ so } y = 2e^{5x}$$

A1

$$\text{Answer: } y = 2e^{5x}$$

3 marks

4 (3 marks)

$$\int \frac{1}{y} \, dy = \int 3x^2 \, dx$$

M1

$$\ln|y| = x^3 + c$$

A1

$$y = Ae^{x^3}$$

A1

$$\text{Answer: } y = Ae^{x^3}$$

3 marks

5 (2 marks)

$$y = \int \cos 2x \, dx$$

M1

$$y = \frac{1}{2} \sin 2x + C$$

A1

$$\text{Answer: } y = \frac{1}{2} \sin 2x + C$$

2 marks

6 (4 marks)

$$\int y \, dy = \int x \, dx$$

M1

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

A1

$$x = 0, y = 3: c = \frac{9}{2}$$

M1

$$y = \sqrt{x^2 + 9}$$

A1

$$\text{Answer: } y = \sqrt{x^2 + 9}$$

4 marks

7 (4 marks)

$$\int \frac{1}{y+1} \, dy = \int \frac{1}{x} \, dx$$

M1

$$\ln|y+1| = \ln x + c$$

A1

$$y+1 = Ax$$

A1

$$x = 2, y = 3: A = 2, \text{ so } y = 2x - 1$$

A1

$$\text{Answer: } y = 2x - 1$$

4 marks

8 (4 marks)

$$(a) \frac{dy}{dx} = 3Ax^2 \quad \text{M1}$$

$$\text{Replace A by } \frac{y}{x^3}: x \frac{dy}{dx} = 3y \quad \text{A1}$$

$$\text{Answer: } x \frac{dy}{dx} = 3y \quad \text{2 marks}$$

$$(b) \frac{dy}{dx} = 2Ae^{2x} = 2(y - 1) \quad \text{M1 A1}$$

$$\text{Answer: } \frac{dy}{dx} = 2(y - 1) \quad \text{2 marks}$$

9 (4 marks)

$$e^y dy = e^x dx \quad \text{M1}$$

$$e^y = e^x + c \quad \text{A1}$$

$$x = 0, y = \ln 2: 2 = 1 + c, \text{ so } c = 1 \quad \text{M1}$$

$$y = \ln(e^x + 1) \quad \text{A1}$$

$$\text{Answer: } y = \ln(e^x + 1) \quad \text{4 marks}$$

10 (4 marks)

$$\int \frac{1}{1+y^2} dy = \int 2x dx \quad \text{M1}$$

$$\tan^{-1} y = x^2 + c \quad \text{A1}$$

$$x = 0, y = 0: c = 0 \quad \text{M1}$$

$$y = \tan(x^2) \quad \text{A1}$$

$$\text{Answer: } y = \tan(x^2) \quad \text{4 marks}$$

11 (3 marks)

$$\int \frac{1}{y} dy = \int \frac{3}{x} dx \quad \text{M1}$$

$$\ln|y| = 3 \ln|x| + c, \text{ so } y = Ax^3 \quad \text{A1}$$

$$(1, 2): A = 2, \text{ so } y = 2x^3 \quad \text{A1}$$

$$\text{Answer: } y = 2x^3 \quad \text{3 marks}$$

12 (6 marks)

$$\frac{1}{y(2-y)} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{2-y} \right) \quad \text{M1 A1}$$

$$\frac{1}{2} (\ln y - \ln(2-y)) = 3x + c \quad \text{M1 A1}$$

$$\frac{y}{2-y} = Ae^{6x}; x = 0, y = 1 \text{ gives } A = 1 \quad \text{M1}$$

$$y = 2e^{6x} - ye^{6x}, \text{ so } y(1 + e^{6x}) = 2e^{6x}, \text{ giving } y = \frac{2}{1 + e^{-6x}} \quad \text{A1}$$

$$\text{Answer: } y = \frac{2}{1 + e^{-6x}} \quad \text{6 marks}$$

13 (5 marks)

$$\int \frac{1}{y} dy = \int \frac{x}{1+x^2} dx \quad \text{M1}$$

$$\ln|y| = \frac{1}{2} \ln(1+x^2) + c \quad \text{M1 A1}$$

$$y = A\sqrt{1+x^2}; x = 0, y = 3 \text{ gives } A = 3 \quad \text{M1}$$

$$y = 3\sqrt{1+x^2} \quad \text{A1}$$

$$\text{Answer: } y = 3\sqrt{1+x^2} \quad \text{5 marks}$$

14 (5 marks)

$$\int y \, dy = \int x e^x \, dx$$

M1

$$\frac{y^2}{2} = x e^x - e^x + c \text{ (integration by parts)}$$

M1 A1

$$x = 0, y = 1: \frac{1}{2} = -1 + c, \text{ so } c = \frac{3}{2}$$

M1

$$y^2 = 2(x - 1)e^x + 3$$

A1

$$\text{Answer: } y^2 = 2(x - 1)e^x + 3$$

5 marks

15 (5 marks)

(a) Gradient of OP = $\frac{y}{x}$, so the gradient of the tangent is $-\frac{x}{y}$

M1 A1

$$\text{Answer: } \frac{dy}{dx} = -\frac{x}{y}$$

2 marks

(b) $\int y \, dy = -\int x \, dx$, so $\frac{y^2}{2} = -\frac{x^2}{2} + c$

M1 A1

(3, 4): $c = \frac{25}{2}$, so $x^2 + y^2 = 25$

A1

$$\text{Answer: } x^2 + y^2 = 25$$

3 marks

16 (5 marks)

(a) At (1, 1): $1 + 1 = 2$; at (-2, 1): $-2 + 1 = -1$

B1 B1

$$\text{Answer: } 2 \text{ and } -1$$

2 marks

(b) $y = -x - 1$ gives $\frac{dy}{dx} = -1$, and $x + y = x - x - 1 = -1$

B1

$$\text{Answer: } \frac{dy}{dx} = -1 = x + y$$

1 mark

(c) Only one solution curve passes through each point, since the gradient there is fixed by the equation

E1

so a crossing would give two different solutions through one point, which is impossible

E1

Answer: Only one solution curve through each point

2 marks

17 (4 marks)

$$\int A^{-1/2} \, dA = \int 2 \, dt$$

M1

$$2\sqrt{A} = 2t + c$$

A1

$$t = 0, A = 4: c = 4$$

M1

$$\sqrt{A} = t + 2, \text{ so } A = (t + 2)^2$$

A1

$$\text{Answer: } A = (t + 2)^2$$

4 marks

18 (5 marks)

$$\int (y^2 + 1) \, dy = \int (2x + 3) \, dx$$

M1

$$\frac{y^3}{3} + y = x^2 + 3x + c$$

A1 A1

$$x = 0, y = 1: c = \frac{4}{3}$$

M1

$$\frac{y^3}{3} + y = x^2 + 3x + \frac{4}{3}$$

A1

$$\text{Answer: } \frac{y^3}{3} + y = x^2 + 3x + \frac{4}{3}$$

5 marks

19 (4 marks)

$$\int \cos^2 y \, dy = \int \sin^2 x \, dx$$

M1

$$\cos^2 y = \frac{1}{2}(1 + \cos 2y) \text{ and } \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

M1

$$\frac{1}{2}y + \frac{1}{4} \sin 2y = \frac{1}{2}x - \frac{1}{4} \sin 2x + c$$

A1

$$x = 0, y = 0 \text{ gives } c = 0, \text{ so } 2y + \sin 2y = 2x - \sin 2x$$

A1

$$\text{Answer: } 2y + \sin 2y = 2x - \sin 2x$$

4 marks

20 (7 marks)

$$(a) \frac{du}{dx} = 2 + \frac{dy}{dx} \quad \text{M1}$$

$$= 2 + (u^2 - 2) = u^2 \quad \text{A1}$$

$$\text{Answer: } \frac{du}{dx} = u^2 \quad \text{2 marks}$$

$$(b) \int u^{-2} du = \int 1 dx, \text{ so } -\frac{1}{u} = x + c \quad \text{M1 A1}$$

$$x = 0: u = 1, \text{ so } c = -1 \quad \text{M1}$$

$$u = \frac{1}{1-x}, \text{ so } y = \frac{1}{1-x} - 2x \quad \text{M1 A1}$$

$$\text{Answer: } y = \frac{1}{1-x} - 2x \quad \text{5 marks}$$

21 (7 marks)

$$(a) \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \text{M1}$$

$$v + x \frac{dv}{dx} = 1 + v + v^2 \quad \text{M1}$$

$$x \frac{dv}{dx} = 1 + v^2 \quad \text{A1}$$

$$\text{Answer: } x \frac{dv}{dx} = 1 + v^2 \quad \text{3 marks}$$

$$(b) \int \frac{1}{1+v^2} dv = \int \frac{1}{x} dx, \text{ so } \tan^{-1} v = \ln x + c \quad \text{M1 A1}$$

$$x = 1, v = \sqrt{3}: c = \frac{\pi}{3} \quad \text{M1}$$

$$v = \tan(\ln x + \frac{\pi}{3}), \text{ so } y = x \tan(\ln x + \frac{\pi}{3}) \quad \text{A1}$$

$$\text{Answer: } y = x \tan(\ln x + \frac{\pi}{3}) \quad \text{4 marks}$$

22 (5 marks)

$$(a) \text{ The error is in line 3: when exponentiating, the constant should multiply, not add } \quad \text{E1}$$

$$e^{2x+2c} = e^{2c}e^{2x} = Ae^{2x}, \text{ because } e^{x+c} = e^xe^c, \text{ not } e^x + e^c \quad \text{E1}$$

$$\text{Answer: Constant added instead of multiplied} \quad \text{2 marks}$$

$$(b) 2y + 1 = Ae^{2x} \quad \text{M1}$$

$$x = 0, y = 1: A = 3 \quad \text{A1}$$

$$y = \frac{1}{2}(3e^{2x} - 1) \quad \text{A1}$$

$$\text{Answer: } y = \frac{1}{2}(3e^{2x} - 1) \quad \text{3 marks}$$

23 (7 marks)

$$(a) \int y^{-2} dy = \int 1 dx, \text{ so } -\frac{1}{y} = x + c \quad \text{M1 A1}$$

$$x = 0, y = a: c = -\frac{1}{a} \quad \text{M1}$$

$$y = \frac{1}{\frac{1}{a} - x} = \frac{a}{1 - ax} \quad \text{A1}$$

$$\text{Answer: } y = \frac{a}{1 - ax} \quad \text{4 marks}$$

$$(b) y = \frac{2}{1-2x} \text{ is undefined at } x = \frac{1}{2}, \text{ so the solution containing } x = 0 \text{ is valid for } x < \frac{1}{2} \quad \text{B1}$$

$$\text{Answer: } x < \frac{1}{2} \quad \text{1 mark}$$

$$(c) \text{ Separating the variables involves dividing by } y^2, \text{ which assumes } y \neq 0 \quad \text{E1}$$

$$y = 0 \text{ gives } \frac{dy}{dx} = 0 = y^2, \text{ so it is a solution that must be stated separately} \quad \text{E1}$$

$$\text{Answer: Division by } y^2 \text{ loses } y = 0 \quad \text{2 marks}$$

24 (7 marks)

(a) $\frac{dy}{dx} = 2Ax$, and $A = \frac{y}{x^2}$ M1

so $\frac{dy}{dx} = \frac{2y}{x}$, giving $x \frac{dy}{dx} = 2y$ A1

Answer: $x \frac{dy}{dx} = 2y$ 2 marks

(b) The gradients multiply to -1 , so the gradient of O is $-\frac{x}{2y}$ M1 A1

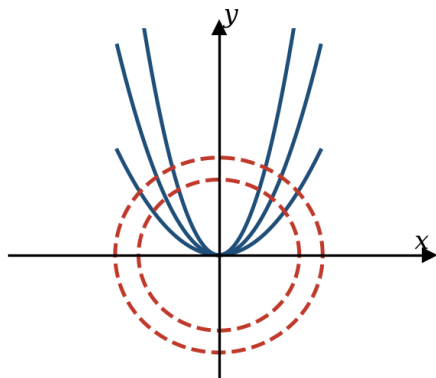
Answer: $\frac{dy}{dx} = -\frac{x}{2y}$ 2 marks

(c) $\int 2y \, dy = -\int x \, dx$ M1

$y^2 = -\frac{x^2}{2} + c$ A1

$x^2 + 2y^2 = 2c = k$, as required A1

Answer: $x^2 + 2y^2 = k$ 3 marks

**25 (8 marks)**

(a) Tangent: $Y - y = \frac{dy}{dx}(X - x)$ M1

At the y-axis $X = 0$, so $Y = y - x \frac{dy}{dx}$ M1 A1

Answer: y-coordinate of Q = $y - x \frac{dy}{dx}$ 3 marks

(b) $y - x \frac{dy}{dx} = x^2$, so $\frac{dy}{dx} = \frac{y}{x} - x$ M1

$\frac{dy}{dx} = v + x \frac{dv}{dx}$, so $v + x \frac{dv}{dx} = v - x$, giving $\frac{dv}{dx} = -1$ M1 A1

$v = -x + c$, so $y = x(c - x)$ M1

(1, 0): $c = 1$, so $y = x(1 - x)$ A1

Answer: $y = x(1 - x)$ 5 marks

26 (8 marks)

(a) $y = e^z$, so $\frac{dy}{dx} = e^z \frac{dz}{dx}$ B1

$xe^z \frac{dz}{dx} + e^z = ze^z$ M1

Dividing by e^z : $x \frac{dz}{dx} = z - 1$ A1

Answer: $x \frac{dz}{dx} = z - 1$ 3 marks

(b) $\int \frac{1}{z-1} \, dz = \int \frac{1}{x} \, dx$ M1

$\ln|z - 1| = \ln x + c$, so $z - 1 = Ax$ A1

$\ln y = 1 + Ax$ M1

$x = 1$, $y = e^3$: $3 = 1 + A$, so $A = 2$ M1

$y = e^{1+2x}$ A1

Answer: $y = e^{1+2x}$

5 marks

27 (11 marks)

(a) $\frac{dy}{dx} = v + x \frac{dv}{dx}$ and $\frac{y^2 - x^2}{2xy} = \frac{v^2 - 1}{2v}$

M1 A1

$$2vx \frac{dv}{dx} = v^2 - 1 - 2v^2$$

M1

$$= -(1 + v^2)$$

A1

Answer: $2vx \frac{dv}{dx} = -(1 + v^2)$

4 marks

(b) $\int \frac{2v}{1+v^2} dv = -\int \frac{1}{x} dx$

M1

$$\ln(1+v^2) = -\ln x + c$$

A1

$$x(1+v^2) = A$$

M1

$$x + \frac{y^2}{x} = A, \text{ so } x^2 + y^2 = Ax$$

A1

Answer: the general solution can be written $x^2 + y^2 = Ax$

4 marks

(c) (1, 1): $A = 2$, so C is $x^2 + y^2 = 2x$

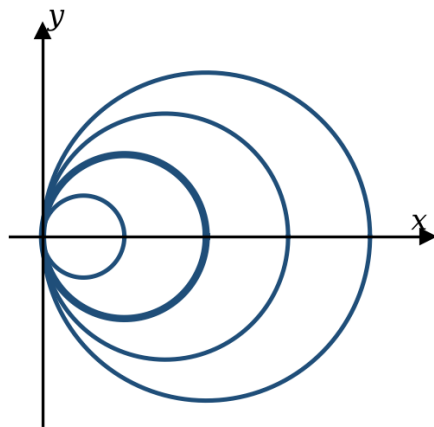
B1

$(x-1)^2 + y^2 = 1$: the circle with centre (1, 0) and radius 1 (upper half, as $y > 0$)

M1 A1

Answer: $x^2 + y^2 = 2x$, a circle centre (1, 0) radius 1

3 marks



28 (11 marks)

(a) $\frac{1}{(y-1)(y-3)} = \frac{1}{2} \left(\frac{1}{y-3} - \frac{1}{y-1} \right)$

M1 A1

$$\frac{1}{2} \ln \left| \frac{y-3}{y-1} \right| = x + c$$

M1 A1

$$\frac{y-3}{y-1} = Ae^{2x}; x=0, y=0: A=3$$

M1 A1

$$y-3 = 3e^{2x}(y-1), \text{ so } y(1-3e^{2x}) = 3-3e^{2x}, \text{ giving } y = \frac{3(e^{2x}-1)}{3e^{2x}-1}$$

A1

Answer: $y = \frac{3(e^{2x}-1)}{3e^{2x}-1}$

7 marks

(b) Denominator zero when $e^{2x} = \frac{1}{3}$, i.e. $x = -\frac{1}{2} \ln 3$

M1

The solution through $x=0$ is valid for $x > -\frac{1}{2} \ln 3$

A1

Answer: $x > -\frac{1}{2} \ln 3$

2 marks

(c) As $x \rightarrow \infty$, $y \rightarrow \frac{3e^{2x}}{3e^{2x}} = 1$

B1

The solution curve approaches the constant solution $y=1$ (an asymptote)

E1

Answer: $y \rightarrow 1$

2 marks

29 (10 marks)

$$(a) \frac{du}{dx} = 1 + \frac{dy}{dx} \quad \text{M1}$$

$$= 1 + \frac{u+1}{u-1} = \frac{2u}{u-1} \quad \text{M1 A1}$$

$$\text{Answer: } \frac{du}{dx} = \frac{2u}{u-1} \quad \text{3 marks}$$

$$(b) \int \frac{u-1}{u} du = \int 2 dx \quad \text{M1}$$

$$u - \ln|u| = 2x + c \quad \text{M1 A1}$$

$$x = 0, y = 2: u = 2, \text{ so } c = 2 - \ln 2 \quad \text{M1}$$

$$x + y - \ln(x + y) = 2x + 2 - \ln 2 \quad \text{A1}$$

$$y - x - 2 = \ln(x + y) - \ln 2 = \ln(\frac{1}{2}(x + y)) \quad \text{M1 A1}$$

$$\text{Answer: } y - x - 2 = \ln(\frac{1}{2}(x + y)) \quad \text{7 marks}$$

30 (12 marks)

$$(a) \text{ From the second equation } y = 2x - 3; x + 2(2x - 3) - 4 = 0, \text{ so } x = 2 \text{ and } y = 1 \quad \text{M1 A1}$$

$$x + 2y - 4 = X + 2Y, 2x - y - 3 = 2X - Y \text{ and } \frac{dy}{dx} = \frac{dY}{dX} \quad \text{A1}$$

$$\text{Answer: } (2, 1); \frac{dY}{dX} = \frac{X + 2Y}{2X - Y} \quad \text{3 marks}$$

$$(b) \frac{dY}{dX} = V + X \frac{dV}{dX} \quad \text{B1}$$

$$V + X \frac{dV}{dX} = \frac{1 + 2V}{2 - V} \quad \text{M1}$$

$$X \frac{dV}{dX} = \frac{1 + 2V - 2V + V^2}{2 - V} = \frac{1 + V^2}{2 - V} \quad \text{A1}$$

$$\text{Answer: } X \frac{dV}{dX} = \frac{1 + V^2}{2 - V} \quad \text{3 marks}$$

$$(c) \int \frac{2 - V}{1 + V^2} dV = \int \frac{1}{X} dX \quad \text{M1}$$

$$2 \tan^{-1} V - \frac{1}{2} \ln(1 + V^2) = \ln X + c \quad \text{M1 A1}$$

$$2 \tan^{-1} V = \ln X + \frac{1}{2} \ln(1 + V^2) + c = \frac{1}{2} \ln(X^2 + Y^2) + c \quad \text{M1}$$

$$x = 3, y = 1 \text{ gives } X = 1, Y = 0, \text{ so } c = 0 \quad \text{M1}$$

$$4 \tan^{-1}\left(\frac{y-1}{x-2}\right) = \ln((x-2)^2 + (y-1)^2) \quad \text{A1}$$

$$\text{Answer: } 4 \tan^{-1}\left(\frac{y-1}{x-2}\right) = \ln((x-2)^2 + (y-1)^2) \quad \text{6 marks}$$

First-Order Linear Differential Equations**1 (2 marks)**

$$I = e^{\int 4 dx} = e^{4x} \quad \text{M1 A1}$$

$$\text{Answer: } e^{4x} \quad \text{2 marks}$$

2 (3 marks)

$$\frac{dy}{dx} + \frac{3}{x}y = 5 \quad \text{B1}$$

$$I = e^{\int 3/x dx} = e^{3 \ln x} = x^3 \quad \text{M1 A1}$$

$$\text{Answer: } \frac{dy}{dx} + \frac{3}{x}y = 5; I = x^3 \quad \text{3 marks}$$

3 (2 marks)

$$I = e^{\int -\tan x dx} = e^{\ln(\cos x)} \quad \text{M1}$$

$$I = \cos x \quad \text{A1}$$

$$\text{Answer: } \cos x \quad \text{2 marks}$$

4 (2 marks)

$$I = e^{\int 2x \, dx}$$

M1

$$= e^{x^2}$$

A1

$$\text{Answer: } e^{\int 2x \, dx} = e^{x^2}$$

2 marks

5 (3 marks)

$$e^{3x}y = \int 6e^{3x} \, dx = 2e^{3x} + C$$

M1 A1

$$y = 2 + Ce^{-3x}$$

A1

$$\text{Answer: } y = 2 + Ce^{-3x}$$

3 marks

6 (5 marks)

$$\text{Integrating factor: } I = e^{\int 3 \, dx} = e^{3x}$$

M1 A1

$$\frac{d}{dx}(e^{3x}y) = 12e^{3x}, \text{ so } e^{3x}y = \int 12e^{3x} \, dx = 4e^{3x} + C$$

M1 A1

$$y = 4 + Ce^{-3x}$$

A1

$$\text{Answer: } y = 4 + Ce^{-3x}$$

5 marks

7 (5 marks)

$$\text{Integrating factor: } I = e^{\int \frac{1}{x} \, dx} = e^{\ln x} = x$$

M1 A1

$$\frac{d}{dx}(xy) = 3x^2, \text{ so } xy = \int 3x^2 \, dx = x^3 + C$$

M1 A1

$$y = x^2 + \frac{C}{x}$$

A1

$$\text{Answer: } y = x^2 + \frac{C}{x}$$

5 marks

8 (5 marks)

$$\text{Integrating factor: } I = e^{\int 2 \, dx} = e^{2x}$$

M1 A1

$$\frac{d}{dx}(e^{2x}y) = e^{5x}, \text{ so } e^{2x}y = \int e^{5x} \, dx = \frac{1}{5}e^{5x} + C$$

M1 A1

$$y = \frac{1}{5}e^{3x} + Ce^{-2x}$$

A1

$$\text{Answer: } y = \frac{1}{5}e^{3x} + Ce^{-2x}$$

5 marks

9 (5 marks)

$$\text{Integrating factor: } I = e^{\int \tan x \, dx} = e^{-\ln(\cos x)} = \sec x$$

M1 A1

$$\frac{d}{dx}(y \sec x) = \sec^2 x, \text{ so } y \sec x = \int \sec^2 x \, dx = \frac{\sin x}{\cos x} + C$$

M1 A1

$$y = \sin x + C \cos x$$

A1

$$\text{Answer: } y = \sin x + C \cos x$$

5 marks

10 (5 marks)

$$\text{Divide by } x: \frac{dy}{dx} - \frac{3}{x}y = x^3$$

$$\text{Integrating factor: } I = e^{\int -\frac{3}{x} \, dx} = e^{-3 \ln x} = \frac{1}{x^3}$$

M1 A1

$$\frac{d}{dx}\left(\frac{1}{x^3}y\right) = 1, \text{ so } \frac{1}{x^3}y = \int 1 \, dx = x + C$$

M1 A1

$$y = x^4 + Cx^3$$

A1

$$\text{Answer: } y = x^4 + Cx^3$$

5 marks

11 (5 marks)Integrating factor: $I = e^{\int \cot x \, dx} = e^{\ln(\sin x)} = \sin x$

M1 A1

$$\frac{d}{dx}(y \sin x) = 3 \sin^3 x \cos x, \text{ so } y \sin x = \int 3 \sin^3 x \cos x \, dx = \frac{3}{4} \sin^4 x + C$$

