

Differential Equations of Motion

1 (2 marks)

$$v = \int (6t - 2) \, dt = 3t^2 - 2t + c, \text{ and } v = 1 \text{ when } t = 0 \text{ so } c = 1$$

M1

$$\text{When } t = 3, v = 27 - 6 + 1 = 22$$

A1

$$\text{Answer: } 22 \, \text{m s}^{-1}$$

2 marks

2 (3 marks)

$$F = ma \text{ gives } a = \frac{8t}{4} = 2t$$

M1

$$v = \int 2t \, dt = t^2 \text{ (} v = 0 \text{ when } t = 0 \text{)}$$

A1

$$\text{When } t = 2, v = 4 \, \text{m s}^{-1}$$

A1

$$\text{Answer: } 4 \, \text{m s}^{-1}$$

3 marks

3 (3 marks)

$$v = \int 12e^{-2t} \, dt = -6e^{-2t} + c$$

M1

$$v = 0 \text{ when } t = 0, \text{ so } c = 6 \text{ and } v = 6(1 - e^{-2t})$$

A1

$$\text{When } t = \ln 2, e^{-2t} = \frac{1}{4}, \text{ so } v = 6 \times \frac{3}{4} = 4.5 \, \text{m s}^{-1}$$

A1

$$\text{Answer: } 4.5 \, \text{m s}^{-1}$$

3 marks

4 (3 marks)

$$\int \frac{1}{v} \, dv = \int -2 \, dt, \text{ so } \ln v = -2t + c$$

M1

$$v = 10 \text{ when } t = 0 \text{ gives } c = \ln 10$$

M1

$$v = 10e^{-2t}$$

A1

$$\text{Answer: } v = 10e^{-2t}$$

3 marks

5 (3 marks)

$$a = v \frac{dv}{dx} = 2x$$

B1

$$\int v \, dv = \int 2x \, dx, \text{ so } \frac{1}{2}v^2 = x^2 + c$$

M1

$$v = 3 \text{ at } x = 0 \text{ gives } c = \frac{9}{2}, \text{ so } v^2 = 2x^2 + 9$$

A1

$$\text{Answer: shown}$$

3 marks

6 (3 marks)

$$\frac{dx}{dt} = 3t^2 - 4t + 2$$

M1

$$x = t^3 - 2t^2 + 2t \text{ (using } x = 0 \text{ when } t = 0 \text{)}$$

A1

$$\text{When } t = 2, x = 8 - 8 + 4 = 4$$

A1

$$\text{Answer: } 4 \, \text{m}$$

3 marks

7 (4 marks)

$$a = \frac{F}{m} = \frac{6}{2t+1}$$

M1

$$v = \int \frac{6}{2t+1} \, dt = 3 \ln(2t+1) + c$$

A1

$$v = 1 \text{ when } t = 0 \text{ so } c = 1$$

M1

$$\text{When } t = 4, v = 1 + 3 \ln 9 = 1 + 6 \ln 3 \, \text{m s}^{-1}$$

A1

$$\text{Answer: } 1 + 6 \ln 3 \, \text{m s}^{-1}$$

4 marks

8 (5 marks)

$$\text{(a) } v = \int 4 \cos 2t \, dt = 2 \sin 2t + c$$

M1

$$v = 0 \text{ when } t = 0 \text{ so } c = 0, v = 2 \sin 2t$$

A1

$$\text{Answer: } v = 2 \sin 2t$$

2 marks

$$(b) x = \int 2 \sin 2t \, dt = -\cos 2t + k$$

M1

$$x = 0 \text{ when } t = 0 \text{ so } k = 1, x = 1 - \cos 2t$$

A1

$$\text{Answer: } x = 1 - \cos 2t$$

2 marks

$$(c) \text{ Greatest value of } 1 - \cos 2t \text{ is } 2 \text{ (when } \cos 2t = -1), \text{ so the greatest distance is } 2 \text{ m}$$

B1

$$\text{Answer: } 2 \text{ m}$$

1 mark

9 (5 marks)

$$(a) \int \frac{1}{6-2v} \, dv = \int 1 \, dt, \text{ so } -\frac{1}{2} \ln(6-2v) = t + c$$

M1 A1

$$v = 0 \text{ when } t = 0 \text{ gives } c = -\frac{1}{2} \ln 6, \text{ so } \ln \frac{6-2v}{6} = -2t$$

M1

$$6-2v = 6e^{-2t}, \text{ so } v = 3(1 - e^{-2t})$$

A1

$$\text{Answer: shown}$$

4 marks

$$(b) \text{ As } t \rightarrow \infty, e^{-2t} \rightarrow 0 \text{ so } v \rightarrow 3 \text{ m s}^{-1}$$

B1

$$\text{Answer: } 3 \text{ m s}^{-1}$$

1 mark

10 (5 marks)

$$v \frac{dv}{dx} = \frac{8}{(x+1)^2}$$

M1

$$\frac{1}{2}v^2 = -\frac{8}{x+1} + c$$

A1

$$v = 0 \text{ when } x = 0: c = 8, \text{ so } v^2 = 16 - \frac{16}{x+1} = \frac{16x}{x+1}$$

M1 A1

$$\text{When } x = 3, v^2 = 12, \text{ so } v = 2\sqrt{3} \approx 3.46 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 2\sqrt{3} \approx 3.46 \text{ m s}^{-1}$$

5 marks

11 (5 marks)

$$a = \frac{F}{m} = 2(t^2 - 5t + 6)$$

M1

$$v = \int 2(t^2 - 5t + 6) \, dt = \frac{2}{3}t^3 - 5t^2 + 12t \text{ (} v = 0 \text{ at } t = 0)$$

M1 A1

$$a = 0 \text{ first when } (t-2)(t-3) = 0, \text{ i.e. } t = 2$$

M1

$$v = \frac{16}{3} - 20 + 24 = \frac{28}{3} \text{ m s}^{-1}$$

A1

$$\text{Answer: } \frac{28}{3} \text{ m s}^{-1}$$

5 marks

12 (4 marks)

$$v = \int (4 - 2t) \, dt = 4t - t^2$$

M1 A1

$$x = \int (4t - t^2) \, dt = 2t^2 - \frac{t^3}{3}$$

A1

$$x = 0 \text{ when } t^2(2 - \frac{t}{3}) = 0, \text{ so } t = 6 \text{ s}$$

A1

$$\text{Answer: } t = 6 \text{ s}$$

4 marks

13 (5 marks)

$$v \frac{dv}{dx} = 12 - 3x$$

M1

$$\frac{1}{2}v^2 = 12x - \frac{3}{2}x^2 + c, \text{ and } c = 0 \text{ since } v = 0 \text{ at } x = 0$$

M1 A1

$$\text{So } v^2 = 24x - 3x^2. \text{ Greatest when } a = 0, \text{ i.e. } x = 4$$

M1

$$v^2 = 96 - 48 = 48, \text{ so the maximum speed is } 4\sqrt{3} \approx 6.93 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 4\sqrt{3} \approx 6.93 \text{ m s}^{-1}$$

5 marks

14 (6 marks)

$$(a) v \frac{dv}{dx} = 4e^{-x/2}, \text{ so } \frac{1}{2}v^2 = -8e^{-x/2} + c$$

M1 A1

$$v = 0 \text{ at } x = 0 \text{ gives } c = 8, \text{ so } v^2 = 16(1 - e^{-x/2