M1 A1

$$y = \frac{3}{4} \sin^3 x + \frac{C}{\sin x}$$

A1

$$\text{Answer: } y = \frac{3}{4} \sin^3 x + \frac{C}{\sin x}$$

5 marks

12 (7 marks)Integrating factor: $I = e^{\int -1 \, dx} = e^{-x}$

M1 A1

$$\frac{d}{dx}(e^{-x}y) = e^x, \text{ so } e^{-x}y = \int e^x \, dx = e^x + C$$

M1 A1

$$y = e^{2x} + Ce^x$$

A1

$$x = 0, y = 3: C = 2$$

M1

$$y = e^{2x} + 2e^x$$

A1

$$\text{Answer: } y = e^{2x} + 2e^x$$

7 marks

13 (6 marks)Integrating factor: $I = e^{\int \frac{3}{x} \, dx} = e^{3 \ln x} = x^3$

M1 A1

$$\frac{d}{dx}(x^3y) = x^3 \ln x, \text{ so } x^3y = \int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

M2 A1

$$y = \frac{1}{4}x \ln x - \frac{1}{16}x + \frac{C}{x^3}$$

A1

$$\text{Answer: } y = \frac{1}{4}x \ln x - \frac{1}{16}x + \frac{C}{x^3}$$

6 marks

14 (7 marks)Integrating factor: $I = e^{\int \tan x \, dx} = e^{-\ln(\cos x)} = \sec x$

M1 A1

$$\frac{d}{dx}(y \sec x) = 2 \cos x, \text{ so } y \sec x = \int 2 \cos x \, dx = 2 \sin x + C$$

M1 A1

$$y = 2 \sin x \cos x + C \cos x$$

A1

$$x = 0, y = 3: C = 3$$

M1

$$y = 2 \sin x \cos x + 3 \cos x = \sin 2x + 3 \cos x$$

A1

$$\text{Answer: } y = \sin 2x + 3 \cos x$$

7 marks

15 (7 marks)

$$\frac{dx}{dy} = x + y, \text{ so } \frac{dx}{dy} - x = y$$

M1

Integrating factor: $I = e^{\int -1 \, dy} = e^{-y}$

M1 A1

$$\frac{d}{dy}(e^{-y}x) = y e^{-y}, \text{ so } e^{-y}x = \int y e^{-y} \, dy = (-y - 1)e^{-y} + C$$

M2 A1

$$x = -y - 1 + Ce^y$$

A1

$$\text{Answer: } x = -y - 1 + Ce^y$$

7 marks

16 (7 marks)(a) Integrating factor: $I = e^{\int 2 \, dx} = e^{2x}$

M1 A1

$$\frac{d}{dx}(e^{2x}y) = 6x e^{2x}, \text{ so } e^{2x}y = \int 6x e^{2x} \, dx = (3x - \frac{3}{2})e^{2x} + C$$

M2 A1

$$y = 3x - \frac{3}{2} + Ce^{-2x}$$

A1

$$\text{Answer: } y = 3x - \frac{3}{2} + Ce^{-2x}$$

6 marks

$$(b) \text{ As } x \rightarrow \infty, e^{-2x} \rightarrow 0, \text{ so } y \rightarrow 3x - \frac{3}{2}$$

B1

$$\text{Answer: } y = 3x - \frac{3}{2}$$

1 mark

17 (5 marks)

$$(a) \frac{d}{dx}[(x^2 + 1)y] = (x^2 + 1)\frac{dy}{dx} + 2xy \text{ (product rule)}$$

B1

$$\text{Answer: } \frac{d}{dx}[(x^2 + 1)y]$$

1 mark

$$(b) (x^2 + 1)y = \int (3x^2 + 1) dx = x^3 + x + C$$

M1 A1

$$x = 0, y = 2: C = 2$$

M1

$$y = \frac{x^3 + x + 2}{x^2 + 1} = x + \frac{2}{x^2 + 1}$$

A1

$$\text{Answer: } y = x + \frac{2}{x^2 + 1}$$

4 marks**18 (5 marks)**

$$\text{Integrating factor: } e^{\int \coth x \, dx} = e^{\ln \sinh x} = \sinh x$$

M1 A1

$$\frac{d}{dx}(y \sinh x) = 2 \sinh x \cosh x = \sinh 2x$$

M1

$$y \sinh x = \frac{1}{2} \cosh 2x + C$$

A1

$$y = \frac{\cosh 2x + A}{2 \sinh x}, \text{ where } A = 2C$$

A1

$$\text{Answer: } y = \frac{\cosh 2x + A}{2 \sinh x}$$

5 marks**19 (6 marks)**

$$\text{Integrating factor: } e^{\int (1 + 1/x) \, dx} = e^{x + \ln x} = xe^x$$

M1 A1

$$\frac{d}{dx}(xe^x y) = xe^x \times e^{-x} = x$$

M1

$$xe^x y = \frac{1}{2} x^2 + C$$

A1

$$x = 1, y = \frac{3}{2e}: \frac{3}{2} = \frac{1}{2} + C, \text{ so } C = 1$$

M1

$$y = \frac{x^2 + 2}{2x} e^{-x}$$

A1

$$\text{Answer: } y = \frac{x^2 + 2}{2x} e^{-x}$$

6 marks**20 (9 marks)**

$$(a) \frac{dz}{dx} = -y^{-2} \frac{dy}{dx}$$

M1

$$= -y^{-2}(xy^2 - \frac{y}{x}) = -x + \frac{1}{xy} = -x + \frac{z}{x}$$

A1

$$\text{Answer: } \frac{dz}{dx} - \frac{z}{x} = -x$$

2 marks

$$(b) \text{ Integrating factor: } I = e^{\int -\frac{1}{x} \, dx} = e^{-\ln x} = \frac{1}{x}$$

M1 A1

$$\frac{d}{dx}(\frac{1}{x}z) = -1, \text{ so } \frac{1}{x}z = \int -1 \, dx = -x + C$$

M1 A1

$$z = -x^2 + Cx$$

A1

$$\text{Answer: } z = Cx - x^2$$

5 marks

$$(c) x = 1, z = 2: 2 = C - 1, \text{ so } C = 3$$

M1

$$y = \frac{1}{z} = \frac{1}{3x - x^2}$$

A1

$$\text{Answer: } y = \frac{1}{3x - x^2}$$

2 marks

21 (9 marks)

$$(a) \frac{dz}{dx} = 2y \frac{dy}{dx}, \text{ so } y \frac{dy}{dx} = \frac{1}{2} \frac{dz}{dx} \quad \text{M1}$$

$$\frac{1}{2} \frac{dz}{dx} + xz = \frac{x}{2}, \text{ giving } \frac{dz}{dx} + 2xz = x \quad \text{A1}$$

$$\text{Answer: } \frac{dz}{dx} + 2xz = x \quad \text{2 marks}$$

$$(b) \text{ Integrating factor: } I = e^{\int 2x \, dx} = e^{x^2} \quad \text{M1 A1}$$

$$\frac{d}{dx}(e^{x^2} z) = x e^{x^2}, \text{ so } e^{x^2} z = \int x e^{x^2} \, dx = \frac{1}{2} e^{x^2} + C \quad \text{M1 A1}$$

$$z = \frac{1}{2} + C e^{-x^2} \quad \text{A1}$$

$$\text{Answer: } z = \frac{1}{2} + C e^{-x^2} \quad \text{5 marks}$$

$$(c) x = 0, z = 1: C = \frac{1}{2} \quad \text{M1}$$

$$y = \sqrt{\frac{1}{2}(1 + e^{-x^2})} \quad \text{A1}$$

$$\text{Answer: } y = \sqrt{\frac{1}{2}(1 + e^{-x^2})} \quad \text{2 marks}$$

22 (7 marks)

$$(a) \text{ The equation must first be divided by } x \text{ to give } \frac{dy}{dx} + \frac{3}{x}y = 6x \quad \text{E1}$$

$$\text{so } P(x) = \frac{3}{x}, \text{ not } 3, \text{ and } I = e^{\int 3/x \, dx} \quad \text{E1}$$

$$\text{Answer: } P(x) \text{ should be } \frac{3}{x} \quad \text{2 marks}$$

$$(b) \frac{dy}{dx} + \frac{3}{x}y = 6x$$

$$\text{Integrating factor: } I = e^{\int \frac{3}{x} \, dx} = e^{3 \ln x} = x^3 \quad \text{M1 A1}$$

$$\frac{d}{dx}(x^3 y) = 6x^4, \text{ so } x^3 y = \int 6x^4 \, dx = \frac{6}{5}x^5 + C \quad \text{M1 A1}$$

$$y = \frac{6}{5}x^2 + \frac{C}{x^3} \quad \text{A1}$$

$$\text{Answer: } y = \frac{6}{5}x^2 + \frac{C}{x^3} \quad \text{5 marks}$$

23 (9 marks)

$$(a) \text{ Integrating factor: } I = e^{\int k \, dx} = e^{kx} \quad \text{M1 A1}$$

$$\frac{d}{dx}(e^{kx} y) = 1, \text{ so } e^{kx} y = \int 1 \, dx = x + C \quad \text{M1 A1}$$

$$y = x e^{-kx} + C e^{-kx} \quad \text{A1}$$

$$\text{Answer: } y = (x + C)e^{-kx} \quad \text{5 marks}$$

$$(b) x = 0, y = 1 \text{ gives } C = 1, \text{ so } y = (x + 1)e^{-kx} \quad \text{B1}$$

$$\frac{dy}{dx} = e^{-kx}(1 - k(x + 1)) = 0 \quad \text{M1}$$

$$x + 1 = \frac{1}{k}, \text{ so } x = \frac{1 - k}{k} \quad \text{A1}$$

$$y = \frac{1}{k} e^{-(1 - k)} \quad \text{A1}$$

$$\text{Answer: } \left(\frac{1 - k}{k}, \frac{e^{k-1}}{k} \right) \quad \text{4 marks}$$

24 (8 marks)

(a) Integrating factor: $I = e^{\int 2 dx} = e^{2x}$

M1 A1

$$\frac{d}{dx}(e^{2x}y) = 5e^{2x} \sin x, \text{ so } e^{2x}y = \int 5e^{2x} \sin x dx = 2e^{2x} \sin x - e^{2x} \cos x + C$$

M2 A1

$$y = 2 \sin x - \cos x + Ce^{-2x}$$

A1

Answer: $y = 2 \sin x - \cos x + Ce^{-2x}$

6 marks

(b) As $x \rightarrow \infty$, $Ce^{-2x} \rightarrow 0$, so $y \approx 2 \sin x - \cos x = R \sin(x - \alpha)$, $R \cos \alpha = 2$, $R \sin \alpha = 1$

M1

$$R = \sqrt{5}, \alpha = \tan^{-1}\left(\frac{1}{2}\right) = 0.464$$

A1

Answer: $R = \sqrt{5}, \alpha = 0.464$

2 marks

25 (7 marks)

$$I = e^{\int 2 \cot 2x dx} = e^{\ln(\sin 2x)} = \sin 2x$$

M1 A1

$$\frac{d}{dx}(y \sin 2x) = 4 \sin 2x \cos 2x = 2 \sin 4x$$

M1

$$y \sin 2x = -\frac{1}{2} \cos 4x + C$$

A1

$$x = \frac{\pi}{4}, y = 1: 1 = \frac{1}{2} + C, \text{ so } C = \frac{1}{2}$$

M1

$$y \sin 2x = \frac{1}{2}(1 - \cos 4x) = \sin^2 2x, \text{ using } 1 - \cos 4x = 2 \sin^2 2x$$

M1

$$y = \sin 2x$$

A1

Answer: $y = \sin 2x$

7 marks

26 (10 marks)

(a) The gradient of OP is $\frac{y}{x}$, so $\frac{dy}{dx} = \frac{y}{x} + x^2$

B1

Integrating factor: $e^{-\int 1/x dx} = \frac{1}{x}$

M1 A1

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{x^2}{x} = x, \text{ so } \frac{y}{x} = \frac{1}{2}x^2 + C$$

M1

$$x = 2, y = 2: 1 = 2 + C, \text{ so } C = -1$$

M1

$$y = \frac{1}{2}x^3 - x$$

A1

Answer: $y = \frac{1}{2}x^3 - x$

6 marks

(b) $\frac{dy}{dx} = \frac{3}{2}x^2 - 1 = 0$, so $x = \sqrt{\frac{2}{3}}$

M1 A1

$$y = x\left(\frac{1}{2}x^2 - 1\right) = -\frac{2\sqrt{6}}{9}$$

A1

$$\frac{d^2y}{dx^2} = 3x > 0, \text{ so it is a minimum}$$

B1

Answer: $\left(\sqrt{\frac{2}{3}}, -\frac{2\sqrt{6}}{9}\right)$, a minimum

4 marks

27 (12 marks)

(a) $I = e^{3x}$, so $\frac{d}{dx}(e^{3x}z) = 0$, $e^{3x}z = C$

M1

$$x = 0, z = 4: z = 4e^{-3x}$$

A1

Answer: $z = 4e^{-3x}$

2 marks

(b) Substitute z : $\frac{dy}{dx} + 3y = 4e^{-3x}$

Integrating factor: $I = e^{\int 3 dx} = e^{3x}$

M1 A1

$$\frac{d}{dx}(e^{3x}y) = 4, \text{ so } e^{3x}y = \int 4 dx = 4x + C$$

M1 A1

$$y = (4x + C)e^{-3x}$$

A1

$$x = 0, y = -1: C = -1$$

M1

$$y = 4x e^{-3x} - e^{-3x}$$

A1

Answer: $y = (4x - 1)e^{-3x}$

7 marks

(c) $\frac{dy}{dx} = e^{-3x}[4 - 3(4x - 1)] = e^{-3x}(7 - 12x)$

M1 A1

$\frac{dy}{dx} = 0$ at $x = \frac{7}{12}$, $y = \frac{4}{3}e^{-7/4}$

A1

Answer: $(\frac{7}{12}, \frac{4}{3}e^{-7/4})$

3 marks

28 (12 marks)

(a) Integrating factor: $I = e^{\int \frac{1}{x \ln x} dx} = e^{\ln(\ln x)} = \ln x$

M1 A1

$\frac{d}{dx}(y \ln x) = \frac{6 \ln x}{x}$, so $y \ln x = \int \frac{6 \ln x}{x} dx = 3 \ln^2 x + C$

M2 A1

$y = \frac{C}{\ln x} + 3 \ln x$

$x = e$, $y = 5$: $5 = C + 3$, so $C = 2$

M1

$y = 3 \ln x + \frac{2}{\ln x}$

A1

Answer: $y = 3 \ln x + \frac{2}{\ln x}$

7 marks

(b) $\frac{dy}{dx} = \frac{3}{x} - \frac{2}{x(\ln x)^2} = 0$, so $(\ln x)^2 = \frac{2}{3}$

M1 A1

$\ln x = \sqrt{\frac{2}{3}}$: $y = 3\sqrt{\frac{2}{3}} + 2\sqrt{\frac{3}{2}} = \sqrt{6} + \sqrt{6} = 2\sqrt{6}$

M1 A1

$\frac{dy}{dx}$ changes sign from negative to positive (as $\ln x$ increases through $\sqrt{\frac{2}{3}}$), so a minimum

E1

Answer: $2\sqrt{6}$

5 marks

29 (9 marks)

(a) Subtracting the two equations: $\frac{d}{dx}(y_1 - y_2) + 2(y_1 - y_2) = f(x) - f(x)$

M1

so $\frac{du}{dx} + 2u = 0$

A1

Answer: $\frac{du}{dx} + 2u = 0$

2 marks

(b) $I = e^{2x}$, so $\frac{d}{dx}(e^{2x}u) = 0$ and $e^{2x}u = A$

M1

$u = Ae^{-2x}$

A1

$u \rightarrow 0$ as $x \rightarrow \infty$, so any two solution curves get arbitrarily close together

E1

Answer: Any two solutions converge

3 marks

(c) Solution: $y = 2x - 1 + Ce^{-2x}$ (by an integrating factor)

M1

y_1 : $C = 1$; y_2 : $C = 2$

A1

$y_1 - y_2 = -e^{-2x}$

A1

When $x = \ln 2$: $-e^{-2 \ln 2} = -\frac{1}{4}$

A1

Answer: $y_1 - y_2 = -e^{-2x}$; $-\frac{1}{4}$

4 marks

30 (8 marks)

(a) $I = e^x$, so $\frac{d}{dx}(e^x y) = e^x g(x)$

M1 A1

Integrating from 0 to x : $e^x y - 0 = \int_0^x e^t g(t) dt$ ($y = 0$ when $x = 0$)

M1

Divide by e^x

A1

Answer: $y = e^{-x} \int_0^x e^t g(t) dt$

4 marks

(b) $\int_0^x e^t \cos t dt$: integrate by parts twice, or use the known result $\int e^t \cos t dt = \frac{1}{2}e^t(\cos t + \sin t)$

M1 M1

$= \frac{1}{2}e^x(\cos x + \sin x) - \frac{1}{2}$

A1

$$y = \frac{1}{2}(\cos x + \sin x - e^{-x})$$

A1

Answer: $y = \frac{1}{2}(\cos x + \sin x - e^{-x})$

4 marks

Second-Order Differential Equations

1 (2 marks)

$$\frac{dy}{dx} = 3e^{3x}, \quad \frac{d^2y}{dx^2} = 9e^{3x}$$

M1

$$9e^{3x} - 6e^{3x} - 3e^{3x} = 0$$

A1

Answer: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$

2 marks

2 (2 marks)

$$m^2 + 5m + 6 = 0$$

B1

$$(m + 2)(m + 3) = 0, \text{ so } m = -2 \text{ or } m = -3$$

B1

Answer: $m^2 + 5m + 6 = 0; m = -2, -3$

2 marks

3 (3 marks)

$$\frac{dy}{dx} = -6 \sin 2x + 2 \cos 2x$$

B1

$$\frac{d^2y}{dx^2} = -12 \cos 2x - 4 \sin 2x = -4y, \text{ so } \frac{d^2y}{dx^2} + 4y = 0$$

M1 A1

Answer: $\frac{d^2y}{dx^2} + 4y = 0$

3 marks

4 (2 marks)

$$\frac{dy}{dx} = 3x^2 + A$$

M1

$$y = x^3 + Ax + B$$

A1

Answer: $y = x^3 + Ax + B$

2 marks

5 (2 marks)

$$m^2 - 9 = 0, \text{ so } m = \pm 3$$

M1

$$y = Ae^{3x} + Be^{-3x}$$

A1

Answer: $y = Ae^{3x} + Be^{-3x}$

2 marks

6 (3 marks)

(a) Second order, linear

B1

Answer: Second order, linear

1 mark

(b) First order, non-linear (the derivative is squared)

B1

Answer: First order, non-linear

1 mark

(c) Third order, linear

B1

Answer: Third order, linear

1 mark

7 (2 marks)

$$\text{Auxiliary equation: } m^2 - 5m + 6 = 0$$

M1

$$m = 2 \text{ or } m = 3, \text{ so the complementary function is } y = Ae^{2x} + Be^{3x}$$

A1

Answer: $y = Ae^{2x} + Be^{3x}$

2 marks

8 (2 marks)

$$\text{Auxiliary equation: } m^2 + 6m + 9 = 0$$

M1

$$\text{Repeated root } m = -3, \text{ so the complementary function is } y = (A + Bx)e^{-3x}$$

A1

Answer: $y = (A + Bx)e^{-3x}$

2 marks

9 (2 marks)Auxiliary equation: $m^2 + 2m + 5 = 0$

M1

 $m = -1 - 2i$ or $-1 + 2i$, so the complementary function is $y = e^{-x}(A \cos 2x + B \sin 2x)$

A1

Answer: $y = e^{-x}(A \cos 2x + B \sin 2x)$

2 marks

10 (4 marks)

$$\frac{dy}{dx} = 2Ae^{2x} - Be^{-x}, \quad \frac{d^2y}{dx^2} = 4Ae^{2x} + Be^{-x}$$

M1 A1

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2Ae^{2x} + 2Be^{-x}$$

M1

$$= 2y, \text{ so } \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

A1

Answer: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

4 marks

11 (4 marks)

$$\frac{dy}{dx} = 4x^3 - 2x + A; \quad x = 0: A = 2$$

M1 A1

$$y = x^4 - x^2 + 2x + B; \quad x = 0: B = 1$$

M1

$$y = x^4 - x^2 + 2x + 1$$

A1

Answer: $y = x^4 - x^2 + 2x + 1$

4 marks

12 (3 marks)

$$\frac{dy}{dx} = me^{mx}, \quad \frac{d^2y}{dx^2} = m^2e^{mx}$$

M1

$$(m^2 - 4m + 3)e^{mx} = 0, \text{ so } m^2 - 4m + 3 = 0$$

M1

$$(m - 1)(m - 3) = 0, \text{ so } m = 1 \text{ or } m = 3$$

A1

Answer: $m = 1$ or $m = 3$

3 marks

13 (4 marks)

The left-hand side is $\frac{d}{dx}\left(x \frac{dy}{dx}\right)$

M1

$$x \frac{dy}{dx} = 2x^2 + A$$

A1

$$\frac{dy}{dx} = 2x + \frac{A}{x}$$

M1

$$y = x^2 + A \ln x + B$$

A1

Answer: $y = x^2 + A \ln x + B$

4 marks

14 (4 marks)

$$y'' = \alpha y_1'' + \beta y_2'' \text{ and } y' = \alpha y_1' + \beta y_2'$$

M1

$$y'' - 4y' + 3y = \alpha(y_1'' - 4y_1' + 3y_1) + \beta(y_2'' - 4y_2' + 3y_2)$$

M1 A1

$$= \alpha \cdot 0 + \beta \cdot 0 = 0$$

A1

Answer: $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 3y = 0$ for $y = \alpha y_1 + \beta y_2$

4 marks

15 (6 marks)

$$\frac{dy}{dx} = e^x[(A + 2B) \cos 2x + (B - 2A) \sin 2x]$$

M1 A1

$$\frac{d^2y}{dx^2} = e^x[(A + 2B + 2B - 4A) \cos 2x + (B - 2A - 2A - 4B) \sin 2x] = e^x[(4B - 3A) \cos 2x - (3B + 4A) \sin 2x]$$

M1 A1

Then $y'' - 2y' + 5y = 0$ (checking coefficients of $\cos 2x$ and $\sin 2x$)

M1 A1

Answer: $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 5y = 0$

6 marks

16 (5 marks)

$$\frac{dy}{dx} = \frac{1}{2}e^{2x} - \cos x + A$$

M1 A1

$$x = 0: 3 = \frac{1}{2} - 1 + A, \text{ so } A = \frac{7}{2}$$

M1

$$y = \frac{1}{4}e^{2x} - \sin x + \frac{7}{2}x + B; x = 0: B = -\frac{1}{4}$$

M1

$$y = \frac{1}{4}e^{2x} - \sin x + \frac{7}{2}x - \frac{1}{4}$$

A1

$$\text{Answer: } y = \frac{1}{4}e^{2x} - \sin x + \frac{7}{2}x - \frac{1}{4}$$

5 marks

17 (5 marks)

$$(a) m^2 + 16 = 0, \text{ so } m = \pm 4i \text{ and } x = A \cos 4t + B \sin 4t$$

M1 A1

$$t = 0, x = 0.5: A = 0.5; \frac{dx}{dt} = -4A \sin 4t + 4B \cos 4t, \text{ so } B = 0$$

M1 A1

$$\text{Answer: } x = 0.5 \cos 4t$$

4 marks

$$(b) \text{ Period} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ seconds}$$

B1

$$\text{Answer: } \frac{\pi}{2} \text{ s}$$

1 mark

18 (4 marks)

$$x \frac{dp}{dx} = 2p, \text{ so } \int \frac{1}{p} dp = \int \frac{2}{x} dx$$

M1

$$\ln|p| = 2 \ln x + c, \text{ so } p = Ax^2$$

A1

$$\frac{dy}{dx} = Ax^2, \text{ so } y = \frac{A}{3}x^3 + B$$

M1 A1

$$\text{Answer: } y = Px^3 + Q$$

4 marks

19 (6 marks)

$$(a) \text{ Auxiliary equation } m^2 + km + 4 = 0 \text{ has discriminant } k^2 - 16$$

M1

$$\text{Distinct real roots need } k^2 - 16 > 0, \text{ so } k > 4$$

A1

$$\text{Answer: } k > 4$$

2 marks

$$(b) \text{ Repeated root needs } k^2 = 16, \text{ so } k = 4$$

A1

$$\text{Answer: } k = 4$$

1 mark

$$(c) k^2 - 16 < 0, \text{ so } 0 < k < 4$$

A1

$$m = -\frac{k}{2} \pm i\sqrt{4 - \frac{k^2}{4}}, \text{ so the angular frequency is } \sqrt{4 - \frac{k^2}{4}}$$

M1 A1

$$\text{Answer: } 0 < k < 4; \sqrt{4 - \frac{k^2}{4}}$$

3 marks

20 (6 marks)

$$\text{Substituting: } \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 2\frac{dy}{dt} + 2y = 0$$

M1

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$$

A1

$$m^2 - 3m + 2 = 0, \text{ so } m = 1 \text{ or } m = 2$$

M1 A1

$$y = Ae^t + Be^{2t}$$

A1

$$e^t = x: y = Ax + Bx^2$$

A1

$$\text{Answer: } y = Ax + Bx^2$$

6 marks

21 (6 marks)

$$(a) \frac{d}{dx}\left(y \frac{dy}{dx}\right) = \frac{dy}{dx} \frac{dy}{dx} + y \frac{d^2y}{dx^2} \text{ (product rule)}$$

M1 A1

$$\text{Answer: } \frac{d}{dx}\left(y \frac{dy}{dx}\right)$$

2 marks

(b) $y \frac{dy}{dx} = A$; $x = 0$: $A = 2$

M1 A1

$\int y \, dy = \int 2 \, dx$, so $\frac{y^2}{2} = 2x + c$, with $c = \frac{1}{2}$

M1

$y = \sqrt{4x + 1}$

A1

Answer: $y = \sqrt{4x + 1}$

4 marks

22 (6 marks)

(a) $Ae^{-2x} + Be^{-2x} = (A + B)e^{-2x}$, which has only one arbitrary constant

E1

A second-order equation needs two independent arbitrary constants

E1

Answer: Only one independent constant

2 marks

(b) $\frac{dy}{dx} = (1 - 2x)e^{-2x}$, $\frac{d^2y}{dx^2} = (-4 + 4x)e^{-2x}$

M1 A1

$(-4 + 4x + 4 - 8x + 4x)e^{-2x} = 0$

A1

Answer: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

3 marks

(c) $y = (A + Bx)e^{-2x}$

B1

Answer: $y = (A + Bx)e^{-2x}$

1 mark

23 (7 marks)

$\frac{dy}{dx} = (u' + 2u)e^{2x}$

M1 A1

$\frac{d^2y}{dx^2} = (u'' + 4u' + 4u)e^{2x}$

A1

$(u'' + 4u' + 4u - 4u' - 8u + 4u)e^{2x} = u''e^{2x} = 0$, so $u'' = 0$

M1 A1

$u = A + Bx$, so $y = (A + Bx)e^{2x}$

M1 A1

Answer: $y = (A + Bx)e^{2x}$

7 marks

24 (7 marks)

(a) $\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v \frac{dv}{dx}$

M1 A1

Answer: $\frac{d^2x}{dt^2} = v \frac{dv}{dx}$

2 marks

(b) $v \frac{dv}{dx} = -4x$, so $\int v \, dv = -\int 4x \, dx$

M1

$\frac{v^2}{2} = -2x^2 + c$; $x = 3$, $v = 0$: $c = 18$

M1 A1

$v^2 = 4(9 - x^2)$

A1

Answer: $v^2 = 4(9 - x^2)$

4 marks

(c) Greatest when $x = 0$: $v^2 = 36$, so $v = 6 \, \text{m s}^{-1}$

A1

Answer: $6 \, \text{m s}^{-1}$

1 mark

25 (8 marks)

(a) $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \times \frac{dy}{dx} = p \frac{dp}{dy}$

M1 A1

$p \frac{dp}{dy} = 2y^3$, so $\int p \, dp = \int 2y^3 \, dy$, giving $\frac{1}{2}p^2 = \frac{1}{2}y^4 + c$

M1

$p = -1$ and $y = 1$: $c = 0$, so $p^2 = y^4$

A1

Answer: $p^2 = y^4$

4 marks

(b) $\frac{dy}{dx} = \pm y^2$; at $x = 0$, $y = 1$ and $\frac{dy}{dx} = -1$, so $\frac{dy}{dx} = -y^2$

M1 A1

$\int \frac{1}{y^2} \, dy = -\int 1 \, dx$, so $-\frac{1}{y} = -x + c$

M1

$x = 0$, $y = 1$: $c = -1$, so $y = \frac{1}{x + 1}$

A1

Answer: $y = \frac{1}{x+1}$

4 marks

26 (9 marks)

(a) $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = (\frac{d^2y}{dt^2} - \frac{dy}{dt}) + 3\frac{dy}{dt} + y$

M1 A1

$= \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y$, and $\ln x = t$, as required

A1

Answer: $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = t$

3 marks

(b) $m^2 + 2m + 1 = 0$, $m = -1$ twice: $y = (A + Bt)e^{-t}$

M1 A1

Try $y = pt + q$: $2p + pt + q = t$, so $p = 1$, $q = -2$

M1 A1

$y = (A + Bt)e^{-t} + t - 2$, so $y = \frac{A + B \ln x}{x} + \ln x - 2$

A1 A1

Answer: $y = \frac{A + B \ln x}{x} + \ln x - 2$

6 marks

27 (10 marks)

(a) $y = x$: $y' = 1$, $y'' = 0$, so $0 - 2x + 2x = 0$

B1

Answer: $(x^2 + 1)(0) - 2x(1) + 2x = 0$

1 mark

(b) $y' = v + xv'$, $y'' = 2v' + xv''$

M1

$(x^2 + 1)(2v' + xv'') - 2x(v + xv') + 2xv$

M1

$= x(x^2 + 1)v'' + (2x^2 + 2 - 2x^2)v' = x(x^2 + 1)v'' + 2v'$

A1

Answer: $x(x^2 + 1)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0$

3 marks

(c) Let $w = v'$: $\frac{w'}{w} = -\frac{2}{x(x^2 + 1)} = -\frac{2}{x} + \frac{2x}{x^2 + 1}$ (partial fractions)

M1 A1

$\ln w = -2 \ln x + \ln(x^2 + 1) + c$, so $w = A(1 + x^{-2})$

M1 A1

$v = A(x - \frac{1}{x}) + B$

A1

$y = xv = A(x^2 - 1) + Bx$

A1

Answer: $y = A(x^2 - 1) + Bx$

6 marks

28 (9 marks)

(a) $\frac{dy}{dx} = e^{kx}(k \sin 3x + 3 \cos 3x)$

M1 A1

$\frac{d^2y}{dx^2} = e^{kx}[(k^2 - 9) \sin 3x + 6k \cos 3x]$

A1

Substituting: $e^{kx}[(k^2 - 9 - 2k^2 + k^2 + 9) \sin 3x + (6k - 6k) \cos 3x] = 0$

M1 A1

Answer: $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + (k^2 + 9)y = 0$

5 marks

(b) $m = k \pm 3i$, so $y = e^{kx}(A \cos 3x + B \sin 3x)$

M1 A1

Answer: $y = e^{kx}(A \cos 3x + B \sin 3x)$

2 marks

(c) $\frac{dy}{dx} = 0$ gives $k \sin 3x + 3 \cos 3x = 0$, so $\tan 3x = -\frac{3}{k}$

M1

$x = \frac{\pi}{12}$: $\tan(\frac{\pi}{4}) = 1$, so $k = -3$

A1

Answer: $k = -3$

2 marks

29 (8 marks)

(a) $\frac{dp}{dx} = p^2$, so $\int p^{-2} dp = \int 1 dx$

M1

$-\frac{1}{p} = x + c$; $x = 0$, $p = 1$: $c = -1$

M1 A1

$p = \frac{1}{1-x}$, so $y = -\ln|1-x| + d$

M1 A1

$x = 0$, $y = 0$: $d = 0$, so $y = -\ln(1-x)$

A1

Answer: $y = -\ln(1-x)$

6 marks

(b) Need $1 - x > 0$, so $x < 1$

B1

As $x \rightarrow 1^-$, $y \rightarrow \infty$ (vertical asymptote at $x = 1$)

E1

Answer: $x < 1$; $y \rightarrow \infty$ as $x \rightarrow 1$

2 marks

30 (11 marks)

(a) $\frac{dz}{dx} = e^x = z$, so $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = z \frac{dy}{dz}$

M1 A1

$\frac{d^2y}{dx^2} = \frac{d}{dx}(z \frac{dy}{dz}) = z \frac{d}{dz}(z \frac{dy}{dz})$, because $\frac{d}{dx} = z \frac{d}{dz}$

M1

$= z(\frac{dy}{dz} + z \frac{d^2y}{dz^2}) = z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz}$

A1

Answer: $\frac{dy}{dx} = z \frac{dy}{dz}$ and $\frac{d^2y}{dx^2} = z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz}$

4 marks

(b) $z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz} - z \frac{dy}{dz} - 4z^2y = 0$

M1

$z^2(\frac{d^2y}{dz^2} - 4y) = 0$, so $\frac{d^2y}{dz^2} - 4y = 0$

A1

$y = Ae^{2z} + Be^{-2z}$

M1

$y = Ae^{2e^x} + Be^{-2e^x}$

A1

Answer: $y = Ae^{2e^x} + Be^{-2e^x}$

4 marks

(c) $x = 0$ gives $z = 1$: $Ae^2 + Be^{-2} = 1$

M1

$\frac{dy}{dx} = z \frac{dy}{dz} = 0$ at $z = 1$: $2Ae^2 - 2Be^{-2} = 0$, so $Ae^2 = Be^{-2} = \frac{1}{2}$

M1 A1

Answer: $y = \cosh(2e^x - 2)$

3 marks

Homogeneous Differential Equations

1 (2 marks)

$(m - 5)(m + 3) = 0$, so $m = 5$ or $m = -3$

M1

$y = Ae^{5x} + Be^{-3x}$

A1

Answer: $y = Ae^{5x} + Be^{-3x}$

2 marks

2 (2 marks)

$m = \pm 5i$

M1

$y = A \cos 5x + B \sin 5x$

A1

Answer: $y = A \cos 5x + B \sin 5x$

2 marks

3 (2 marks)

$(m - 2)(m + 5) = m^2 + 3m - 10$

M1

$\frac{d^2y}{dx^2} + 3 \frac{dy}{dx} - 10y = 0$

A1

Answer: $\frac{d^2y}{dx^2} + 3 \frac{dy}{dx} - 10y = 0$

2 marks

4 (2 marks)

Auxiliary equation: $m^2 - 6m + 8 = 0$

M1

$m = 2$ or $m = 4$, so the complementary function is $y = Ae^{2x} + Be^{4x}$

A1

Answer: $y = Ae^{2x} + Be^{4x}$

2 marks

5 (2 marks)

Auxiliary equation: $4m^2 - 4m + 1 = 0$

M1

Repeated root $m = \frac{1}{2}$, so the complementary function is $y = (A + Bx)e^{\frac{1}{2}x}$

A1

Answer: $y = (A + Bx)e^{\frac{1}{2}x}$

2 marks

6 (3 marks)

(a) Discriminant $16 - 12 = 4 > 0$, so the roots are real and distinct

B1

Answer: No: real distinct roots

1 mark

(b) Discriminant $16 - 32 < 0$, so the roots are complex $(-2 \pm 2i)$

B1

Answer: Yes: complex roots $-2 \pm 2i$

1 mark

(c) Discriminant $16 - 16 = 0$, so there is a repeated real root

B1

Answer: No: repeated real root

1 mark

7 (2 marks)

Auxiliary equation: $3m^2 - 5m - 2 = 0$

M1

$m = -\frac{1}{3}$ or $m = 2$, so the complementary function is $y = Ae^{-\frac{1}{3}x} + Be^{2x}$

A1

Answer: $y = Ae^{-\frac{1}{3}x} + Be^{2x}$

2 marks

8 (2 marks)

Auxiliary equation: $m^2 + 2m + 10 = 0$

M1

$m = -1 - 3i$ or $-1 + 3i$, so the complementary function is $y = e^{-x}(A \cos 3x + B \sin 3x)$

A1

Answer: $y = e^{-x}(A \cos 3x + B \sin 3x)$

2 marks

9 (3 marks)

$(m + 1)^2 + 4 = 0$, so $m^2 + 2m + 5 = 0$

M1 A1

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$; $y = e^{-x}(A \cos 2x + B \sin 2x)$

A1

Answer: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$; $y = e^{-x}(A \cos 2x + B \sin 2x)$

3 marks

10 (2 marks)

Auxiliary equation: $m^2 - 4m = 0$

M1

$m = 0$ or $m = 4$, so the complementary function is $y = A + Be^{4x}$

A1

Answer: $y = A + Be^{4x}$

2 marks

11 (4 marks)

$m^2 + 9 = 0$, so $m = \pm 3i$ and $x = A \cos 3t + B \sin 3t$

M1 A1

$t = 0$: $x = 2$ gives $A = 2$; $\frac{dx}{dt} = -3A \sin 3t + 3B \cos 3t = 0$ at $t = 0$ gives $B = 0$

M1

$x = 2 \cos 3t$

A1

Answer: $x = 2 \cos 3t$

4 marks

12 (4 marks)

$x \frac{dv}{dx} = 1$, so $\int 1 dv = \int \frac{1}{x} dx$

M1

$v = \ln x + c$

A1

$x = 1$, $v = 2$: $c = 2$

M1

$y = x(\ln x + 2)$

A1

Answer: $y = x(\ln x + 2)$

4 marks

13 (3 marks)

$e^x = \cosh x + \sinh x$ and $e^{-x} = \cosh x - \sinh x$

M1

$y = (A + B) \cosh x + (A - B) \sinh x$

M1

So $C = A + B$ and $D = A - B$

A1

Answer: $C = A + B$, $D = A - B$

3 marks

14 (6 marks)Auxiliary equation: $m^2 - m - 6 = 0$

M1

 $m = -2$ or $m = 3$, so the complementary function is $y = Ae^{-2x} + Be^{3x}$

A1

 $y(0) = 1: A + B = 1$

M1

 $y'(0) = 8: -2A + 3B = 8$

M1

Solving: $A = -1, B = 2$

A1

 $y = 2e^{3x} - e^{-2x}$

A1

Answer: $y = 2e^{3x} - e^{-2x}$

6 marks

15 (6 marks)Auxiliary equation: $m^2 + 6m + 9 = 0$

M1

Repeated root $m = -3$, so the complementary function is $y = (A + Bx)e^{-3x}$

A1

 $y(0) = 2: A = 2$

M1

 $y'(0) = -1: -3A + B = -1$

M1

Solving: $A = 2, B = 5$

A1

 $y = (5x + 2)e^{-3x}$

A1

Answer: $y = (5x + 2)e^{-3x}$

6 marks

16 (6 marks)Auxiliary equation: $m^2 + 2m + 5 = 0$

M1

 $m = -1 - 2i$ or $-1 + 2i$, so the complementary function is $y = e^{-x}(A \cos 2x + B \sin 2x)$

A1

 $y(0) = 1: A = 1$

M1

 $y'(0) = 3: -A + 2B = 3$

M1

Solving: $A = 1, B = 2$

A1

 $y = e^{-x}(2 \sin 2x + \cos 2x)$

A1

Answer: $y = e^{-x}(2 \sin 2x + \cos 2x)$

6 marks

17 (6 marks) $v + x \frac{dv}{dx} = \frac{1 + 3v^2}{2v}$, so $x \frac{dv}{dx} = \frac{1 + v^2}{2v}$

M1 A1

 $\int \frac{2v}{1 + v^2} dv = \int \frac{1}{x} dx$, so $\ln(1 + v^2) = \ln x + c$

M1 A1

 $1 + \frac{y^2}{x^2} = Ax$, so $x^2 + y^2 = Ax^3$

A1

 $(1, 1): A = 2$, so $x^2 + y^2 = 2x^3$

A1

Answer: $x^2 + y^2 = 2x^3$

6 marks

18 (6 marks)**(a)** $m^2 + 10m + 9 = 0$, so $m = -1$ or $m = -9$

M1

 $x = Ae^{-t} + Be^{-9t}$: overdamped

A1

Answer: $x = Ae^{-t} + Be^{-9t}$, overdamped

2 marks

(b) $m^2 + 6m + 9 = 0$, so $m = -3$ (repeated)

M1

 $x = (A + Bt)e^{-3t}$: critically damped

A1

Answer: $x = (A + Bt)e^{-3t}$, critically damped

2 marks

(c) $m^2 + 2m + 9 = 0$, so $m = -1 \pm 2\sqrt{2}i$

M1

 $x = e^{-t}(A \cos 2\sqrt{2}t + B \sin 2\sqrt{2}t)$: underdamped

A1

Answer: $x = e^{-t}(A \cos 2\sqrt{2}t + B \sin 2\sqrt{2}t)$, underdamped

2 marks

19 (5 marks)

$$m^2 + am + (a + 3) = 0, \text{ equal roots: } a^2 - 4(a + 3) = 0$$

M1

$$a^2 - 4a - 12 = 0, \text{ so } (a - 6)(a + 2) = 0, \text{ and } a = 6 \text{ as } a > 0$$

M1 A1

$$m^2 + 6m + 9 = 0, \text{ so } m = -3 \text{ (repeated)}$$

M1

$$y = (A + Bx)e^{-3x}$$

A1

$$\text{Answer: } a = 6; y = (A + Bx)e^{-3x}$$

5 marks

20 (5 marks)

$$\frac{dy}{dx} = e^{-2x}(\cos x - 2 \sin x)$$

M1 A1

$$\frac{dy}{dx} = 0: \tan x = \frac{1}{2}, \text{ so } x = \tan^{-1}\left(\frac{1}{2}\right)$$

M1 A1

$$\sin x = \frac{1}{\sqrt{5}}, \text{ so } y = \frac{1}{\sqrt{5}}e^{-2\tan^{-1}(\frac{1}{2})}$$

A1

$$\text{Answer: } \left(\tan^{-1}\left(\frac{1}{2}\right), \frac{1}{\sqrt{5}}e^{-2\tan^{-1}(\frac{1}{2})}\right)$$

5 marks

21 (7 marks)

$$(a) m^2 + 4m + 29 = 0, \text{ so } m = -2 \pm 5i$$

M1 A1

$$x = e^{-2t}(A \cos 5t + B \sin 5t); t = 0, x = 1: A = 1$$

M1

$$\frac{dx}{dt} = e^{-2t}[(5B - 2A) \cos 5t + (-5A - 2B) \sin 5t] = 0 \text{ at } t = 0: 5B = 2, \text{ so } B = \frac{2}{5}$$

M1 A1

$$x = e^{-2t}\left(\cos 5t + \frac{2}{5} \sin 5t\right)$$

A1

$$\text{Answer: } x = e^{-2t}\left(\cos 5t + \frac{2}{5} \sin 5t\right)$$

6 marks

$$(b) \text{ The factor } e^{-2t} \rightarrow 0, \text{ so the oscillations decay and } x \rightarrow 0 \text{ (underdamped motion)}$$

B1

$$\text{Answer: Decaying oscillations, } x \rightarrow 0$$

1 mark

22 (7 marks)

$$y = vx: v + x \frac{dv}{dx} = v + \tan v, \text{ so } x \frac{dv}{dx} = \tan v$$

M1 A1

$$\int \cot v \, dv = \int \frac{1}{x} \, dx, \text{ so } \ln(\sin v) = \ln x + c$$

M1 A1

$$\sin v = Ax; x = 1, v = \frac{\pi}{6}: A = \frac{1}{2}$$

M1

$$\sin\left(\frac{y}{x}\right) = \frac{x}{2}$$

A1

$$y = x \sin^{-1}\left(\frac{x}{2}\right)$$

A1

$$\text{Answer: } y = x \sin^{-1}\left(\frac{x}{2}\right)$$

7 marks

23 (6 marks)

$$\text{Auxiliary equation: } m^3 - 3m^2 + 4 = 0$$

M1

$$(m + 1)(m^2 - 4m + 4) = 0$$

M1 A1

$$(m + 1)(m - 2)^2 = 0, \text{ so } m = -1 \text{ or } m = 2 \text{ (repeated)}$$

A1

$$y = Ae^{-x} + (B + Cx)e^{2x}$$

M1 A1

$$\text{Answer: } y = Ae^{-x} + (B + Cx)e^{2x}$$

6 marks

24 (5 marks)

$$(a) A \cos 2x + B \sin 2x = R \cos(2x - \varphi) = R \cos \varphi \cos 2x + R \sin \varphi \sin 2x$$

M1

$$R \cos \varphi = A, R \sin \varphi = B$$

A1

$$R = \sqrt{A^2 + B^2}, \tan \varphi = \frac{B}{A}$$

A1

$$\text{Answer: } R = \sqrt{A^2 + B^2}, \tan \varphi = \frac{B}{A}$$

3 marks

$$(b) \text{ Period } \pi, \text{ so the interval is } \pi$$

B1

Amplitude multiplied by e^{-x}

B1

Answer: π ; e^{-x}

2 marks

25 (5 marks)

$$y = x^m: x^2 \cdot m(m-1)x^{m-2} + 5x \cdot mx^{m-1} + 4x^m = 0$$

M1

$$m(m-1) + 5m + 4 = 0, \text{ so } m^2 + 4m + 4 = 0$$

M1 A1

$$(m+2)^2 = 0, m = -2 \text{ only}$$

A1

The root is repeated, so a second independent solution has the extra factor $\ln x$ ($y = x^{-2} \ln x$), which is not a power of x

E1

Answer: $m = -2$ only

5 marks

26 (7 marks)

$$(a) m^2 + 2m + k = 0$$

M1

$$m = -1 \pm \sqrt{1-k}$$

A1

Answer: $m = -1 \pm \sqrt{1-k}$

2 marks

(b) Real roots need $1-k \geq 0$, so $k \leq 1$

B1

Both roots negative: $-1 + \sqrt{1-k} < 0$, so $\sqrt{1-k} < 1$, giving $k > 0$

M1

$$0 < k \leq 1$$

A1

Answer: $0 < k \leq 1$

3 marks

$$(c) m = -1 \pm i\sqrt{k-1}, \text{ so } y = e^{-x}(A \cos \sqrt{k-1}x + B \sin \sqrt{k-1}x)$$

B1

Oscillations whose amplitude decays to 0

B1

Answer: Decaying oscillations

2 marks

27 (9 marks)

$$(a) \frac{dy}{dx} = e^{-ax}(\omega \cos \omega x - a \sin \omega x)$$

M1 A1

$$\frac{d^2y}{dx^2} = e^{-ax}[(a^2 - \omega^2) \sin \omega x - 2a\omega \cos \omega x]$$

A1

$$\text{Sum: } e^{-ax}[(a^2 - \omega^2 - 2a^2 + a^2 + \omega^2) \sin \omega x + (-2a\omega + 2a\omega) \cos \omega x] = 0$$

M1 A1

$$\text{Answer: } \frac{d^2y}{dx^2} + 2a \frac{dy}{dx} + (a^2 + \omega^2)y = 0$$

5 marks

$$(b) \frac{dy}{dx} = 0: \tan \omega x = \frac{\omega}{a}$$

M1

tan has period π , so stationary points are $\frac{\pi}{\omega}$ apart

A1

$$\text{At the first, } \omega x = \tan^{-1}\left(\frac{\omega}{a}\right) = \alpha, \text{ with } \sin \alpha = \frac{\omega}{\sqrt{a^2 + \omega^2}}$$

M1

$$y = \frac{\omega}{\sqrt{a^2 + \omega^2}} e^{-(a/\omega)\alpha}$$

A1

$$\text{Answer: Spacing } \frac{\pi}{\omega}; y = \frac{\omega}{\sqrt{a^2 + \omega^2}} e^{-(a/\omega)\tan^{-1}(\omega/a)}$$

4 marks

28 (9 marks)

$$m^3 + m^2 + 4m + 4 = 0, \text{ so } m^2(m+1) + 4(m+1) = (m+1)(m^2+4) = 0$$

M1 A1

$$m = -1 \text{ or } m = \pm 2i, \text{ so } y = Ae^{-x} + B \cos 2x + C \sin 2x$$

A1

$$y' = -Ae^{-x} - 2B \sin 2x + 2C \cos 2x, y'' = Ae^{-x} - 4B \cos 2x - 4C \sin 2x$$

M1

$$x = 0: A + B = 3; -A + 2C = 0; A - 4B = -4$$

M1 A1

$$\text{Solving: } A = \frac{8}{5}, B = \frac{7}{5}, C = \frac{4}{5}$$

M1 A1

$$y = \frac{4}{5} \sin 2x + \frac{7}{5} \cos 2x + \frac{8}{5} e^{-x}$$

A1

$$\text{Answer: } y = \frac{4}{5} \sin 2x + \frac{7}{5} \cos 2x + \frac{8}{5} e^{-x}$$

9 marks

29 (9 marks)

$$(a) \frac{dy}{dx} = v + x \frac{dv}{dx}, \text{ so } x(v + x \frac{dv}{dx}) = vx + \sqrt{x^2 + v^2 x^2}$$

M1 A1

$$x^2 \frac{dv}{dx} = x\sqrt{1 + v^2} \quad (x > 0), \text{ so } x \frac{dv}{dx} = \sqrt{1 + v^2}$$

A1

$$\text{Answer: } x \frac{dv}{dx} = \sqrt{1 + v^2}$$

3 marks

$$(b) \int \frac{1}{\sqrt{1 + v^2}} dv = \int \frac{1}{x} dx, \text{ so } \sinh^{-1} v = \ln x + c$$

M1 A1

$$x = 1, v = 0: c = 0, \text{ so } v = \sinh(\ln x)$$

M1

$$v = \frac{1}{2}(x - \frac{1}{x})$$

M1 A1

$$y = vx = \frac{1}{2}(x^2 - 1)$$

A1

$$\text{Answer: } y = \frac{1}{2}(x^2 - 1)$$

6 marks

30 (8 marks)

$$(a) \text{ Roots } \alpha \text{ and } \beta: y = Ae^{\alpha x} + Be^{\beta x}$$

M1

$$A + B = 0 \text{ and } \alpha A + \beta B = 1$$

M1

$$A = \frac{1}{\alpha - \beta}, B = -\frac{1}{\alpha - \beta}, \text{ so } y_1 = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$$

A1 A1

$$\text{Answer: } y_1 = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$$

4 marks

$$(b) \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} = e^{\alpha x} \frac{1 - e^{(\beta - \alpha)x}}{\alpha - \beta}$$

M1

$$\text{Let } h = \beta - \alpha \rightarrow 0: \frac{1 - e^{hx}}{-h} \rightarrow x \text{ (since } e^{hx} \approx 1 + hx)$$

M1 A1

$$\text{So } y_1 \rightarrow xe^{\alpha x}, \text{ which is the second solution when the root } \alpha \text{ is repeated}$$

E1

$$\text{Answer: } xe^{\alpha x}$$

4 marks

Non-Homogeneous Differential Equations**1 (2 marks)**

$$y = 2: y' = y'' = 0, \text{ so } 0 + 0 + 12 = 12$$

B1

$$\text{General solution: } y = 2 + Ae^{-2x} + Be^{-3x}$$

B1

$$\text{Answer: } y = 2 + Ae^{-2x} + Be^{-3x}$$

2 marks

2 (2 marks)

$$y = x: y' = 1, y'' = 0$$

M1

$$y'' + y = 0 + x = x, \text{ as required}$$

A1

$$\text{Answer: } \frac{d^2 y}{dx^2} + y = x$$

2 marks

3 (2 marks)

$$y = pe^x: y' = y'' = pe^x, \text{ so } (p - 2p - 3p)e^x = 8e^x$$

M1

$$-4p = 8, \text{ so } p = -2$$

A1

$$\text{Answer: } p = -2$$

2 marks

4 (2 marks)

The complementary function solves the homogeneous equation and the particular integral produces the right-hand side, so their sum satisfies the full equation

E1

A second-order equation requires two independent constants, which come from the complementary function

E1

$$\text{Answer: General solution} = \text{CF} + \text{PI}; \text{ two constants from the CF}$$

2 marks

5 (2 marks)

$$y = CF + PI$$

M1

$$y = e^{-x}(A \cos 2x + B \sin 2x) + 3x - 1$$

A1

$$\text{Answer: } y = e^{-x}(A \cos 2x + B \sin 2x) + 3x - 1$$

2 marks

6 (3 marks)

$$CF: m^2 + 9 = 0, y = A \cos 3x + B \sin 3x$$

B1

$$PI: y = p, 9p = 18, \text{ so } p = 2$$

M1

$$y = 2 + A \cos 3x + B \sin 3x$$

A1

$$\text{Answer: } y = 2 + A \cos 3x + B \sin 3x$$

3 marks

7 (5 marks)

$$\text{Auxiliary equation: } m^2 + 3m + 2 = 0$$

M1

$$m = -2 \text{ or } m = -1, \text{ so the complementary function is } y = Ae^{-2x} + Be^{-x}$$

A1

$$\text{Try } y = p e^x: \text{ substituting gives } 6p e^x = 4e^x, \text{ so } p = \frac{2}{3}$$

M1 A1

$$\text{Particular integral: } y = \frac{2}{3}e^x$$

$$\text{General solution: } y = \frac{2}{3}e^x + Ae^{-2x} + Be^{-x}$$

A1

$$\text{Answer: } y = \frac{2}{3}e^x + Ae^{-2x} + Be^{-x}$$

5 marks

8 (5 marks)

$$\text{Auxiliary equation: } m^2 + 4 = 0$$

M1

$$m = -2i \text{ or } 2i, \text{ so the complementary function is } y = A \cos 2x + B \sin 2x$$

A1

$$\text{Try } y = p + qx: \text{ substituting gives } 4p + 4qx = 8x, \text{ so } p = 0, q = 2$$

M1 A1

$$\text{Particular integral: } y = 2x$$

$$\text{General solution: } y = 2x + A \cos 2x + B \sin 2x$$

A1

$$\text{Answer: } y = 2x + A \cos 2x + B \sin 2x$$

5 marks

9 (5 marks)

$$\text{Auxiliary equation: } m^2 - 1 = 0$$

M1

$$m = -1 \text{ or } m = 1, \text{ so the complementary function is } y = Ae^{-x} + Be^x$$

A1

$$\text{Try } y = p \cos x + q \sin x: \text{ substituting gives } -2p \cos x - 2q \sin x = 2 \sin x, \text{ so } p = 0, q = -1$$

M1 A1

$$\text{Particular integral: } y = -\sin x$$

$$\text{General solution: } y = -\sin x + Ae^{-x} + Be^x$$

A1

$$\text{Answer: } y = -\sin x + Ae^{-x} + Be^x$$

5 marks

10 (5 marks)

$$\text{Auxiliary equation: } m^2 - 3m = 0$$

M1

$$m = 0 \text{ or } m = 3, \text{ so the complementary function is } y = A + Be^{3x}$$

A1

$$\text{The constant 6 is part of the CF (} m = 0 \text{ is a root), so try } y = px:$$

$$\text{Try } y = px: \text{ substituting gives } -3p = 6, \text{ so } p = -2$$

M1 A1

$$\text{Particular integral: } y = -2x$$

$$\text{General solution: } y = -2x + A + Be^{3x}$$

A1

$$\text{Answer: } y = -2x + A + Be^{3x}$$

5 marks

11 (5 marks)

$$\text{Auxiliary equation: } m^2 - 3m + 2 = 0$$

M1

$$m = 1 \text{ or } m = 2, \text{ so the complementary function is } y = Ae^x + Be^{2x}$$

A1

$$\text{Try } y = px e^x: \text{ substituting gives } -p e^x = e^x, \text{ so } p = -1$$

M1 A1

$$\text{Particular integral: } y = -x e^x$$

$$\text{General solution: } y = -x e^x + Ae^x + Be^{2x}$$

A1

$$\text{Answer: } y = -x e^x + Ae^x + Be^{2x}$$

5 marks

12 (5 marks)

$$m^2 - 2m - 3 = 0, \text{ so } m = 3 \text{ or } m = -1; \text{ CF: } y = Ae^{3x} + Be^{-x}$$

M1 A1

$$\text{Try } y = px^2 + qx + r: -3p = 6, -4p - 3q = -4, 2p - 2q - 3r = 0, \text{ so } p = -2, q = 4, r = -4$$

M1 A1

$$y = Ae^{3x} + Be^{-x} - 2x^2 + 4x - 4$$

A1

$$\text{Answer: } y = Ae^{3x} + Be^{-x} - 2x^2 + 4x - 4$$

5 marks

13 (6 marks)

$$\text{Auxiliary equation: } m^2 + 4m + 3 = 0$$

M1

$$m = -3 \text{ or } m = -1, \text{ so the complementary function is } y = Ae^{-3x} + Be^{-x}$$

A1

$$\text{Try } y = p + qx$$

M1

$$y' = q, y'' = 0$$

$$\text{Substituting and comparing coefficients: } p = 0, q = 1$$

M1 A1

$$\text{Particular integral: } y = x$$

$$\text{General solution: } y = x + Ae^{-3x} + Be^{-x}$$

A1

$$\text{Answer: } y = x + Ae^{-3x} + Be^{-x}$$

6 marks

14 (6 marks)

$$\text{Auxiliary equation: } m^2 + 9 = 0$$

M1

$$m = -3i \text{ or } 3i, \text{ so the complementary function is } y = A \cos 3x + B \sin 3x$$

A1

$$\text{Try } y = p \cos 2x + q \sin 2x$$

M1

$$y' = -2p \sin 2x + 2q \cos 2x, y'' = -4p \cos 2x - 4q \sin 2x$$

$$\text{Substituting and comparing coefficients: } p = 1, q = 0$$

M1 A1

$$\text{Particular integral: } y = \cos 2x$$

$$\text{General solution: } y = \cos 2x + A \cos 3x + B \sin 3x$$

A1

$$\text{Answer: } y = \cos 2x + A \cos 3x + B \sin 3x$$

6 marks

15 (6 marks)

$$\text{Auxiliary equation: } m^2 - 4m + 4 = 0$$

M1

$$\text{Repeated root } m = 2, \text{ so the complementary function is } y = (A + Bx)e^{2x}$$

A1

$$\text{Try } y = px^2 e^{2x}$$

M1

$$y' = 2px^2 e^{2x} + 2px e^{2x}, y'' = 4px^2 e^{2x} + 8px e^{2x} + 2p e^{2x}$$

$$\text{Substituting and comparing coefficients: } p = 4$$

M1 A1

$$\text{Particular integral: } y = 4x^2 e^{2x}$$

$$\text{General solution: } y = 4x^2 e^{2x} + (A + Bx)e^{2x}$$

A1

$$\text{Answer: } y = 4x^2 e^{2x} + (A + Bx)e^{2x}$$

6 marks

16 (7 marks)

$$\text{Auxiliary equation: } m^2 - m - 2 = 0$$

M1

$$m = -1 \text{ or } m = 2, \text{ so the complementary function is } y = Ae^{-x} + Be^{2x}$$

A1

$$\text{Try } y = px e^{-x} + q + rx$$

M1

$$y' = -px e^{-x} + p e^{-x} + r, y'' = px e^{-x} - 2p e^{-x}$$

$$\text{Substituting and comparing coefficients: } p = -\frac{2}{3}, q = \frac{3}{2}, r = -3$$

M1 A2

$$\text{Particular integral: } y = -3x - \frac{2}{3}x e^{-x} + \frac{3}{2}$$

$$\text{General solution: } y = -3x - \frac{2}{3}x e^{-x} + \frac{3}{2} + Ae^{-x} + Be^{2x}$$

A1

$$\text{Answer: } y = -3x - \frac{2}{3}x e^{-x} + \frac{3}{2} + Ae^{-x} + Be^{2x}$$

7 marks

17 (5 marks)

$$y' = a, y'' = 0, \text{ so } 2a - 3(ax + b) = 6x - 1$$

M1

$$-3a = 6, 2a - 3b = -1, \text{ so } a = -2 \text{ and } b = -1$$

M1 A1

$$\text{CF: } m^2 + 2m - 3 = 0, m = 1 \text{ or } m = -3, \text{ so } y = Ae^x + Be^{-3x}$$

M1

$$y = Ae^x + Be^{-3x} - 2x - 1$$

A1

$$\text{Answer: } y = Ae^x + Be^{-3x} - 2x - 1$$

5 marks

18 (6 marks)

$$(a) y = pe^{2x}: y' = 2pe^{2x}, y'' = 4pe^{2x}$$

M1

$$(4p - 6p + 2p)e^{2x} = 0 \neq e^{2x}, \text{ since } e^{2x} \text{ is part of the complementary function}$$

E1

$$\text{Answer: LHS is 0 for every } p$$

2 marks

$$(b) \text{ CF: } m = 1 \text{ or } 2, \text{ so } y = Ae^x + Be^{2x}$$

M1

$$\text{Try } y = pxe^{2x}: y' = p(1 + 2x)e^{2x}, y'' = p(4 + 4x)e^{2x}$$

M1

$$(4 + 4x - 3 - 6x + 2x)p e^{2x} = pe^{2x} = e^{2x}, \text{ so } p = 1$$

A1

$$y = Ae^x + Be^{2x} + xe^{2x}$$

A1

$$\text{Answer: } y = Ae^x + Be^{2x} + xe^{2x}$$

4 marks

19 (6 marks)

$$\text{Auxiliary equation: } m^2 + 4 = 0$$

M1

$$m = -2i \text{ or } 2i, \text{ so the complementary function is } y = A \cos 2x + B \sin 2x$$

A1

$$\text{Try } y = x(p \cos 2x + q \sin 2x)$$

M1

$$y' = -2px \sin 2x + p \cos 2x + 2qx \cos 2x + q \sin 2x, y'' = -4px \cos 2x - 4p \sin 2x - 4qx \sin 2x + 4q \cos 2x$$

$$\text{Substituting and comparing coefficients: } p = -2, q = 0$$

M1 A1

$$\text{Particular integral: } y = -2x \cos 2x$$

$$\text{General solution: } y = -2x \cos 2x + A \cos 2x + B \sin 2x$$

A1

$$\text{Answer: } y = -2x \cos 2x + A \cos 2x + B \sin 2x$$

6 marks

20 (9 marks)

$$\text{Auxiliary equation: } m^2 - 3m + 2 = 0$$

M1

$$m = 1 \text{ or } m = 2, \text{ so the complementary function is } y = Ae^x + Be^{2x}$$

A1

$$\text{Try } y = p e^{-x}: \text{substituting gives } 6p e^{-x} = 6e^{-x}, \text{ so } p = 1$$

M1 A1

$$\text{Particular integral: } y = e^{-x}$$

$$\text{General solution: } y = e^{-x} + Ae^x + Be^{2x}$$

A1

$$y(0) = 2: 1 + A + B = 2$$

M1

$$y'(0) = 1: -1 + A + 2B = 1$$

M1

$$\text{Solving: } A = 0, B = 1$$

A1

$$y = e^{2x} + e^{-x}$$

A1

$$\text{Answer: } y = e^{2x} + e^{-x}$$

9 marks

21 (9 marks)

$$\text{Auxiliary equation: } m^2 + 4m + 13 = 0$$

M1

$$m = -2 - 3i \text{ or } -2 + 3i, \text{ so the complementary function is } y = e^{-2x}(A \cos 3x + B \sin 3x)$$

A1

$$\text{Try } y = p: \text{substituting gives } 13p = 26, \text{ so } p = 2$$

M1 A1

$$\text{Particular integral: } y = 2$$

$$\text{General solution: } y = 2 + e^{-2x}(A \cos 3x + B \sin 3x)$$

A1

$$y(0) = 3: 2 + A = 3$$

M1

$$y'(0) = 1: -2A + 3B = 1$$

M1

$$\text{Solving: } A = 1, B = 1$$

A1

$$y = e^{-2x}(\sin 3x + \cos 3x) + 2$$

A1

$$\text{Answer: } y = e^{-2x}(\sin 3x + \cos 3x) + 2$$

9 marks

22 (9 marks)

Auxiliary equation: $m^2 + 6m + 9 = 0$

M1

Repeated root $m = -3$, so the complementary function is $y = (A + Bx)e^{-3x}$

A1

Try $y = px^2e^{-3x}$: substituting gives $2p e^{-3x} = 12e^{-3x}$, so $p = 6$

M1 A1

Particular integral: $y = 6x^2e^{-3x}$

General solution: $y = 6x^2e^{-3x} + (A + Bx)e^{-3x}$

A1

$y(0) = 1$: $A = 1$

M1

$y'(0) = -2$: $-3A + B = -2$

M1

Solving: $A = 1$, $B = 1$

A1

$y = 6x^2e^{-3x} + (x + 1)e^{-3x}$

A1

Answer: $y = (6x^2 + x + 1)e^{-3x}$

9 marks

23 (8 marks)

(a) Try $y = p \cos kx + q \sin kx$: $(9 - k^2)(p \cos kx + q \sin kx) = \sin kx$

M1 A1

$p = 0$, $q = \frac{1}{9 - k^2}$, so $y = \frac{\sin kx}{9 - k^2}$

A1

Answer: $y = \frac{\sin kx}{9 - k^2}$

3 marks

(b) $\sin 3x$ is in the CF ($m = \pm 3i$), so try $y = x(p \cos 3x + q \sin 3x)$

E1 M1

$y'' + 9y = -6p \sin 3x + 6q \cos 3x = \sin 3x$, so $p = -\frac{1}{6}$, $q = 0$

A1

$y = -\frac{1}{6}x \cos 3x$

A1

Answer: $y = -\frac{1}{6}x \cos 3x$

4 marks

(c) The amplitude of the term $-\frac{1}{6}x \cos 3x$ grows without bound in proportion to x (resonance)

B1

Answer: Amplitude grows linearly

1 mark

24 (9 marks)

Auxiliary equation: $m^2 + 2m + 2 = 0$

M1

$m = -1 - i$ or $-1 + i$, so the complementary function is $y = e^{-x}(A \cos x + B \sin x)$

A1

Try $y = p \cos x + q \sin x$: substituting gives $-2p \sin x + p \cos x + q \sin x + 2q \cos x = 5 \sin x$, so

M1 A1

$p = -2$, $q = 1$

Particular integral: $y = \sin x - 2 \cos x$

General solution: $y = \sin x - 2 \cos x + e^{-x}(A \cos x + B \sin x)$

A1

$y(0) = 0$: $-2 + A = 0$

M1

$y'(0) = 1$: $1 - A + B = 1$

M1

Solving: $A = 2$, $B = 2$

A1

$y = e^{-x}(2 \sin x + 2 \cos x) + \sin x - 2 \cos x$

A1

Answer: $y = e^{-x}(2 \sin x + 2 \cos x) + \sin x - 2 \cos x$

9 marks

25 (6 marks)

Auxiliary equation: $m^2 - 3m + 2 = 0$

M1

$m = 1$ or $m = 2$, so the complementary function is $y = Ae^x + Be^{2x}$

A1

Try $y = p e^{3x} + qx e^{3x}$

M1

$y' = 3p e^{3x} + 3qx e^{3x} + q e^{3x}$, $y'' = 9p e^{3x} + 9qx e^{3x} + 6q e^{3x}$

Substituting and comparing coefficients: $p = -\frac{3}{2}$, $q = 1$

M1 A1

Particular integral: $y = x e^{3x} - \frac{3}{2}e^{3x}$

General solution: $y = x e^{3x} - \frac{3}{2}e^{3x} + Ae^x + Be^{2x}$

A1

Answer: $y = x e^{3x} - \frac{3}{2}e^{3x} + Ae^x + Be^{2x}$

6 marks

26 (10 marks)

(a) At the origin $y = 0$ and $\frac{dy}{dx} = 0$, so $\frac{d^2y}{dx^2} = 10$

M1

$\frac{d^2y}{dx^2} > 0$, so a local minimum

A1

Answer: $\frac{d^2y}{dx^2} = 10 > 0$

2 marks

(b) $m^2 + 4m + 5 = 0$, so $m = -2 \pm i$; CF: $e^{-2x}(A \cos x + B \sin x)$

M1 A1

Try $y = px^2 + qx + r$: $5p = 5$, $8p + 5q = 18$, $2p + 4q + 5r = 10$

M1 A1

$p = 1$, $q = 2$, $r = 0$, so $y = e^{-2x}(A \cos x + B \sin x) + x^2 + 2x$

A1

$y(0) = 0$ gives $A = 0$; $\frac{dy}{dx} = e^{-2x}(B \cos x - 2B \sin x) + 2x + 2$, and $\frac{dy}{dx} = 0$ at $x = 0$ gives $B + 2 = 0$

M1 M1

$y = x^2 + 2x - 2e^{-2x} \sin x$

A1

Answer: $y = x^2 + 2x - 2e^{-2x} \sin x$

8 marks

27 (12 marks)

(a) The CF is $e^{-x}(A \cos 2x + B \sin 2x)$, which already contains $e^{-x} \cos 2x$, so the usual trial form fails and a factor x is needed

E1

Substituting $y = xe^{-x}(p \cos 2x + q \sin 2x)$ gives $y'' + 2y' + 5y = e^{-x}(-4p \sin 2x + 4q \cos 2x)$

M1 A1

$= 4e^{-x} \cos 2x$, so $p = 0$ and $q = 1$

M1 A1

Answer: $p = 0$, $q = 1$

5 marks

(b) General solution $y = e^{-x}(A \cos 2x + B \sin 2x) + xe^{-x} \sin 2x$

B1

$y(0) = 0$: $A = 0$

M1

With $A = 0$: $y = (B + x)e^{-x} \sin 2x$, so $\frac{dy}{dx} = e^{-x}[(1 - B - x) \sin 2x + 2(B + x) \cos 2x]$; at $x = 0$:

M1 A1

$2B = 0$, so $B = 0$

$y = xe^{-x} \sin 2x$

A1

Answer: $y = xe^{-x} \sin 2x$

5 marks

(c) The factor $\sin 2x$ gives oscillation

E1

xe^{-x} has a maximum at $x = 1$ and tends to 0 as $x \rightarrow \infty$ (exponential decay beats the linear growth)

E1

Answer: Exponential decay dominates

2 marks

28 (10 marks)

$8 \cosh 2x = 4e^{2x} + 4e^{-2x}$; CF: $m = \pm 2$, $y = Ae^{2x} + Be^{-2x}$

M1 A1

Both exponentials in the right-hand side are in the CF, so try $y = x(pe^{2x} + qe^{-2x})$

M1

$y'' - 4y = 4pe^{2x} - 4qe^{-2x}$

M1 A1

$p = 1$, $q = -1$, so a PI is $x(e^{2x} - e^{-2x}) = 2x \sinh 2x$

A1

$y = C \cosh 2x + D \sinh 2x + 2x \sinh 2x$

B1

$y(0) = 0$: $C = 0$; $y'(0) = 2D = 0$, so $D = 0$

M1 A1

$y = 2x \sinh 2x$

A1

Answer: $y = 2x \sinh 2x$

10 marks

29 (10 marks)

Auxiliary equation: $m^2 - m - 2 = 0$

M1

$m = -1$ or $m = 2$, so the complementary function is $y = Ae^{-x} + Be^{2x}$

A1

Try $y = px e^{-x} + qx^2 e^{-x}$

M1

$y' = -px e^{-x} + p e^{-x} - qx^2 e^{-x} + 2qx e^{-x}$, $y'' = px e^{-x} - 2p e^{-x} + qx^2 e^{-x} - 4qx e^{-x} + 2q e^{-x}$

Substituting and comparing coefficients: $p = -\frac{2}{3}$, $q = -1$

M1 A1

Particular integral: $y = -x^2 e^{-x} - \frac{2}{3}x e^{-x}$

General solution: $y = -x^2 e^{-x} - \frac{2}{3}x e^{-x} + Ae^{-x} + Be^{2x}$

A1

$y(0) = 0$: $A + B = 0$

M1

$y'(0) = 0$: $-\frac{2}{3} - A + 2B = 0$

M1

Solving: $A = -\frac{2}{9}$, $B = \frac{2}{9}$

A1

$y = -x^2 e^{-x} - \frac{2}{3}x e^{-x} + \frac{2}{9}e^{2x} - \frac{2}{9}e^{-x}$

A1

Answer: $y = -x^2 e^{-x} - \frac{2}{3}x e^{-x} + \frac{2}{9}e^{2x} - \frac{2}{9}e^{-x}$

10 marks

30 (10 marks)

(a) The difference of any two solutions of the non-homogeneous equation satisfies the homogeneous equation

E1

$(x + e^x) - x = e^x$ and $(x + e^{2x}) - x = e^{2x}$

E1

Answer: Differences of solutions are CF solutions

2 marks

(b) The auxiliary equation has roots 1 and 2: $m^2 - 3m + 2 = 0$

M1

$a = -3$, $b = 2$

A1

Answer: $a = -3$, $b = 2$

2 marks

(c) $y = x$: $0 - 3 + 2x = f(x)$, so $f(x) = 2x - 3$

M1 A1

$y = x + Ae^x + Be^{2x}$

A1

Answer: $f(x) = 2x - 3$; $y = x + Ae^x + Be^{2x}$

3 marks

(d) $A + B = 1$ and $1 + A + 2B = 2$, so $A + 2B = 1$

M1

$B = 0$, $A = 1$

A1

$y = x + e^x$

A1

Answer: $y = x + e^x$

3 marks

Complementary Functions**1 (2 marks)**

Auxiliary equation: $m^2 - 3m - 4 = 0$

M1

$m = -1$ or $m = 4$, so the complementary function is $y = Ae^{-x} + Be^{4x}$

A1

Answer: $y = Ae^{-x} + Be^{4x}$

2 marks

2 (2 marks)

Auxiliary equation: $m^2 + 49 = 0$

M1

$m = -7i$ or $7i$, so the complementary function is $y = A \cos 7x + B \sin 7x$

A1

Answer: $y = A \cos 7x + B \sin 7x$

2 marks

3 (2 marks)

Auxiliary equation: $m^2 + 4m + 4 = 0$

M1

Repeated root $m = -2$, so the complementary function is $y = (A + Bx)e^{-2x}$

A1

Answer: $y = (A + Bx)e^{-2x}$

2 marks

4 (2 marks)

The complementary function solves $\frac{dy}{dx} + 4y = 0$

M1

$$y = Ae^{-4x}$$

A1

Answer: $y = Ae^{-4x}$

2 marks

5 (2 marks)

Repeated root $m = 2$: $(m - 2)^2 = m^2 - 4m + 4$

M1

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

A1

Answer: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

2 marks

6 (3 marks)

(a) $m = 3$: $9 - 3 - 6 = 0$, so yes

B1

Answer: Yes

1 mark

(b) $m = -2$: $4 + 2 - 6 = 0$, so yes

B1

Answer: Yes

1 mark

(c) $m = 2$: $4 - 2 - 6 = -4 \neq 0$, so no

B1

Answer: No

1 mark

7 (2 marks)

Auxiliary equation: $3m^2 + 10m + 3 = 0$

M1

$m = -3$ or $m = -\frac{1}{3}$, so the complementary function is $y = Ae^{-3x} + Be^{-\frac{1}{3}x}$

A1

Answer: $y = Ae^{-3x} + Be^{-\frac{1}{3}x}$

2 marks

8 (2 marks)

Auxiliary equation: $m^2 + 6m + 25 = 0$

M1

$m = -3 - 4i$ or $-3 + 4i$, so the complementary function is $y = e^{-3x}(A \cos 4x + B \sin 4x)$

A1

Answer: $y = e^{-3x}(A \cos 4x + B \sin 4x)$

2 marks

9 (2 marks)

Auxiliary equation: $4m^2 + 12m + 9 = 0$

M1

Repeated root $m = -\frac{3}{2}$, so the complementary function is $y = (A + Bx)e^{-\frac{3}{2}x}$

A1

Answer: $y = (A + Bx)e^{-\frac{3}{2}x}$

2 marks

10 (2 marks)

Auxiliary equation: $m^2 + 2m + 3 = 0$

M1

$m = -1 - i\sqrt{2}$ or $-1 + i\sqrt{2}$, so the complementary function is $y = e^{-x}(A \cos(x\sqrt{2}) + B \sin(x\sqrt{2}))$

A1

Answer: $y = e^{-x}(A \cos(x\sqrt{2}) + B \sin(x\sqrt{2}))$

2 marks

11 (4 marks)

(a) Auxiliary equation: $m^2 + 6m + 5 = 0$

M1

$m = -5$ or $m = -1$, so the complementary function is $y = Ae^{-5x} + Be^{-x}$

A1

Answer: $y = Ae^{-5x} + Be^{-x}$

2 marks

(b) Auxiliary equation: $m^2 + 6m + 9 = 0$

M1

Repeated root $m = -3$, so the complementary function is $y = (A + Bx)e^{-3x}$

A1

Answer: $y = (A + Bx)e^{-3x}$

2 marks

12 (3 marks)

Roots $m = -1$ and $m = 3$: $(m + 1)(m - 3) = m^2 - 2m - 3$

M1 A1

$a = -2$, $b = -3$

A1

Answer: $a = -2$, $b = -3$

3 marks

13 (3 marks)

Solve $\frac{dy}{dx} = \frac{2y}{x}$: $\int \frac{1}{y} dy = \int \frac{2}{x} dx$

M1

$\ln|y| = 2 \ln x + c$

A1

$y = Cx^2$

A1

Answer: $y = Cx^2$

3 marks

14 (4 marks)

(a) $m^2 + 6m + 25 = 0$, so $m = -3 \pm 4i$

M1 A1

$x = e^{-3t}(A \cos 4t + B \sin 4t)$

A1

Answer: $x = e^{-3t}(A \cos 4t + B \sin 4t)$

3 marks

(b) The factor $e^{-3t} \rightarrow 0$ as $t \rightarrow \infty$, so this part dies away whatever the values of A and B

E1

Answer: It tends to 0 as $t \rightarrow \infty$

1 mark

15 (4 marks)

$y' = y_c' + y_p'$ and $y'' = y_c'' + y_p''$

M1

$y'' + py' + qy = (y_c'' + py_c' + qy_c) + (y_p'' + py_p' + qy_p)$

M1 A1

$= 0 + f(x) = f(x)$

A1

Answer: $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x)$

4 marks

16 (6 marks)

(a) Auxiliary equation $m^2 + 2km + (k + 6) = 0$, discriminant $4k^2 - 4(k + 6)$

M1

Complex roots need $4k^2 - 4k - 24 < 0$, i.e. $k^2 - k - 6 < 0$

A1

Answer: Complex roots when $4k^2 - 4(k + 6) < 0$, i.e. $k^2 - k - 6 < 0$

2 marks

(b) $(k - 3)(k + 2) < 0$

M1

$-2 < k < 3$

A1

Answer: $-2 < k < 3$

2 marks

(c) $m^2 + 2m + 7 = 0$, so $m = -1 \pm i\sqrt{6}$

M1

$y = e^{-x}(A \cos \sqrt{6}x + B \sin \sqrt{6}x)$

A1

Answer: $y = e^{-x}(A \cos \sqrt{6}x + B \sin \sqrt{6}x)$

2 marks

17 (4 marks)

Auxiliary equation: $m^3 - 4m = 0$

M1

$m(m - 2)(m + 2) = 0$, so $m = 0, 2, -2$

M1 A1

$y = A + Be^{2x} + Ce^{-2x}$

A1

Answer: $y = A + Be^{2x} + Ce^{-2x}$

4 marks

18 (7 marks)

(a) $m = -2 \pm 3i$, so $y = e^{-2x}(A \cos 3x + B \sin 3x)$

M1 A1

$A = 3$; $\frac{dy}{dx} = 0$ at $x = 0$: $-2A + 3B = 0$, so $B = 2$

M1 A1

$y = e^{-2x}(3 \cos 3x + 2 \sin 3x)$

Answer: $y = e^{-2x}(3 \cos 3x + 2 \sin 3x)$

4 marks

(b) $R = \sqrt{13}$

B1

$\tan \varphi = \frac{2}{3}$, $\varphi = 0.588$

M1 A1

Answer: $R = \sqrt{13}$, $\varphi = 0.588$

3 marks

19 (4 marks)

(a) $\frac{d\theta}{dt} + 0.2\theta = 0$, so $\theta = Ce^{-0.2t}$

M1 A1

Answer: $\theta = Ce^{-0.2t}$

2 marks

(b) $e^{-0.2t} = \frac{1}{2}$

M1

$$t = \frac{\ln 2}{0.2} = 3.47 \text{ minutes}$$

A1

Answer: 3.47 minutes

2 marks

20 (5 marks)(a) The auxiliary equation is $m^2 + 9 = 0$, not $m^2 - 9 = 0$

E1

The roots are $m = \pm 3i$ (complex), so the complementary function is oscillatory, not exponential

E1

Answer: Roots are $\pm 3i$

2 marks

(b) $y = A \cos 3x + B \sin 3x$

B1

Answer: $y = A \cos 3x + B \sin 3x$

1 mark

(c) Try $y = pe^{3x}$: $9p + 9p = 5$, so $18pe^{3x} = 5e^{3x}$

M1

$$p = \frac{5}{18}, \text{ so } y = \frac{5}{18}e^{3x}$$

A1

Answer: $y = \frac{5}{18}e^{3x}$

2 marks

21 (6 marks)Complementary function has roots $m = -2 \pm i$

M1

 $(m + 2)^2 + 1 = 0$, so $m^2 + 4m + 5 = 0$, giving $a = 4$ and $b = 5$

M1 A1

Particular integral $y = 2x - 1$: $y' = 2$, $y'' = 0$

M1

$$f(x) = 0 + 4(2) + 5(2x - 1)$$

M1

$$f(x) = 10x + 3$$

A1

Answer: $a = 4$, $b = 5$, $f(x) = 10x + 3$

6 marks

22 (6 marks)Auxiliary equation: $m^3 - 2m^2 + 5m = 0$

M1

$$m(m^2 - 2m + 5) = 0$$

M1 A1

$$m = 0 \text{ or } m = 1 \pm 2i$$

M1 A1

$$y = A + e^x(B \cos 2x + C \sin 2x)$$

A1

Answer: $y = A + e^x(B \cos 2x + C \sin 2x)$

6 marks

23 (5 marks)

$$y = x^m: x^2m(m-1)x^{m-2} - xmx^{m-1} - 3x^m = 0$$

M1

$$m(m-1) - m - 3 = 0, \text{ so } m^2 - 2m - 3 = 0$$

M1 A1

$$(m-3)(m+1) = 0, \text{ so } m = 3 \text{ or } m = -1$$

A1

$$y = Ax^3 + \frac{B}{x}$$

A1

Answer: $m = 3 \text{ or } -1$; $y = Ax^3 + \frac{B}{x}$

5 marks

24 (5 marks)Suppose $Ae^{2x} + Be^{-x} = 0$ for all x

M1

$$x = 0: A + B = 0$$

A1

Differentiating: $2Ae^{2x} - Be^{-x} = 0$, and at $x = 0$: $2A - B = 0$

M1 A1

Adding: $3A = 0$, so $A = 0$ and then $B = 0$

A1

Answer: $A = B = 0$

5 marks

25 (7 marks)(a) $m^2 + 2m + 5 = 0$, so $m = -1 \pm 2i$

M1

$$x = e^{-t}(A \cos 2t + B \sin 2t)$$

A1

Answer: $x = e^{-t}(A \cos 2t + B \sin 2t)$

2 marks

(b) Try $x = p \cos t + q \sin t$: $(4p + 2q) \cos t + (4q - 2p) \sin t = 10 \sin t$

M1 A1

$$p = -1, q = 2, \text{ so } x = 2 \sin t - \cos t$$

A1

Answer: $x = 2 \sin t - \cos t$

3 marks

(c) Amplitude $= \sqrt{2^2 + 1^2} = \sqrt{5}$

B1

Answer: $\sqrt{5}$

1 mark

(d) $e^{-t} < 0.01$, so $t > \ln 100 = 4.61$

Answer: $t > 4.61$

B1

1 mark

26 (5 marks)

$m^2 + 2m - 2 = 0$, so $m = -1 \pm \sqrt{3}$

M1 A1

$y = e^{-x}(Pe^{\sqrt{3}x} + Qe^{-\sqrt{3}x})$

A1

$e^{\pm\sqrt{3}x} = \cosh \sqrt{3}x \pm \sinh \sqrt{3}x$

M1

$y = e^{-x}[(P + Q) \cosh \sqrt{3}x + (P - Q) \sinh \sqrt{3}x]$, so $C = P + Q$ and $D = P - Q$

A1

Answer: $y = e^{-x}(C \cosh \sqrt{3}x + D \sinh \sqrt{3}x)$

5 marks

27 (8 marks)

(a) $m^4 - 16 = (m^2 - 4)(m^2 + 4)$

M1

$= (m - 2)(m + 2)(m^2 + 4)$

A1

$m = \pm 2$ or $m = \pm 2i$

A1

$y = Ae^{2x} + Be^{-2x} + C \cos 2x + D \sin 2x$

A1

Answer: $y = Ae^{2x} + Be^{-2x} + C \cos 2x + D \sin 2x$

4 marks

(b) e^{2x} is unbounded as $x \rightarrow \infty$ and e^{-2x} is unbounded as $x \rightarrow -\infty$, so $A = B = 0$

E1 E1

$y = C \cos 2x + D \sin 2x$; $y(0) = 1$ gives $C = 1$

M1

$\frac{dy}{dx} = 2D$ at $x = 0$, so $D = 1$ and $y = \cos 2x + \sin 2x$

A1

Answer: $y = \cos 2x + \sin 2x$

4 marks

28 (8 marks)

(a) Substituting $y = u + iv$ and separating real and imaginary parts: $(au'' + bu' + cu) + i(av'' + bv' + cv) = 0$

M1 A1

A complex number is zero only if both its real and imaginary parts are zero, so u and v are both solutions

E1

Answer: $a \frac{d^2u}{dx^2} + b \frac{du}{dx} + cu = 0$ and $a \frac{d^2v}{dx^2} + b \frac{dv}{dx} + cv = 0$

3 marks

(b) $y' = (-1 + 2i)y$, $y'' = (-1 + 2i)^2y = (-3 - 4i)y$

M1

$(-3 - 4i + (-2 + 4i) + 5)y = 0$

A1

Answer: $\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 5y = 0$

2 marks

(c) $e^{(-1 + 2i)x} = e^{-x}(\cos 2x + i \sin 2x)$

M1

Real and imaginary parts: $y = e^{-x} \cos 2x$ and $y = e^{-x} \sin 2x$

A1 A1

Answer: $y = e^{-x} \cos 2x$ and $y = e^{-x} \sin 2x$

3 marks

29 (9 marks)

(a) $y' = 2x$, $y'' = 2$: $2x^2 - 6x^2 + 4x^2 = 0$

M1 A1

Answer: $2x^2 - 6x^2 + 4x^2 = 0$

2 marks

(b) $y' = 2xv + x^2v'$, $y'' = 2v + 4xv' + x^2v''$

M1 A1

$x^2(2v + 4xv' + x^2v'') - 3x(2xv + x^2v') + 4x^2v$

M1

$= x^4v'' + x^3v' = x^3(xv'' + v') = 0$

A1

Answer: $x \frac{d^2v}{dx^2} + \frac{dv}{dx} = 0$

4 marks

(c) $\frac{d}{dx}(xv') = xv'' + v' = 0$, so $xv' = A$

M1

$v = A \ln x + B$

A1

$y = x^2(A \ln x + B)$

A1

Answer: $y = x^2(A \ln x + B)$

3 marks

30 (9 marks)

(a) $W' = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' = y_1y_2'' - y_1''y_2$ M1 A1

Using $y'' = -py' - qy$ for each: $y_1(-py_2' - qy_2) - (-py_1' - qy_1)y_2$ M1

$= -p(y_1y_2' - y_1'y_2) = -pW$ A1

Answer: $\frac{dW}{dx} = -pW$ 4 marks

(b) $\int \frac{1}{W} dW = \int -p dx$, so $W = Ce^{-px}$ M1 A1

e^{-px} is never zero, so W is zero everywhere only if $C = 0$, otherwise it is never zero E1

Answer: $W = Ce^{-px}$ 3 marks

(c) $W = e^x \cdot 3e^{3x} - e^x \cdot e^{3x} = 2e^{4x}$ M1

$p = -4$, so $Ce^{-px} = Ce^{4x}$ with $C = 2$ A1

Answer: $W = 2e^{4x}$ 2 marks

Particular Integrals

1 (2 marks)

Try $y = p$: substituting gives $2p = 10$, so $p = 5$ M1 A1

Particular integral: $y = 5$

Answer: $y = 5$ 2 marks

2 (3 marks)

(a) $y = px + q$ B1

Answer: $y = px + q$ 1 mark

(b) $y = pe^{2x}$ B1

Answer: $y = pe^{2x}$ 1 mark

(c) $y = p \cos 3x + q \sin 3x$ B1

Answer: $y = p \cos 3x + q \sin 3x$ 1 mark

3 (2 marks)

Try $y = p e^{2x}$: substituting gives $5p e^{2x} = 5e^{2x}$, so $p = 1$ M1 A1

Particular integral: $y = e^{2x}$

Answer: $y = e^{2x}$ 2 marks

4 (3 marks)

Try $y = p + qx$ M1

$y' = q$, $y'' = 0$

Substituting and comparing coefficients: $p = 0$, $q = 2$ M1 A1

Particular integral: $y = 2x$

Answer: $y = 2x$ 3 marks

5 (3 marks)

Try $y = p \cos x + q \sin x$ M1

$y' = -p \sin x + q \cos x$, $y'' = -p \cos x - q \sin x$

Substituting and comparing coefficients: $p = -\frac{6}{5}$, $q = 0$ M1 A1

Particular integral: $y = -\frac{6}{5} \cos x$

Answer: $y = -\frac{6}{5} \cos x$ 3 marks

6 (2 marks)

Try $y = p e^{-2x}$: substituting gives $3p e^{-2x} = 6e^{-2x}$, so $p = 2$ M1 A1

Particular integral: $y = 2e^{-2x}$

Answer: $y = 2e^{-2x}$ 2 marks

7 (4 marks)

Try $y = p + qx + rx^2$

M1

$y' = q + 2rx, y'' = 2r$

Substituting and comparing coefficients: $p = 0, q = 0, r = 1$

M1 A2

Particular integral: $y = x^2$

Answer: $y = x^2$

4 marks

8 (3 marks)

Try $y = p \cos x + q \sin x$

M1

$y' = -p \sin x + q \cos x, y'' = -p \cos x - q \sin x$

Substituting and comparing coefficients: $p = -2, q = 1$

M1 A1

Particular integral: $y = \sin x - 2 \cos x$

Answer: $y = \sin x - 2 \cos x$

3 marks

9 (3 marks) e^x is part of the complementary function ($m = 1$ is a root), so multiply the trial form by x .

Try $y = px e^x$

M1

$y' = px e^x + p e^x, y'' = px e^x + 2p e^x$

Substituting and comparing coefficients: $p = -3$

M1 A1

Particular integral: $y = -3x e^x$

Answer: $y = -3x e^x$

3 marks

10 (3 marks) $\sin 3x$ is part of the complementary function ($m = \pm 3i$), so include a factor x .

Try $y = x(p \cos 3x + q \sin 3x)$

M1

$y' = -3px \sin 3x + p \cos 3x + 3qx \cos 3x + q \sin 3x, y'' = -9px \cos 3x - 6p \sin 3x - 9qx \sin 3x + 6q \cos 3x$

Substituting and comparing coefficients: $p = -1, q = 0$

M1 A1

Particular integral: $y = -x \cos 3x$

Answer: $y = -x \cos 3x$

3 marks

11 (3 marks)The constant term is part of the complementary function ($m = 0$ is a root), so try a polynomial with a factor x .

Try $y = px + qx^2$

M1

$y' = p + 2qx, y'' = 2q$

Substituting and comparing coefficients: $p = -\frac{2}{3}, q = -1$

M1 A1

Particular integral: $y = -x^2 - \frac{2}{3}x$

Answer: $y = -x^2 - \frac{2}{3}x$

3 marks

12 (3 marks)

Try $y = p + q e^{2x}$

M1

$y' = 2q e^{2x}, y'' = 4q e^{2x}$

Substituting and comparing coefficients: $p = -3, q = \frac{2}{3}$

M1 A1

Particular integral: $y = \frac{2}{3}e^{2x} - 3$

Answer: $y = \frac{2}{3}e^{2x} - 3$

3 marks

13 (6 marks)

Auxiliary equation: $m^2 + 2m + 5 = 0$

M1

$m = -1 - 2i$ or $-1 + 2i$, so the complementary function is $y = e^{-x}(A \cos 2x + B \sin 2x)$

A1

Try $y = p \cos 2x + q \sin 2x$

M1

$y' = -2p \sin 2x + 2q \cos 2x$, $y'' = -4p \cos 2x - 4q \sin 2x$

Substituting and comparing coefficients: $p = 1$, $q = 4$

M1 A1

Particular integral: $y = 4 \sin 2x + \cos 2x$

General solution: $y = 4 \sin 2x + \cos 2x + e^{-x}(A \cos 2x + B \sin 2x)$

A1

Answer: $y = 4 \sin 2x + \cos 2x + e^{-x}(A \cos 2x + B \sin 2x)$

6 marks

14 (5 marks)

The auxiliary equation has the repeated root $m = -2$, so e^{-2x} and xe^{-2x} are both in the complementary function

E1

Hence try $y = px^2e^{-2x}$

E1

Try $y = px^2e^{-2x}$

M1

$y' = -2px^2e^{-2x} + 2px e^{-2x}$, $y'' = 4px^2e^{-2x} - 8px e^{-2x} + 2p e^{-2x}$

Substituting and comparing coefficients: $p = 3$

M1 A1

Particular integral: $y = 3x^2e^{-2x}$

Answer: $y = 3x^2e^{-2x}$

5 marks

15 (4 marks)

Try $y = p + qx + r e^x$

M1

$y' = q + r e^x$, $y'' = r e^x$

Substituting and comparing coefficients: $p = 2$, $q = 2$, $r = 2$

M1 A2

Particular integral: $y = 2x + 2e^x + 2$

Answer: $y = 2x + 2e^x + 2$

4 marks

16 (6 marks)

Auxiliary equation: $m^2 + 2m + 2 = 0$

M1

$m = -1 - i$ or $-1 + i$, so the complementary function is $y = e^{-x}(A \cos x + B \sin x)$

A1

Try $y = e^x(p \cos x + q \sin x)$

M1

$y' = -p e^x \sin x + p e^x \cos x + q e^x \sin x + q e^x \cos x$, $y'' = -2p e^x \sin x + 2q e^x \cos x$

Substituting and comparing coefficients: $p = 1$, $q = 1$

M1 A1

Particular integral: $y = e^x(\sin x + \cos x)$

General solution: $y = e^x(\sin x + \cos x) + e^{-x}(A \cos x + B \sin x)$

A1

Answer: $y = e^x(\sin x + \cos x) + e^{-x}(A \cos x + B \sin x)$

6 marks

17 (6 marks)

Auxiliary equation: $m^2 + 4 = 0$

M1

$m = -2i$ or $2i$, so the complementary function is $y = A \cos 2x + B \sin 2x$

A1

Try $y = p e^x + qx e^x$

M1

$y' = p e^x + qx e^x + q e^x$, $y'' = p e^x + qx e^x + 2q e^x$

Substituting and comparing coefficients: $p = 1$, $q = 1$

M1 A1

Particular integral: $y = x e^x + e^x$

General solution: $y = x e^x + e^x + A \cos 2x + B \sin 2x$

A1

Answer: $y = x e^x + e^x + A \cos 2x + B \sin 2x$

6 marks

18 (4 marks)

$$y' = (x + \frac{3}{2}x^2)e^{3x} \quad \text{M1}$$

$$y'' = (1 + 6x + \frac{9}{2}x^2)e^{3x} \quad \text{A1}$$

$$y'' - 6y' + 9y = [1 + 6x + \frac{9}{2}x^2 - 6x - 9x^2 + \frac{9}{2}x^2]e^{3x} = e^{3x} \quad \text{M1 A1}$$

$$\text{Answer: } \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x} \quad \text{4 marks}$$

19 (6 marks)

$$y = x^2: 2 + 2ax + bx^2 \equiv 6x^2 + 8x + 2 \quad \text{M1}$$

$$\text{Comparing coefficients: } b = 6 \text{ and } a = 4 \quad \text{A1 A1}$$

$$m^2 + 4m + 6 = 0, \text{ so } m = -2 \pm i\sqrt{2} \quad \text{M1 A1}$$

$$y = x^2 + e^{-2x}(A \cos \sqrt{2}x + B \sin \sqrt{2}x) \quad \text{A1}$$

$$\text{Answer: } a = 4, b = 6; y = x^2 + e^{-2x}(A \cos \sqrt{2}x + B \sin \sqrt{2}x) \quad \text{6 marks}$$

20 (6 marks)

$$(a) m = 1 \text{ is a root of the auxiliary equation } (m = 1 \text{ or } 2), \text{ so } e^x \text{ is part of the CF} \quad \text{B1}$$

$$\text{Try } y = pxe^x \quad \text{B1}$$

$$\text{Answer: } y = pxe^x \quad \text{2 marks}$$

$$(b) m = \pm 2i, \text{ so } \cos 2x \text{ and } \sin 2x \text{ are part of the CF} \quad \text{B1}$$

$$\text{Try } y = x(p \cos 2x + q \sin 2x) \quad \text{B1}$$

$$\text{Answer: } y = x(p \cos 2x + q \sin 2x) \quad \text{2 marks}$$

$$(c) m = 2 \text{ is a repeated root, so } e^{2x} \text{ and } xe^{2x} \text{ are both in the CF} \quad \text{B1}$$

$$\text{Try } y = px^2e^{2x} \quad \text{B1}$$

$$\text{Answer: } y = px^2e^{2x} \quad \text{2 marks}$$

21 (6 marks)

$$\text{Auxiliary equation: } m^2 + m - 2 = 0 \quad \text{M1}$$

$$m = -2 \text{ or } m = 1, \text{ so the complementary function is } y = Ae^{-2x} + Be^x \quad \text{A1}$$

$$m = 1 \text{ is a simple root of } m^2 + m - 2 = 0, \text{ so multiply the usual form } (px + q)e^x \text{ by } x.$$

$$\text{Try } y = px e^x + qx^2 e^x \quad \text{M1}$$

$$y' = px e^x + p e^x + qx^2 e^x + 2qx e^x, y'' = px e^x + 2p e^x + qx^2 e^x + 4qx e^x + 2q e^x$$

$$\text{Substituting and comparing coefficients: } p = -\frac{2}{3}, q = 1 \quad \text{M1 A1}$$

$$\text{Particular integral: } y = x^2e^x - \frac{2}{3}x e^x$$

$$\text{General solution: } y = x^2e^x - \frac{2}{3}x e^x + Ae^{-2x} + Be^x \quad \text{A1}$$

$$\text{Answer: } y = x^2e^x - \frac{2}{3}x e^x + Ae^{-2x} + Be^x \quad \text{6 marks}$$

22 (7 marks)

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \text{ so } \frac{d^2y}{dx^2} + 4y = \frac{1}{2} - \frac{1}{2}\cos 2x \quad \text{M1}$$

$$\text{For } \frac{1}{2}: \text{ try } y = p, 4p = \frac{1}{2}, p = \frac{1}{8} \quad \text{M1 A1}$$

$$\cos 2x \text{ is in the complementary function, so for } -\frac{1}{2}\cos 2x \text{ try } y = x(q \cos 2x + r \sin 2x) \quad \text{M1}$$

$$y'' + 4y = -4q \sin 2x + 4r \cos 2x = -\frac{1}{2}\cos 2x, \text{ so } q = 0, r = -\frac{1}{8} \quad \text{M1 A1}$$

$$y = \frac{1}{8} - \frac{1}{8}x \sin 2x \quad \text{A1}$$

$$\text{Answer: } y = \frac{1}{8} - \frac{1}{8}x \sin 2x \quad \text{7 marks}$$

23 (6 marks)

(a) Try $y = pe^{ax}$: $p(a^2 - 2a + 1)e^{ax} = e^{ax}$ M1 A1

$$p = \frac{1}{(a-1)^2}, \text{ valid since } a \neq 1$$
 A1

Answer: $y = \frac{e^{ax}}{(a-1)^2}$ 3 marks

(b) e^x and xe^x are both in the complementary function (repeated root $m = 1$) E1

Try $y = px^2e^x$: $y'' - 2y' + y = 2pe^x$ M1

$$p = \frac{1}{2}, \text{ so } y = \frac{1}{2}x^2e^x$$
 A1

Answer: $y = \frac{1}{2}x^2e^x$ 3 marks

24 (6 marks)

Auxiliary equation: $m^2 + 6m + 13 = 0$ M1

$m = -3 - 2i \text{ or } -3 + 2i$, so the complementary function is $y = e^{-3x}(A \cos 2x + B \sin 2x)$ A1

The right-hand side has the same form as the complementary function, so include a factor x in the trial.

Try $y = x e^{-3x}(p \cos 2x + q \sin 2x)$ M1

$$y' = -2px e^{-3x} \sin 2x - 3px e^{-3x} \cos 2x + p e^{-3x} \cos 2x - 3qx e^{-3x} \sin 2x + 2qx e^{-3x} \cos 2x + q$$

$$e^{-3x} \sin 2x, y'' = 12px e^{-3x} \sin 2x + 5px e^{-3x} \cos 2x - 4p e^{-3x} \sin 2x - 6p e^{-3x} \cos 2x + 5qx$$

$$e^{-3x} \sin 2x - 12qx e^{-3x} \cos 2x - 6q e^{-3x} \sin 2x + 4q e^{-3x} \cos 2x$$

Substituting and comparing coefficients: $p = -2, q = 0$ M1 A1

Particular integral: $y = -2x e^{-3x} \cos 2x$

General solution: $y = -2x e^{-3x} \cos 2x + e^{-3x}(A \cos 2x + B \sin 2x)$ A1

Answer: $y = -2x e^{-3x} \cos 2x + e^{-3x}(A \cos 2x + B \sin 2x)$ 6 marks

25 (7 marks)

(a) $y' = ipe^{ix}, y'' = -pe^{ix}$ M1

$$(-1 + 2i + 5)p = 10, \text{ so } p = \frac{10}{4 + 2i} = \frac{10(4 - 2i)}{20} = 2 - i$$
 M1 A1

Answer: $y = (2 - i)e^{ix}$ 3 marks

(b) $e^{ix} = \cos x + i \sin x$, so $(2 - i)e^{ix} = (2 \cos x + \sin x) + i(2 \sin x - \cos x)$ M1

Taking real parts: $y = 2 \cos x + \sin x$ A1

Answer: $y = 2 \cos x + \sin x$ 2 marks

(c) Taking imaginary parts of the result in part (a): $y = 2 \sin x - \cos x$ M1 A1

Answer: $y = 2 \sin x - \cos x$ 2 marks

26 (6 marks)

(a) e^{-3x} is part of the complementary function Ae^{-3x} , so try $y = (px^2 + qx)e^{-3x}$ M1

$$y' = (2px + q - 3px^2 - 3qx)e^{-3x}$$
 M1

$$y' + 3y = (2px + q)e^{-3x} = 4xe^{-3x}$$
 M1 A1

$$p = 2, q = 0, \text{ so } y = 2x^2e^{-3x}$$
 A1

Answer: $y = 2x^2e^{-3x}$ 5 marks

(b) $y = 2x^2e^{-3x} + Ae^{-3x}$ A1

Answer: $y = (2x^2 + A)e^{-3x}$ 1 mark

27 (9 marks)

(a) $y' = (u' + u)e^x, y'' = (u'' + 2u' + u)e^x$ M1 A1

$$(u'' + 2u' + u - 2u' - 2u + u)e^x = u''e^x = 12xe^x$$
 M1 A1

Answer: $\frac{d^2u}{dx^2} = 12x$ 4 marks

(b) $u' = 6x^2 + A$ M1

$$u = 2x^3 + Ax + B$$
 A1

$$y = (2x^3 + Ax + B)e^x$$

A1

Answer: $y = (2x^3 + Ax + B)e^x$

3 marks

(c) A particular integral is $y = 2x^3e^x$

B1

e^x and xe^x are both in the complementary function (repeated root), so the trial must start at x^2

E1

Answer: $y = 2x^3e^x$

2 marks

28 (11 marks)

Auxiliary equation: $m^2 - 1 = 0$

M1

$m = -1$ or $m = 1$, so the complementary function is $y = Ae^{-x} + Be^x$

A1

$m = 1$ is a simple root of $m^2 - 1 = 0$, so try $y = x(px^2 + qx + r)e^x$.

Try $y = px^3e^x + qx^2e^x + rx^3e^x$

M1

$y' = px^3e^x + p^2e^x + qx^2e^x + 2qx^2e^x + rx^3e^x + 3rx^2e^x$, $y'' = px^3e^x + 2p^2e^x + qx^2e^x + 4qx^2e^x + 2q^2e^x + rx^3e^x + 6rx^2e^x + 6rx^2e^x$

Substituting and comparing coefficients: $p = \frac{1}{4}$, $q = -\frac{1}{4}$, $r = \frac{1}{6}$

M1 A2

Particular integral: $y = \frac{1}{6}x^3e^x - \frac{1}{4}x^2e^x + \frac{1}{4}xe^x$

General solution: $y = \frac{1}{6}x^3e^x - \frac{1}{4}x^2e^x + \frac{1}{4}xe^x + Ae^{-x} + Be^x$

A1

$y(0) = 0$: $A + B = 0$

M1

$y'(0) = 0$: $\frac{1}{4} - A + B = 0$

M1

Solving: $A = \frac{1}{8}$, $B = -\frac{1}{8}$

A1

$y = \frac{1}{6}x^3e^x - \frac{1}{4}x^2e^x + \frac{1}{4}xe^x - \frac{1}{8}e^x + \frac{1}{8}e^{-x}$

A1

Answer: $y = \frac{1}{6}x^3e^x - \frac{1}{4}x^2e^x + \frac{1}{4}xe^x - \frac{1}{8}e^x + \frac{1}{8}e^{-x}$

11 marks

29 (9 marks)

(a) Try $y = pe^{ax}$: $p(a^2 - 3a + 2)e^{ax} = e^{ax}$, so $p = \frac{1}{(a-1)(a-2)}$

M1 A1

e^{2x} is a solution of the homogeneous equation, so subtracting $\frac{e^{2x}}{(a-1)(a-2)}$ still gives a particular integral

E1

Answer: $y_a = \frac{e^{ax} - e^{2x}}{(a-1)(a-2)}$

3 marks

(b) $\frac{e^{ax} - e^{2x}}{a-2} = e^{2x} \frac{e^{(a-2)x} - 1}{a-2}$

M1

As $a \rightarrow 2$: $\frac{e^{hx} - 1}{h} \rightarrow x$ where $h = a - 2$

M1 A1

And $\frac{1}{a-1} \rightarrow 1$, so $y_a \rightarrow xe^{2x}$

A1

Answer: $y = xe^{2x}$

4 marks

(c) $y' = (1 + 2x)e^{2x}$, $y'' = (4 + 4x)e^{2x}$

M1

$(4 + 4x - 3 - 6x + 2x)e^{2x} = e^{2x}$

A1

Answer: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$

2 marks

30 (10 marks)

(a) $y'' + 2y' + 5y = [(5 - \omega^2)p + 2\omega q] \cos \omega x + [(5 - \omega^2)q - 2\omega p] \sin \omega x$

M1 A1

$(5 - \omega^2)p + 2\omega q = 1$ and $(5 - \omega^2)q - 2\omega p = 0$

M1

$q = \frac{2\omega p}{5 - \omega^2}$, so $p[(5 - \omega^2) + \frac{4\omega^2}{5 - \omega^2}] = 1$, giving $p = \frac{5 - \omega^2}{D}$ and $q = \frac{2\omega}{D}$

A1

Answer: $p = \frac{5 - \omega^2}{D}$, $q = \frac{2\omega}{D}$

4 marks

$$(b) \text{ Amplitude} = \sqrt{p^2 + q^2} = \sqrt{(5 - \omega^2)^2 + 4\omega^2/D^2} = \frac{\sqrt{D}}{D} = \frac{1}{\sqrt{D}}$$

M1 A1

$$\text{Answer: Amplitude} = \frac{1}{\sqrt{D}}$$

2 marks

$$(c) D = \omega^4 - 6\omega^2 + 25, \text{ so } \frac{dD}{d\omega} = 4\omega^3 - 12\omega = 0 \text{ gives } \omega^2 = 3$$

M1 A1

$$D = 9 - 18 + 25 = 16, \text{ so the greatest amplitude is } \frac{1}{4}$$

M1 A1

$$\text{Answer: } \omega = \sqrt{3}; \text{ amplitude } \frac{1}{4}$$

4 marks

Initial and Boundary Conditions

1 (2 marks)

$$5 = Ae^0, \text{ so } A = 5$$

M1

$$y = 5e^{2x}$$

A1

$$\text{Answer: } y = 5e^{2x}$$

2 marks

2 (3 marks)

$$x = 0, y = 3: A = 3$$

B1

$$y' = -2A \sin 2x + 2B \cos 2x; \text{ at } x = 0, 2B = 4, \text{ so } B = 2$$

M1 A1

$$\text{Answer: } A = 3, B = 2$$

3 marks

3 (3 marks)

$$A + B = 3 \text{ and } A - B = 1$$

M1 A1

$$A = 2, B = 1$$

A1

$$\text{Answer: } A = 2, B = 1$$

3 marks

4 (3 marks)

(a) Initial conditions: both are given at the same value of x

B1

Answer: Initial

1 mark

(b) Boundary conditions: given at two different values of x

B1

Answer: Boundary

1 mark

(c) Initial conditions: both given at $x = 1$

B1

Answer: Initial

1 mark

5 (2 marks)

A second-order equation has a general solution with two arbitrary constants

E1

A first-order equation has only one, and each condition fixes one constant

E1

Answer: Number of conditions = number of arbitrary constants

2 marks

6 (2 marks)

$$y = x^2 + C$$

M1

$$(0, 3): C = 3, \text{ so } y = x^2 + 3$$

A1

$$\text{Answer: } y = x^2 + 3$$

2 marks

7 (3 marks)

$$1 = 3 + C, \text{ so } C = -2$$

M1

$$y = 3 - 2e^{-2x}$$

A1

$$\text{As } x \rightarrow \infty, e^{-2x} \rightarrow 0, \text{ so } y \rightarrow 3$$

A1

$$\text{Answer: } y = 3 - 2e^{-2x}; y \rightarrow 3$$

3 marks

8 (4 marks)

$A + B = 1$

M1

$2A + 3B = 0$

M1

$B = -2, A = 3$

A1

$y = 3e^{2x} - 2e^{3x}$

A1

Answer: $y = 3e^{2x} - 2e^{3x}$

4 marks

9 (4 marks)

$y(0) = 1: A = 1$

M1

$y(\frac{\pi}{2}) = 3: B = 3$

M1 A1

$y = \cos x + 3 \sin x$

A1

Answer: $y = \cos x + 3 \sin x$

4 marks

10 (4 marks)

$y(0) = 0: A + B = 0, \text{ so } B = -A$

M1

$y(1) = Ae^2 - Ae^{-2} = e^2 - e^{-2}, \text{ so } A = 1$

M1 A1

$y = e^{2x} - e^{-2x}$

A1

Answer: $y = e^{2x} - e^{-2x}$

4 marks

11 (4 marks)

$\frac{dy}{dx} = 3x^2 + A; x = 1: 3 + A = 2, \text{ so } A = -1$

M1 A1

$y = x^3 - x + B; (1, 5): 5 = 1 - 1 + B, \text{ so } B = 5$

M1

$y = x^3 - x + 5$

A1

Answer: $y = x^3 - x + 5$

4 marks

12 (3 marks)

$e^{2x} \rightarrow \infty \text{ as } x \rightarrow \infty, \text{ so } B = 0 \text{ for a bounded solution}$

E1

$y(0) = 3: A = 3$

M1

$y = 3e^{-x}$

A1

Answer: $y = 3e^{-x}$

3 marks

13 (4 marks)

$y(0) = 3: A + B = 3$

M1

$\frac{dy}{dx} = 2Ae^{2x} - Be^{-x} = 0 \text{ at } x = 0: 2A - B = 0$

M1

$A = 1, B = 2$

A1 A1

Answer: $A = 1, B = 2$

4 marks

14 (6 marks)

Auxiliary equation: $m^2 + 4 = 0$

M1

$m = -2i \text{ or } 2i, \text{ so the complementary function is } y = A \cos 2x + B \sin 2x$

A1

$y(0) = 1: A = 1$

M1

$y(\frac{1}{4}\pi) = 3: B = 3$

M1

Solving: $A = 1, B = 3$

A1

$y = 3 \sin 2x + \cos 2x$

A1

Answer: $y = 3 \sin 2x + \cos 2x$

6 marks

15 (6 marks)

Auxiliary equation: $m^2 - 2m + 1 = 0$

M1

Repeated root $m = 1$, so the complementary function is $y = (A + Bx)e^x$

A1

$y(0) = 1$: $A = 1$

M1

$y(1) = 3e$: $(A + B)e = 3e$

M1

Solving: $A = 1$, $B = 2$

A1

$y = (2x + 1)e^x$

A1

Answer: $y = (2x + 1)e^x$

6 marks

16 (6 marks)

Auxiliary equation: $m^2 + 2m + 5 = 0$

M1

$m = -1 - 2i$ or $-1 + 2i$, so the complementary function is $y = e^{-x}(A \cos 2x + B \sin 2x)$

A1

$y(0) = 0$: $A = 0$

M1

$y(\frac{1}{4}\pi) = e^{-\frac{1}{4}\pi}$: $Be^{-\frac{1}{4}\pi} = e^{-\frac{1}{4}\pi}$

M1

Solving: $A = 0$, $B = 1$

A1

$y = e^{-x} \sin 2x$

A1

Answer: $y = e^{-x} \sin 2x$

6 marks

17 (7 marks)

Divide by x : $\frac{dy}{dx} + \frac{y}{x} = 2$

Integrating factor: $I = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

M1 A1

$\frac{d}{dx}(xy) = 2x$, so $xy = \int 2x dx = x^2 + C$

M1 A1

$y = x + \frac{C}{x}$

A1

$x = 1$, $y = 3$: $C = 2$

M1

$y = x + \frac{2}{x}$

A1

Answer: $y = x + \frac{2}{x}$

7 marks

18 (4 marks)

(a) $y = Ae^{3x} + Be^{-3x}$; bounded as $x \rightarrow \infty$ requires $A = 0$

M1 E1

$y(0) = 4$: $B = 4$, so $y = 4e^{-3x}$

A1

Answer: $y = 4e^{-3x}$

3 marks

(b) $\frac{dy}{dx} = -12e^{-3x} = -12$ at $x = 0$

B1

Answer: -12

1 mark

19 (5 marks)

(a) $y = A \cos x + B \sin x$; $y(0) = 0$ gives $A = 0$

M1

$y(\pi) = B \sin \pi = 0 \neq 1$

M1

so no solution exists

E1

Answer: No solution

3 marks

(b) $y = B \sin x$ satisfies $y(0) = 0$ and $y(\pi) = 0$ for every value of B

E1

so there are infinitely many solutions

E1

Answer: $y = B \sin x$, infinitely many

2 marks

20 (5 marks)

$y = (A + Bx)e^{-3x}$; $y(0) = 2$: $A = 2$

M1 A1

$\frac{dy}{dx} = (B - 3A - 3Bx)e^{-3x} = 0$ at $x = 1$: $B - 6 - 3B = 0$, so $B = -3$

M1 A1

$y = (2 - 3x)e^{-3x}$; at $x = 1$, $y = -e^{-3}$

A1

Answer: $y = (2 - 3x)e^{-3x}$; $y = -e^{-3}$

5 marks

21 (9 marks)

Auxiliary equation: $m^2 - 6m + 9 = 0$

M1

Repeated root $m = 3$, so the complementary function is $y = (A + Bx)e^{3x}$

A1

Try $y = p$: substituting gives $9p = 18$, so $p = 2$

M1 A1

Particular integral: $y = 2$

General solution: $y = 2 + (A + Bx)e^{3x}$

A1

$y(0) = 3$: $2 + A = 3$

M1

$y'(0) = 1$: $3A + B = 1$

M1

Solving: $A = 1$, $B = -2$

A1

$y = (1 - 2x)e^{3x} + 2$

A1

Answer: $y = (1 - 2x)e^{3x} + 2$

9 marks

22 (6 marks)

$y = A \cos \sqrt{k}x + B \sin \sqrt{k}x$; $y(0) = 0$ gives $A = 0$

M1 A1

$y' = B\sqrt{k} \cos \sqrt{k}x$; $y'(1) = 0$: $B\sqrt{k} \cos \sqrt{k} = 0$

M1

$B \neq 0$ for a non-zero solution, so $\cos \sqrt{k} = 0$

A1

$\sqrt{k} = (n - \frac{1}{2})\pi$, $n = 1, 2, 3, \dots$

M1

$k = (n - \frac{1}{2})^2 \pi^2$

A1

Answer: $k = (n - \frac{1}{2})^2 \pi^2$, $n = 1, 2, 3, \dots$

6 marks

23 (9 marks)

Auxiliary equation: $m^2 + 1 = 0$

M1

$m = -i$ or i , so the complementary function is $y = A \cos x + B \sin x$

A1

Try $y = p$: substituting gives $p = 2$, so $p = 2$

M1 A1

Particular integral: $y = 2$

General solution: $y = 2 + A \cos x + B \sin x$

A1

$y(0) = 0$: $2 + A = 0$

M1

$y(\frac{1}{2}\pi) = 3$: $2 + B = 3$

M1

Solving: $A = -2$, $B = 1$

A1

$y = \sin x - 2 \cos x + 2$

A1

Answer: $y = \sin x - 2 \cos x + 2$

9 marks

24 (7 marks)

(a) Differentiate: $y''' + 2y'' + 5y' = 0$, so $y''' = -2y'' - 5y'$

M1

$y'''' = -2y''' - 5y''$

A1

Answer: $y''' = -2y'' - 5y'$; $y'''' = -2y''' - 5y''$

2 marks

(b) $y'(0) = 1$; $y''(0) = 10 - 2(1) - 5(0) = 8$

M1

$y'''(0) = -16 - 5 = -21$; $y''''(0) = 42 - 40 = 2$

A1

Answer: 1, 8, -21, 2

2 marks

(c) $y = x + 4x^2 - \frac{7}{2}x^3 + \frac{1}{12}x^4 + \dots$

M1 A1

$y(0.1) \approx 0.1 + 0.04 - 0.0035 + 0.0000083 = 0.1365$

A1

Answer: $y(0.1) \approx 0.1365$

3 marks

25 (7 marks)

(a) $\int \frac{1}{4-2y} dy = \int 1 dx$, so $-\frac{1}{2} \ln|4-2y| = x + c$

M1 A1

$4 - 2y = Ke^{-2x}$, and $x = 0$ gives $K = 4 - 2a$

M1

$y = 2 - \frac{1}{2}(4 - 2a)e^{-2x} = 2 + (a - 2)e^{-2x}$

A1

Answer: $y = 2 + (a - 2)e^{-2x}$

4 marks

(b) $\frac{dy}{dx} = -2(a - 2)e^{-2x} > 0$ requires $a < 2$

M1 A1

When $a = 2$, $y = 2$ for all x (a constant solution)

Answer: $a < 2$; $y = 2$

B1

3 marks

26 (6 marks)

$y = e^{-x}(A \cos x + B \sin x)$; $y(0) = 0$ gives $A = 0$

M1 A1

$\frac{dy}{dx} = Be^{-x}(\cos x - \sin x) = 0$, so $\tan x = 1$ and $x = \frac{\pi}{4}$

M1 A1

$y(\frac{\pi}{4}) = B \cdot \frac{\sqrt{2}}{2} e^{-\pi/4} = 1$

M1

$B = \sqrt{2}e^{\pi/4}$, so $y = \sqrt{2}e^{\pi/4 - x} \sin x$

A1

Answer: $y = \sqrt{2}e^{\pi/4 - x} \sin x$

6 marks

27 (9 marks)

(a) $\lambda = 0$: $y = Ax + B$; $y(0) = 0$ gives $B = 0$ and $y(\pi) = 0$ gives $A = 0$

M1 A1

$\lambda = -\mu^2 < 0$: $y = Ae^{\mu x} + Be^{-\mu x}$; $y(0) = 0$ gives $B = -A$, $y(\pi) = A(e^{\mu\pi} - e^{-\mu\pi}) = 0$ so $A = 0$

M1 A1

Answer: Only $y = 0$

4 marks

(b) $y = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x$; $y(0) = 0$ gives $A = 0$

M1

$y(\pi) = B \sin(\sqrt{\lambda}\pi) = 0$ with $B \neq 0$, so $\sqrt{\lambda}\pi = n\pi$

M1 A1

$\lambda = n^2$, $n = 1, 2, 3, \dots$, with $y = B \sin nx$

A1 A1

Answer: $\lambda = n^2$; $y = B \sin nx$

5 marks

28 (10 marks)

(a) CF: $y = A \cos x + B \sin x$

M1 A1

$\sin x$ is in the CF, so try $y = x(p \cos x + q \sin x)$: $y'' + y = -2p \sin x + 2q \cos x = \sin x$

M1 A1

$p = -\frac{1}{2}$, $q = 0$, so $y = A \cos x + B \sin x - \frac{1}{2}x \cos x$

A1

Answer: $y = A \cos x + B \sin x - \frac{1}{2}x \cos x$

5 marks

(b) $y(0) = 0$ gives $A = 0$

B1

$y(\pi) = B \sin \pi - \frac{1}{2}\pi \cos \pi = \frac{\pi}{2} \neq 0$

M1 A1

Answer: $y(\pi) = \frac{\pi}{2} \neq 0$ for every B

3 marks

(c) $y(0) = 0$: $A = 0$; $y(\pi) = \frac{\pi}{2}$ holds for every B

M1

so $y = B \sin x - \frac{1}{2}x \cos x$, for any B

A1

Answer: $y = B \sin x - \frac{1}{2}x \cos x$

2 marks

29 (7 marks)

(a) $y = Ae^x + Be^{2x}$

B1

$A + B = 0$ and $A + 2B = 0$

M1

so $A = B = 0$

A1

Answer: $y = 0$

3 marks

(b) Let $u = y_1 - y_2$. Subtracting the equations gives $u'' - 3u' + 2u = 0$

M1 A1

$u(0) = 0$ and $u'(0) = 0$, so by part (a) $u = 0$ for all x

M1

Hence $y_1 = y_2$

A1

Answer: $y_1 = y_2$ for all x

4 marks

30 (10 marks)

(a) $m^2 + m - 2 = 0$, so $m = 1$ or $m = -2$, and $y = Ae^x + Be^{-2x}$

M1 A1

$A + B = 3$ and $A - 2B = k$

M1

$y = \frac{6+k}{3}e^x + \frac{3-k}{3}e^{-2x}$

A1

Answer: $y = \frac{6+k}{3}e^x + \frac{3-k}{3}e^{-2x}$

4 marks

(b) Need the coefficient of e^x to be zero: $6 + k = 0$, so $k = -6$

M1 A1

Answer: $k = -6$; $y = 3e^{-2x}$

2 marks

(c) $\frac{dy}{dx} = \frac{6+k}{3}e^x - 2\left(\frac{3-k}{3}\right)e^{-2x}$

M1

At $x = \ln 2$: $e^x = 2$, $e^{-2x} = \frac{1}{4}$: $\frac{2(6+k)}{3} - \frac{3-k}{6} = 0$

M1

$4(6+k) - (3-k) = 0$, so $5k = -21$ and $k = -\frac{21}{5}$

A1 A1

Answer: $k = -\frac{21}{5}$

4 marks

Coupled Differential Equations

1 (3 marks)

$x = Ae^{3t}$, with $x(0) = 4$ giving $x = 4e^{3t}$

M1 A1

$y = Be^{-2t}$, with $y(0) = 5$ giving $y = 5e^{-2t}$

A1

Answer: $x = 4e^{3t}$, $y = 5e^{-2t}$

3 marks

2 (2 marks)

$\frac{dx}{dt} = 2e^t$ and $3x - 4y = 6e^t - 4e^t = 2e^t$

M1

$\frac{dy}{dt} = e^t$ and $x - y = 2e^t - e^t = e^t$

A1

Answer: $\frac{dx}{dt} = 2e^t = 3x - 4y$ and $\frac{dy}{dt} = e^t = x - y$

2 marks

3 (1 mark)

$y = \frac{dx}{dt} - 2x$

B1

Answer: $y = \frac{dx}{dt} - 2x$

1 mark

4 (2 marks)

$M = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

B1 B1

Answer: $M = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

2 marks

5 (2 marks)

Differentiate the first equation: $\frac{d^2x}{dt^2} = 5\frac{dx}{dt} + 2\frac{dy}{dt}$

M1

Substitute $\frac{dy}{dt} = x + 4y$

A1

Answer: $\frac{d^2x}{dt^2} = 5\frac{dx}{dt} + 2(x + 4y)$

2 marks

6 (3 marks)

$x + 4 = Ae^t$; $x(0) = 1$: $A = 5$, so $x = 5e^t - 4$

M1 A1

$y = 3e^{2t}$

A1

Answer: $x = 5e^t - 4$, $y = 3e^{2t}$

3 marks

7 (5 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = 0$ A1

Auxiliary equation: $m^2 - 4m + 3 = 0$ M1

$m = 1$ or $m = 3$, so $x = Ae^t + Be^{3t}$ A1

$$y = \frac{dx}{dt} - 2x = -Ae^t + Be^{3t}$$

M1 A1

Answer: $x = Ae^t + Be^{3t}$, $y = -Ae^t + Be^{3t}$ 5 marks

8 (5 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 0$ A1

Auxiliary equation: $m^2 - 2m - 3 = 0$ M1

$m = -1$ or $m = 3$, so $x = Ae^{-t} + Be^{3t}$ A1

$$y = \frac{dx}{dt} - x = -2Ae^{-t} + 2Be^{3t}$$

M1 A1

Answer: $x = Ae^{-t} + Be^{3t}$, $y = -2Ae^{-t} + 2Be^{3t}$ 5 marks

9 (5 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 5x = 0$ A1

Auxiliary equation: $m^2 - 6m + 5 = 0$ M1

$m = 1$ or $m = 5$, so $x = Ae^t + Be^{5t}$ A1

$$y = \frac{dx}{dt} - 4x = -3Ae^t + Be^{5t}$$

M1 A1

Answer: $x = Ae^t + Be^{5t}$, $y = -3Ae^t + Be^{5t}$ 5 marks

10 (5 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} + 4x = 0$ A1

Auxiliary equation: $m^2 + 4 = 0$ M1

$m = -2i$ or $2i$, so $x = A \cos 2t + B \sin 2t$ A1

$$y = \frac{dx}{dt} = -2A \sin 2t + 2B \cos 2t$$

M1 A1

Answer: $x = A \cos 2t + B \sin 2t$, $y = -2A \sin 2t + 2B \cos 2t$ 5 marks

11 (5 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$ A1

Auxiliary equation: $m^2 - 3m + 2 = 0$ M1

$m = 1$ or $m = 2$, so $x = Ae^t + Be^{2t}$ A1

$$y = \frac{1}{2}\left(\frac{dx}{dt} - 3x\right) = -Ae^t - \frac{1}{2}Be^{2t}$$

M1 A1

Answer: $x = Ae^t + Be^{2t}$, $y = -Ae^t - \frac{1}{2}Be^{2t}$ 5 marks

12 (5 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} + 9x = 0$ A1

Auxiliary equation: $m^2 + 9 = 0$ M1

$m = -3i$ or $3i$, so $x = A \cos 3t + B \sin 3t$ A1

$$y = -\frac{1}{2}\left(\frac{dx}{dt} - x\right) = \frac{3}{2}A \sin 3t + \frac{1}{2}A \cos 3t + \frac{1}{2}B \sin 3t - \frac{3}{2}B \cos 3t$$

M1 A1

Answer: $x = A \cos 3t + B \sin 3t$, $y = \frac{3}{2}A \sin 3t + \frac{1}{2}A \cos 3t + \frac{1}{2}B \sin 3t - \frac{3}{2}B \cos 3t$ 5 marks

13 (5 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 9\frac{dx}{dt} + 14x = 0$ A1

Auxiliary equation: $m^2 - 9m + 14 = 0$ M1

$m = 2$ or $m = 7$, so $x = Ae^{2t} + Be^{7t}$ A1

$y = \frac{1}{3}(\frac{dx}{dt} - 5x) = -Ae^{2t} + \frac{2}{3}Be^{7t}$ M1 A1

Answer: $x = Ae^{2t} + Be^{7t}$, $y = -Ae^{2t} + \frac{2}{3}Be^{7t}$ 5 marks

14 (7 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0$ A1

Auxiliary equation: $m^2 - 2m + 1 = 0$

Repeated root $m = 1$, so $x = (A + Bt)e^t$ A1

$x(0) = 2$: $A = 2$ M1

$x'(0) = 2$: $A + B = 2$ M1

Solving: $A = 2$, $B = 0$ A1

$x = 2e^t$ A1

$y = -\frac{1}{4}(\frac{dx}{dt} - 3x) = e^t$ A1

Answer: $x = 2e^t$, $y = e^t$ 7 marks

15 (7 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 8e^t$ A1

Auxiliary equation: $m^2 - 2m - 3 = 0$

$m = -1$ or $m = 3$, so $x = Ae^{-t} + Be^{3t}$ A1

Try $x = pe^t$: substituting gives $-4pe^t = 8e^t$, so $p = -2$ A1

Particular integral: $x = -2e^t$

$x(0) = 1$: $-2 + A + B = 1$

$x'(0) = 1$: $-2 - A + 3B = 1$ M1

Solving: $A = \frac{3}{2}$, $B = \frac{3}{2}$ A1

$x = \frac{3}{2}e^{3t} - 2e^t + \frac{3}{2}e^{-t}$ A1

$y = \frac{1}{2}(\frac{dx}{dt} - x) = \frac{3}{2}e^{3t} - \frac{3}{2}e^{-t}$ A1

Answer: $x = \frac{3}{2}e^{3t} - 2e^t + \frac{3}{2}e^{-t}$, $y = \frac{3}{2}e^{3t} - \frac{3}{2}e^{-t}$ 7 marks

16 (7 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = -12$ A1

Auxiliary equation: $m^2 - 4m + 3 = 0$

$m = 1$ or $m = 3$, so $x = Ae^t + Be^{3t}$ A1

Try $x = p$: substituting gives $3p = -12$, so $p = -4$ A1

Particular integral: $x = -4$

$x(0) = 0$: $-4 + A + B = 0$

$x'(0) = 6$: $A + 3B = 6$ M1

Solving: $A = 3$, $B = 1$ A1

$x = e^{3t} + 3e^t - 4$ A1

$y = \frac{dx}{dt} - 2x - 6 = e^{3t} - 3e^t + 2$ A1

Answer: $x = e^{3t} + 3e^t - 4$, $y = e^{3t} - 3e^t + 2$ 7 marks

17 (5 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$ A1

Auxiliary equation: $m^2 - 5m + 6 = 0$ M1

$m = 2$ or $m = 3$, so $x = Ae^{2t} + Be^{3t}$ A1

$y = -\frac{1}{2}\left(\frac{dx}{dt} - 4x\right) = Ae^{2t} + \frac{1}{2}Be^{3t}$ M1 A1

Answer: $x = Ae^{2t} + Be^{3t}$, $y = Ae^{2t} + \frac{1}{2}Be^{3t}$ 5 marks

18 (4 marks)

(a) $\frac{dx}{dt} = 4Ae^{4t} + 2Be^{2t}$ M1

$3x + y = 4Ae^{4t} + 4Be^{2t} \neq \frac{dx}{dt}$ unless $B = 0$ A1

Answer: Only works if $B = 0$ 2 marks

(b) $y = \frac{dx}{dt} - 3x = (4A - 3A)e^{4t} + (2B - 3B)e^{2t}$ M1

$y = Ae^{4t} - Be^{2t}$ A1

Answer: $y = Ae^{4t} - Be^{2t}$ 2 marks

19 (9 marks)

(a) $\frac{d^2x}{dt^2} = -2\frac{dx}{dt} + 3\frac{dy}{dt} = -2\frac{dx}{dt} + 3(4x - 3y)$ M1 A1

From the first equation $3y = \frac{dx}{dt} + 2x$, so $9y = 3\frac{dx}{dt} + 6x$ M1

$\frac{d^2x}{dt^2} = -2\frac{dx}{dt} + 12x - 3\frac{dx}{dt} - 6x = -5\frac{dx}{dt} + 6x$, giving the result A1

Answer: $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 6x = 0$ 4 marks

(b) $m^2 + 5m - 6 = 0$, so $m = 1$ or $m = -6$ and $x = Ae^t + Be^{-6t}$ M1 A1

$x(0) = 3$: $A + B = 3$; $x'(0) = -2(3) + 3(1) = -3$: $A - 6B = -3$ M1

$A = \frac{15}{7}$, $B = \frac{6}{7}$ A1

$y = \frac{1}{3}\left(\frac{dx}{dt} + 2x\right) = \frac{15}{7}e^t - \frac{8}{7}e^{-6t}$ A1

Answer: $x = \frac{15}{7}e^t + \frac{6}{7}e^{-6t}$, $y = \frac{15}{7}e^t - \frac{8}{7}e^{-6t}$ 5 marks

20 (9 marks)

(a) Rate of change of salt in A = rate in - rate out = $5 \times \frac{y}{100} - 5 \times \frac{x}{100}$, and similarly for B: $\frac{dy}{dt} = 0.05x - 0.05y$ M1 A1

Answer: $\frac{dx}{dt} = -0.05x + 0.05y$ and $\frac{dy}{dt} = 0.05x - 0.05y$ 2 marks

(b) Adding the equations: $\frac{d}{dt}(x + y) = 0$, so $x + y = 20$ M1 A1

$\frac{dx}{dt} = -0.05x + 0.05(20 - x) = 1 - 0.1x$ M1

$x - 10 = Ce^{-0.1t}$; $x(0) = 20$: $C = 10$ M1

$x = 10 + 10e^{-0.1t}$, $y = 10 - 10e^{-0.1t}$ A1

Answer: $x = 10 + 10e^{-0.1t}$, $y = 10 - 10e^{-0.1t}$ 5 marks

(c) As $t \rightarrow \infty$, $x \rightarrow 10$ and $y \rightarrow 10$; $x - y = 20e^{-0.1t} = 5$, so $t = 10 \ln 4 = 13.9$ minutes M1 A1

Answer: 10 kg each; 13.9 minutes 2 marks

21 (8 marks)

(a) $0 = 4p + 2q + 10$ and $0 = p + 3q$, so $p = -3q$ M1

$-12q + 2q + 10 = 0$, so $q = 1$ and $p = -3$ A1

Answer: $x = -3, y = 1$ 2 marks

(b) $\frac{du}{dt} = 4u + 2v$ and $\frac{dv}{dt} = u + 3v$ M1

$\frac{d^2u}{dt^2} - 7\frac{du}{dt} + 10u = 0$, so $m = 2$ or $m = 5$ M1 A1

$u = Ae^{2t} + Be^{5t}$, $v = \frac{1}{2}\left(\frac{du}{dt} - 4u\right) = -Ae^{2t} + \frac{B}{2}e^{5t}$ M1 A1

$x = -3 + Ae^{2t} + Be^{5t}$, $y = 1 - Ae^{2t} + \frac{B}{2}e^{5t}$ A1

Answer: $x = -3 + Ae^{2t} + Be^{5t}$, $y = 1 - Ae^{2t} + \frac{B}{2}e^{5t}$ 6 marks

22 (9 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$ A1

Auxiliary equation: $m^2 - 3m + 2 = 0$ M1

$m = 1$ or $m = 2$, so $x = Ae^t + Be^{2t}$ A1

$x(0) = 1$: $A + B = 1$ M1

$x'(0) = 3$: $A + 2B = 3$ M1

Solving: $A = -1$, $B = 2$ A1

$x = 2e^{2t} - e^t$ A1

$y = \frac{dx}{dt} - 3x = -2e^{2t} + 2e^t$ M1 A1

Answer: $x = 2e^{2t} - e^t$, $y = -2e^{2t} + 2e^t$ 9 marks

23 (9 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 5x = 0$ A1

Auxiliary equation: $m^2 - 2m + 5 = 0$ M1

$m = 1 - 2i$ or $1 + 2i$, so $x = e^t(A \cos 2t + B \sin 2t)$ A1

$x(0) = 2$: $A = 2$ M1

$x'(0) = 6$: $A + 2B = 6$ M1

Solving: $A = 2$, $B = 2$ A1

$x = e^t(2 \sin 2t + 2 \cos 2t)$ A1

$y = \frac{1}{4}\left(\frac{dx}{dt} - x\right) = -e^t \sin 2t + e^t \cos 2t$ M1 A1

Answer: $x = 2e^t \sin 2t + 2e^t \cos 2t$, $y = -e^t \sin 2t + e^t \cos 2t$ 9 marks

24 (9 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 3e^{2t}$ A1

Auxiliary equation: $m^2 - 2m - 3 = 0$ M1

$m = -1$ or $m = 3$, so $x = Ae^{-t} + Be^{3t}$ A1

Try $x = p e^{2t}$: substituting gives $-3p e^{2t} = 3e^{2t}$, so $p = -1$ A1

Particular integral: $x = -e^{2t}$

$x(0) = 0$: $-1 + A + B = 0$ M1

$x'(0) = 5$: $-2 - A + 3B = 5$ M1

Solving: $A = -1$, $B = 2$ A1

$x = 2e^{3t} - e^{2t} - e^{-t}$ A1

$y = \frac{1}{2}\left(\frac{dx}{dt} - x - 3e^{2t}\right) = 2e^{3t} - 2e^{2t} + e^{-t}$ A1

Answer: $x = 2e^{3t} - e^{2t} - e^{-t}$, $y = 2e^{3t} - 2e^{2t} + e^{-t}$ 9 marks

25 (11 marks)

$$(a) \int \frac{1}{x} dx = -0.5 \int 1 dt, \text{ so } x = Ae^{-0.5t} \quad \text{M1}$$

$$x = 40 \text{ when } t = 0, \text{ so } x = 40e^{-0.5t} \quad \text{A1}$$

$$\text{Answer: } x = 40e^{-0.5t} \quad \text{2 marks}$$

$$(b) \frac{dy}{dt} = 0.5(40e^{-0.5t}) - 0.2y = 20e^{-0.5t} - 0.2y \quad \text{B1}$$

$$\text{Integrating factor } e^{0.2t}: \frac{d}{dt}(ye^{0.2t}) = 20e^{-0.3t} \quad \text{M1 A1}$$

$$ye^{0.2t} = -\frac{200}{3}e^{-0.3t} + C \quad \text{A1}$$

$$t = 0, y = 0: C = \frac{200}{3} \quad \text{M1}$$

$$y = \frac{200}{3}(e^{-0.2t} - e^{-0.5t}) \quad \text{A1}$$

$$\text{Answer: } y = \frac{200}{3}(e^{-0.2t} - e^{-0.5t}) \quad \text{6 marks}$$

$$(c) \frac{dy}{dt} = \frac{200}{3}(-0.2e^{-0.2t} + 0.5e^{-0.5t}) = 0 \quad \text{M1}$$

$$e^{0.3t} = 2.5, \text{ so } t = \frac{\ln 2.5}{0.3} = 3.05 \text{ hours} \quad \text{M1 A1}$$

$$\text{Answer: } 3.05 \text{ hours} \quad \text{3 marks}$$

26 (9 marks)

$$(a) \frac{d^2x}{dt^2} = \frac{dy}{dt} \quad \text{M1}$$

$$= 2t - x, \text{ so } \frac{d^2x}{dt^2} + x = 2t \quad \text{A1}$$

$$\text{Answer: } \frac{d^2x}{dt^2} + x = 2t \quad \text{2 marks}$$

$$(b) \text{ CF: } x = A \cos t + B \sin t \quad \text{B1}$$

$$\text{Try } x = pt + q: pt + q = 2t, \text{ so } p = 2 \text{ and } q = 0 \quad \text{M1 A1}$$

$$x = 2t + A \cos t + B \sin t; x = 1 \text{ at } t = 0 \text{ gives } A = 1. \frac{dx}{dt} = y + 2 = 5 \text{ at } t = 0 \text{ gives } 2 + B = 5, \text{ so } B = 3$$

$$x = 2t + \cos t + 3 \sin t \text{ and } y = \frac{dx}{dt} - 2 = 3 \cos t - \sin t \quad \text{A1}$$

$$\text{Answer: } x = 2t + \cos t + 3 \sin t, y = 3 \cos t - \sin t \quad \text{5 marks}$$

$$(c) x - 2t = \cos t + 3 \sin t \text{ and } y = 3 \cos t - \sin t \quad \text{M1}$$

$$(x - 2t)^2 + y^2 = (\cos^2 t + 6 \sin t \cos t + 9 \sin^2 t) + (9 \cos^2 t - 6 \sin t \cos t + \sin^2 t) = 10 \quad \text{A1}$$

$$\text{Answer: } (x - 2t)^2 + y^2 = 10 \quad \text{2 marks}$$

27 (11 marks)

$$(a) \frac{d^2x}{dt^2} = \frac{dy}{dt} = -x - 2y = -x - 2\frac{dx}{dt} \quad \text{M1 A1}$$

$$m = -1 \text{ (repeated): } x = (A + Bt)e^{-t} \quad \text{M1 A1}$$

$$y = \frac{dx}{dt} = (B - A - Bt)e^{-t} \quad \text{M1 A1}$$

$$\text{Answer: } x = (A + Bt)e^{-t}, y = (B - A - Bt)e^{-t} \quad \text{6 marks}$$

$$(b) A = 1, B - A = 1 \text{ so } B = 2 \quad \text{M1 A1}$$

$$x = (1 + 2t)e^{-t}, y = (1 - 2t)e^{-t} \quad \text{A1}$$

$$\frac{dx}{dt} = y = 0 \text{ at } t = \frac{1}{2}, \text{ and } y \text{ changes from positive to negative} \quad \text{M1 A1}$$

$$\text{Answer: } x = (1 + 2t)e^{-t}; \text{ maximum at } t = \frac{1}{2} \quad \text{5 marks}$$

28 (10 marks)

Differentiate the first equation and substitute for $\frac{dy}{dt}$ and y : $\frac{d^2x}{dt^2} + x = 0$

M1 A1

Auxiliary equation: $m^2 + 1 = 0$

M1

$m = -i$ or i , so $x = A \cos t + B \sin t$

A1

$x(0) = 1$: $A = 1$

M1

$x'(0) = 0$: $B = 0$

M1

Solving: $A = 1$, $B = 0$

A1

$x = \cos t$

A1

$y = -\left(\frac{dx}{dt} - 2x\right) = \sin t + 2 \cos t$

M1 A1

Answer: $x = \cos t$, $y = \sin t + 2 \cos t$

10 marks

29 (10 marks)

(a) $\frac{d^2x}{dt^2} = -\frac{dy}{dt} = -4x$, so $x = A \cos 2t + B \sin 2t$

M1 A1

$x(0) = 1$: $A = 1$; $x'(0) = -y(0) = 0$: $B = 0$

M1 A1

$x = \cos 2t$, $y = -\frac{dx}{dt} = 2 \sin 2t$

M1 A1

Answer: $x = \cos 2t$, $y = 2 \sin 2t$

6 marks

(b) $4\cos^2 2t + 4 \sin^2 2t = 4$

M1 A1

The path is the ellipse $4x^2 + y^2 = 4$, so both populations oscillate with period π

B1 E1

Answer: $4x^2 + y^2 = 4$: ellipse, period π

4 marks

30 (12 marks)

(a) $\frac{d^2x}{dt^2} = a \frac{dx}{dt} + \frac{dy}{dt} = a \frac{dx}{dt} - x + ay$

M1 A1

$ay = a\left(\frac{dx}{dt} - ax\right)$, so $\frac{d^2x}{dt^2} = a \frac{dx}{dt} - x + a \frac{dx}{dt} - a^2x$

M1

$= 2a \frac{dx}{dt} - (a^2 + 1)x$, giving the result

A1

Answer: $\frac{d^2x}{dt^2} - 2a \frac{dx}{dt} + (a^2 + 1)x = 0$

4 marks

(b) $m = a \pm i$, so $x = e^{at}(P \cos t + Q \sin t)$

M1 A1

$x(0) = 1$: $P = 1$; $x'(0) = a$: $aP + Q = a$, so $Q = 0$

M1 A1

$x = e^{at} \cos t$, $y = \frac{dx}{dt} - ax = -e^{at} \sin t$

A1

Answer: $x = e^{at} \cos t$, $y = -e^{at} \sin t$

5 marks

(c) $e^{at} \rightarrow 0$ requires $a < 0$

B1

$a = 0$: $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, a circle of radius 1 traversed clockwise

M1 A1

Answer: $a < 0$; unit circle

3 marks

Differential Equation Modelling**1 (3 marks)**

$P = 500e^{0.03t}$

M1 A1

$t = 10$: $P = 500e^{0.3} = 675$

A1

Answer: $P = 500e^{0.03t}$; 675

3 marks

2 (2 marks)

$\int \frac{1}{\theta - 20} d\theta = \int -k dt$

M1

$\ln|\theta - 20| = -kt + c$, so $\theta - 20 = Ae^{-kt}$

A1

Answer: $\theta = 20 + Ae^{-kt}$

2 marks

3 (2 marks)

$$N = N_0 e^{-\lambda t}, \text{ half-life: } \frac{1}{2} = e^{-5\lambda}$$

M1

$$\lambda = \frac{\ln 2}{5}$$

A1

$$\text{Answer: } \lambda = \frac{\ln 2}{5}$$

2 marks

4 (2 marks)

$$v = 9.8t + c, \text{ with } v = 0 \text{ at } t = 0 \text{ so } c = 0$$

M1

$$v(3) = 29.4 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 29.4 \text{ m s}^{-1}$$

2 marks

5 (2 marks)

$$\text{Comparing with } \frac{d^2x}{dt^2} = -\omega^2 x \text{ gives } \omega = 4$$

B1

$$\text{Period} = \frac{2\pi}{\omega} = \frac{\pi}{2} \text{ seconds}$$

B1

$$\text{Answer: } \omega = 4; \text{ period } \frac{\pi}{2} \text{ s}$$

2 marks

6 (4 marks)

$$\theta - 20 = Ae^{-0.1t}, t = 0: A = 70, \text{ so } \theta = 20 + 70e^{-0.1t}$$

M1 A1

$$50 = 20 + 70e^{-0.1t}, \text{ so } e^{-0.1t} = \frac{3}{7}$$

M1

$$t = 10 \ln\left(\frac{7}{3}\right) = 8.47 \text{ minutes}$$

A1

$$\text{Answer: } 8.47 \text{ minutes}$$

4 marks

7 (4 marks)

$$\int \frac{1}{5 - 0.2C} dC = \int 1 dt, \text{ so } -5 \ln|5 - 0.2C| = t + c$$

M1

$$5 - 0.2C = 5e^{-0.2t}$$

A1

$$C = 25(1 - e^{-0.2t})$$

A1

$$\text{As } t \rightarrow \infty, C \rightarrow 25 \text{ mg per litre}$$

A1

$$\text{Answer: } C = 25(1 - e^{-0.2t}); 25$$

4 marks

8 (4 marks)

$$\text{Terminal velocity: } \frac{dv}{dt} = 0, \text{ so } v = 19.6$$

M1 A1

$$\int \frac{1}{9.8 - 0.5v} dv = \int 1 dt, \text{ giving } 9.8 - 0.5v = 9.8e^{-0.5t}$$

M1

$$v = 19.6(1 - e^{-0.5t})$$

A1

$$\text{Answer: } v = 19.6(1 - e^{-0.5t}); 19.6 \text{ m s}^{-1}$$

4 marks

9 (4 marks)

$$x = A \cos 5t + B \sin 5t; A = 0.2, B = 0$$

M1 A1

$$x = 0.2 \cos 5t; \frac{dx}{dt} = -\sin 5t, \text{ so maximum speed } 1 \text{ m s}^{-1}$$

A1

$$\text{Period } \frac{2\pi}{5} \text{ s}$$

A1

$$\text{Answer: } x = 0.2 \cos 5t; 1 \text{ m s}^{-1}; \frac{2\pi}{5} \text{ s}$$

4 marks

10 (4 marks)

$$P - 500 = Ae^{0.1t}; t = 0: A = -100, \text{ so } P = 500 - 100e^{0.1t}$$

M1 A1

$$P = 0: e^{0.1t} = 5, \text{ so } t = 10 \ln 5 = 16.1 \text{ years}$$

M1 A1

$$\text{Answer: } P = 500 - 100e^{0.1t}; 16.1 \text{ years}$$

4 marks

11 (5 marks)

$$200 \frac{dv}{dt} = -40v, \text{ so } \frac{dv}{dt} = -0.2v \quad \text{M1}$$

$$\int \frac{1}{v} dv = -0.2 \int 1 dt, \text{ so } v = Ae^{-0.2t}; v = 8 \text{ at } t = 0, \text{ so } v = 8e^{-0.2t} \quad \text{M1 A1}$$

$$\text{Distance} = \int_0^{\infty} 8e^{-0.2t} dt = [-40e^{-0.2t}] \text{ between } 0 \text{ and } \infty \quad \text{M1}$$

$$= 0 - (-40) = 40 \text{ m} \quad \text{A1}$$

$$\text{Answer: } v = 8e^{-0.2t}; 40 \text{ m} \quad \text{5 marks}$$

12 (6 marks)

$$\text{(a) Rate in} = 4 \times 0.5 = 2; \text{ rate out} = 4 \times \frac{x}{200} = \frac{x}{50} \quad \text{M1 A1}$$

$$\text{Answer: } \frac{dx}{dt} = 2 - \frac{x}{50} \quad \text{2 marks}$$

$$\text{(b) } \int \frac{1}{2 - x/50} dx: -50 \ln|2 - \frac{x}{50}| = t + c \quad \text{M1}$$

$$x = 100(1 - e^{-t/50}) \text{ using } x = 0 \text{ at } t = 0 \quad \text{A1}$$

$$50 = 100(1 - e^{-t/50}), \text{ so } t = 50 \ln 2 = 34.7 \text{ minutes} \quad \text{M1 A1}$$

$$\text{Answer: } x = 100(1 - e^{-t/50}); 34.7 \text{ minutes} \quad \text{4 marks}$$

13 (6 marks)

$$\text{(a) } \frac{1}{P(1 - P/1000)} = \frac{1}{P} + \frac{1}{1000 - P} \quad \text{M1}$$

$$\ln P - \ln(1000 - P) = 0.2t + c, \text{ so } \frac{P}{1000 - P} = Ae^{0.2t}; A = \frac{1}{9} \quad \text{M1 A1}$$

$$P = \frac{1000}{1 + 9e^{-0.2t}} \quad \text{A1}$$

$$\text{Answer: } P = \frac{1000}{1 + 9e^{-0.2t}} \quad \text{4 marks}$$

$$\text{(b) } 1 + 9e^{-0.2t} = 2, \text{ so } e^{-0.2t} = \frac{1}{9} \quad \text{M1}$$

$$t = 5 \ln 9 = 11.0 \quad \text{A1}$$

$$\text{Answer: } 11.0 \quad \text{2 marks}$$

14 (5 marks)

$$m^2 + 6m + 8 = 0, \text{ so } m = -2 \text{ or } -4; \text{PI } x = 2 \quad \text{M1 A1}$$

$$x = 2 + Ae^{-2t} + Be^{-4t} \quad \text{A1}$$

$$A + B = -2, -2A - 4B = 0 \quad \text{M1}$$

$$A = -4, B = 2, \text{ so } x = 2 - 4e^{-2t} + 2e^{-4t}; \text{long-term value } 2 \quad \text{A1}$$

$$\text{Answer: } x = 2 - 4e^{-2t} + 2e^{-4t}; 2 \quad \text{5 marks}$$

15 (5 marks)

$$\frac{d\theta}{dt} + 0.5\theta = 7.5 + 0.5t; \text{ try } \theta = at + b: a + 0.5at + 0.5b = 7.5 + 0.5t \quad \text{M1}$$

$$0.5a = 0.5 \text{ gives } a = 1, \text{ and } a + 0.5b = 7.5 \text{ gives } b = 13 \quad \text{A1}$$

$$\text{CF: } \theta = Ce^{-0.5t}, \text{ so } \theta = t + 13 + Ce^{-0.5t} \quad \text{M1}$$

$$t = 0, \theta = 5: C = -8 \quad \text{M1}$$

$$\theta = t + 13 - 8e^{-\frac{1}{2}t} \quad \text{A1}$$

$$\text{Answer: } \theta = t + 13 - 8e^{-\frac{1}{2}t} \quad \text{5 marks}$$

16 (7 marks)

$$\text{(a) Volume after } t \text{ minutes is } 100 + 2t, \text{ so } \frac{dx}{dt} = 6(0.5) - 4 \times \frac{x}{100 + 2t} = 3 - \frac{2x}{50 + t} \quad \text{M1 A1}$$

$$\text{Answer: } \frac{dx}{dt} + \frac{2x}{50 + t} = 3 \quad \text{2 marks}$$

$$\text{(b) } I = e^{\int 2/(50+t) dt} = (50 + t)^2 \quad \text{M1 A1}$$

$$\frac{d}{dt}[(50 + t)^2 x] = 3(50 + t)^2, \text{ so } (50 + t)^2 x = (50 + t)^3 + C \quad \text{M1}$$

$$t = 0, x = 0: C = -50^3, \text{ so } x = (50 + t) - \frac{125000}{(50 + t)^2} \quad \text{A1}$$

$$t = 10: x = 60 - \frac{125000}{3600} = 25.3 \text{ kg} \quad \text{A1}$$

$$\text{Answer: } x = (50 + t) - \frac{125000}{(50 + t)^2}; 25.3 \text{ kg} \quad \text{5 marks}$$

17 (5 marks)

$$\theta = 20 + 60e^{-kt} \quad \text{M1 A1}$$

$$50 = 20 + 60e^{-5k}, \text{ so } e^{-5k} = \frac{1}{2} \text{ and } k = \frac{\ln 2}{5} \quad \text{M1 A1}$$

$$\theta(10) = 20 + 60 \times \frac{1}{4} = 35^\circ \text{C} \quad \text{A1}$$

$$\text{Answer: } k = \frac{\ln 2}{5}; 35^\circ \text{C} \quad \text{5 marks}$$

18 (5 marks)

$$B - 30000 = Ae^{0.04t} \quad \text{M1 A1}$$

$$t = 0: A = -10000, \text{ so } B = 30000 - 10000e^{0.04t} \quad \text{A1}$$

$$B = 0: e^{0.04t} = 3, \text{ so } t = 25 \ln 3 = 27.5 \text{ years} \quad \text{M1 A1}$$

$$\text{Answer: } B = 30000 - 10000e^{0.04t}; 27.5 \text{ years} \quad \text{5 marks}$$

19 (7 marks)

$$\text{(a) Newton's second law: } 2v \frac{dv}{dx} = \frac{40}{(x + 1)^2} \quad \text{M1}$$

$$\int v \, dv = \int \frac{20}{(x + 1)^2} \, dx, \text{ so } \frac{1}{2}v^2 = -\frac{20}{x + 1} + c \quad \text{M1 A1}$$

$$x = 1, v = 1: \frac{1}{2} = -10 + c, \text{ so } c = \frac{21}{2} \text{ and } v^2 = 21 - \frac{40}{x + 1} \quad \text{A1}$$

$$\text{Answer: } v^2 = 21 - \frac{40}{x + 1} \quad \text{4 marks}$$

$$\text{(b) } v^2 = 21 - \frac{40}{8} = 16 \quad \text{M1}$$

$$v = 4 \text{ m s}^{-1} \quad \text{A1}$$

$$\text{Answer: } 4 \text{ m s}^{-1} \quad \text{2 marks}$$

$$\text{(c) As } x \rightarrow \infty, \frac{40}{x + 1} \rightarrow 0, \text{ so } v^2 \rightarrow 21 \quad \text{B1}$$

$$\text{Answer: } \sqrt{21} \text{ m s}^{-1} \quad \text{1 mark}$$

20 (9 marks)

$$\text{(a) } \frac{dv}{dt} = 0: v^2 = 98, \text{ so } v = 7\sqrt{2} \text{ m s}^{-1} \quad \text{M1 A1}$$

$$\text{Answer: } 7\sqrt{2} \text{ m s}^{-1} \quad \text{2 marks}$$

$$\text{(b) } \frac{dv}{dt} = 0.1(98 - v^2), \text{ so } \int \frac{1}{98 - v^2} \, dv = \int 0.1 \, dt \quad \text{M1}$$

$$\frac{1}{14\sqrt{2}} \ln \left| \frac{7\sqrt{2} + v}{7\sqrt{2} - v} \right| = 0.1t + c, \text{ with } c = 0 \quad \text{M1 A1}$$

$$\frac{7\sqrt{2} + v}{7\sqrt{2} - v} = e^{1.4\sqrt{2}t}, \text{ so } v = 7\sqrt{2} \tanh(0.7\sqrt{2}t) \quad \text{M1 A1}$$

$$\text{Answer: } v = 7\sqrt{2} \tanh(0.7\sqrt{2}t) \quad \text{5 marks}$$

$$\text{(c) } \tanh(0.7\sqrt{2}t) = 0.9, \text{ so } 0.7\sqrt{2}t = \tanh^{-1} 0.9 = 1.472 \quad \text{M1}$$

$$t = 1.49 \text{ s} \quad \text{A1}$$

$$\text{Answer: } 1.49 \text{ s} \quad \text{2 marks}$$

21 (9 marks)

(a) $x' = -2p \sin 2t + 2q \cos 2t$, $x'' = -4p \cos 2t - 4q \sin 2t$ M1

$(p + 4q) \cos 2t + (q - 4p) \sin 2t = 10 \cos 2t$, so $q = 4p$ and $17p = 10$ M1 A1

$$x = \frac{10}{17} \cos 2t + \frac{40}{17} \sin 2t$$
 A1

Answer: $x = \frac{10}{17} \cos 2t + \frac{40}{17} \sin 2t$ 4 marks

(b) CF: $m = -1 \pm 2i$, so $x = e^{-t}(A \cos 2t + B \sin 2t) + \text{steady state}$ M1

$$x(0) = 0: A = -\frac{10}{17}$$
 A1

$$x'(0) = 0: -A + 2B + \frac{80}{17} = 0, \text{ so } 2B = -\frac{90}{17} \text{ and } B = -\frac{45}{17}$$
 M1 A1

Answer: $A = -\frac{10}{17}$, $B = -\frac{45}{17}$ 4 marks

(c) $\sqrt{(10/17)^2 + (40/17)^2} = \frac{10}{\sqrt{17}} = 2.43$ B1

Answer: 2.43 1 mark

22 (8 marks)

(a) $m^2 + 6m + 8 = 0$, so $m = -2$ or -4 ; PI $q = \frac{3}{2}$ M1 A1

$$q = \frac{3}{2} + Ae^{-2t} + Be^{-4t}; A + B = -\frac{3}{2}, -2A - 4B = 0$$
 M1

$$A = -3, B = \frac{3}{2}, \text{ so } q = \frac{3}{2} - 3e^{-2t} + \frac{3}{2}e^{-4t}$$
 A1

Answer: $q = \frac{3}{2} - 3e^{-2t} + \frac{3}{2}e^{-4t}$ 4 marks

(b) $i = 6e^{-2t} - 6e^{-4t}$ B1

$$\frac{di}{dt} = -12e^{-2t} + 24e^{-4t} = 0, \text{ so } e^{-2t} = \frac{1}{2}, t = \frac{1}{2} \ln 2$$
 M1 A1

$$i = 6\left(\frac{1}{2}\right) - 6\left(\frac{1}{4}\right) = 1.5 \text{ A}$$
 A1

Answer: $t = \frac{1}{2} \ln 2$; 1.5 A 4 marks

23 (9 marks)

(a) $0.5 \frac{d^2x}{dt^2} = -16x - 4 \frac{dx}{dt}$ M1 A1

Multiplying by 2 gives the result A1

Answer: $\frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 32x = 0$ 3 marks

(b) $m^2 + 8m + 32 = 0$, so $m = -4 \pm 4i$; $x = e^{-4t}(A \cos 4t + B \sin 4t)$ M1 A1

$$A = 0.1; x'(0) = -4A + 4B = 0, \text{ so } B = 0.1$$
 M1 A1

Answer: $x = 0.1e^{-4t}(\cos 4t + \sin 4t)$ 4 marks

(c) The oscillation has period $\frac{2\pi}{4} = \frac{\pi}{2}$ M1

$$\text{Ratio} = e^{-4 \times \pi/2} = e^{-2\pi} = 0.00187$$
 A1

Answer: $e^{-2\pi} = 0.00187$ 2 marks

24 (7 marks)

(a) $\int h^{-1/2} dh = -\int k dt$, so $2\sqrt{h} = -kt + c$ M1 A1

$$t = 0, h = 4: c = 4; h = 0 \text{ at } t = 10: 0 = -10k + 4, \text{ so } k = 0.4$$
 M1 A1

$$\sqrt{h} = 2 - 0.2t = 2 - \frac{t}{5}, \text{ so } h = \left(2 - \frac{t}{5}\right)^2$$
 A1

Answer: $k = 0.4$; $h = \left(2 - \frac{t}{5}\right)^2$ 5 marks

(b) $h = 2: 2 - \frac{t}{5} = \sqrt{2}$ M1

$$t = 5(2 - \sqrt{2}) = 2.93 \text{ minutes}$$
 A1

Answer: 2.93 minutes**2 marks****25 (7 marks)**

$$(a) \frac{1}{N(1000 - N)} = \frac{1}{1000} \left(\frac{1}{N} + \frac{1}{1000 - N} \right)$$

M1

$$\frac{1}{1000} [\ln N - \ln(1000 - N)] = kt + c, \text{ so } \frac{N}{1000 - N} = Ae^{1000kt}, A = \frac{1}{9}$$

M1 A1

$$N = \frac{1000}{1 + 9e^{-1000kt}}$$

A1

$$\text{Answer: } N = \frac{1000}{1 + 9e^{-1000kt}}$$

4 marks

$$(b) 400 = \frac{1000}{1 + 9e^{-2000k}}, \text{ so } e^{-2000k} = \frac{1}{6}, \text{ and } 1000k = \frac{1}{2} \ln 6$$

M1 A1

$$e^{-4000k} = \frac{1}{36}, \text{ so } N = \frac{1000}{1 + \frac{9}{36}} = 800$$

A1

$$\text{Answer: } 1000k = \frac{1}{2} \ln 6; 800$$

3 marks**26 (9 marks)**

$$(a) \int \frac{1}{\theta - \theta_0} d\theta = -k \int 1 dt, \text{ so } \ln|\theta - \theta_0| = -kt + c$$

M1 A1

$$\theta - \theta_0 = Ae^{-kt}; t = 0, \theta = 90 \text{ gives } A = 90 - \theta_0, \text{ hence the result}$$

A1

$$\text{Answer: } \theta = \theta_0 + (90 - \theta_0)e^{-kt}$$

3 marks

$$(b) e^{-5k} = \frac{60 - \theta_0}{90 - \theta_0} \text{ and } e^{-10k} = \frac{45 - \theta_0}{90 - \theta_0}$$

M1

$$\frac{45 - \theta_0}{90 - \theta_0} = \left(\frac{60 - \theta_0}{90 - \theta_0} \right)^2$$

M1

$$(45 - \theta_0)(90 - \theta_0) = (60 - \theta_0)^2, \text{ so } 4050 - 135\theta_0 = 3600 - 120\theta_0$$

A1

$$\theta_0 = 30$$

A1

$$\text{Answer: } \theta_0 = 30$$

4 marks

$$(c) e^{-5k} = \frac{30}{60} = \frac{1}{2}$$

M1

$$k = \frac{\ln 2}{5}$$

A1

$$\text{Answer: } k = \frac{\ln 2}{5}$$

2 marks**27 (12 marks)**

$$(a) \text{ CF: } x = A \cos 2t + B \sin 2t, \text{ which already contains the form of the right-hand side}$$

M1

$$\text{Try } x = t(p \cos 2t + q \sin 2t): x'' + 4x = -4p \sin 2t + 4q \cos 2t = 8 \cos 2t, \text{ so } p = 0, q = 2$$

M1 A1

$$x(0) = 0: A = 0; x'(0) = 2B = 0, \text{ so } B = 0$$

M1 A1

$$x = 2t \sin 2t$$

A1

$$\text{Answer: } x = 2t \sin 2t$$

6 marks

$$(b) \frac{dx}{dt} = 2 \sin 2t + 4t \cos 2t = 0, \text{ so } \tan 2t = -2t$$

M1 A1

$$\text{Solving numerically in } \left(\frac{\pi}{4}, \frac{\pi}{2} \right): t = 1.014$$

M1

$$x = 2t \sin 2t = 1.82$$

A1

$$\text{Answer: } \tan 2t = -2t; 1.82$$

4 marks

$$(c) \text{ The factor } 2t \text{ in } x = 2t \sin 2t \text{ grows without bound, so the oscillations have an ever-increasing amplitude (resonance)}$$

E1 E1

$$\text{Answer: Amplitude grows without limit}$$

2 marks

28 (9 marks)

$$(a) \frac{1}{(4-x)(2-x)} = \frac{1}{2} \left(\frac{1}{2-x} - \frac{1}{4-x} \right) \quad \text{M1 A1}$$

$$\frac{1}{2} [-\ln(2-x) + \ln(4-x)] = 0.1t + c \quad \text{M1 A1}$$

$$\frac{4-x}{2-x} = 2e^{0.2t} \text{ (using } x=0) \quad \text{M1}$$

$$4-x = 2e^{0.2t}(2-x), \text{ so } x(2e^{0.2t} - 1) = 4e^{0.2t} - 4 \quad \text{A1}$$

$$\text{Answer: } x = \frac{4(e^{0.2t} - 1)}{2e^{0.2t} - 1} \quad \text{6 marks}$$

$$(b) \text{ As } t \rightarrow \infty, x \rightarrow \frac{4e^{0.2t}}{2e^{0.2t}} = 2 \quad \text{B1}$$

$$1.5(2e^{0.2t} - 1) = 4e^{0.2t} - 4, \text{ so } 3e^{0.2t} - 1.5 = 4e^{0.2t} - 4 \text{ and } e^{0.2t} = 2.5 \quad \text{M1}$$

$$t = 5 \ln 2.5 = 4.58 \text{ minutes} \quad \text{A1}$$

Answer: 2 g; 4.58 minutes 3 marks

29 (10 marks)

$$(a) \text{ CF: } \theta = Ce^{-0.5t}; \text{ for the constant try } \theta = 10 \quad \text{M1 A1}$$

$$\text{For } 2 \sin t \text{ try } \theta = p \cos t + q \sin t: (q + 0.5p) \cos t + (-p + 0.5q) \sin t = 2 \sin t \quad \text{M1}$$

$$q = -0.5p \text{ and } -1.25p = 2, \text{ so } p = -\frac{8}{5}, q = \frac{4}{5} \quad \text{M1 A1}$$

$$\theta = 10 - \frac{8}{5} \cos t + \frac{4}{5} \sin t + Ce^{-0.5t}; t = 0: C = -\frac{2}{5} \quad \text{M1 A1}$$

$$\text{Answer: } \theta = 10 - \frac{8}{5} \cos t + \frac{4}{5} \sin t - \frac{2}{5}e^{-t/2} \quad \text{7 marks}$$

$$(b) \text{ The exponential term tends to 0, leaving } \theta \approx 10 - \frac{8}{5} \cos t + \frac{4}{5} \sin t, \text{ so the mean is } 10^\circ \text{C} \quad \text{B1}$$

$$\text{Amplitude of } -\frac{8}{5} \cos t + \frac{4}{5} \sin t \text{ is } \sqrt{\frac{64}{25} + \frac{16}{25}} = \frac{4\sqrt{5}}{5} = 1.789 \quad \text{M1}$$

$$\text{Greatest temperature } \approx 10 + 1.789 = 11.8^\circ \text{C} \quad \text{A1}$$

Answer: Mean 10 °C; greatest 11.8 °C 3 marks

30 (10 marks)

$$(a) \text{ Newton's second law: } \frac{dv}{dt} = -9.8 - 0.1v^2 \quad \text{M1 A1}$$

$$\frac{dv}{dt} = v \frac{dv}{dx} \quad \text{A1}$$

$$\text{Answer: } v \frac{dv}{dx} = -(9.8 + 0.1v^2) \quad \text{3 marks}$$

$$(b) \int \frac{v}{9.8 + 0.1v^2} dv = -\int 1 dx, \text{ so } 5 \ln(9.8 + 0.1v^2) = -x + c \quad \text{M1 A1}$$

$$x = 0, v = 20: c = 5 \ln 49.8 \quad \text{M1}$$

$$v = 0: x = 5 \ln\left(\frac{49.8}{9.8}\right) = 8.13 \text{ m} \quad \text{A1}$$

Answer: 8.13 m 4 marks

$$(c) \text{ Greater, because there is no resistive force opposing the motion} \quad \text{E1}$$

$$\text{Height} = \frac{20^2}{2 \times 9.8} = 20.4 \text{ m} \quad \text{M1 A1}$$

Answer: Greater; 20.4 m 3 marks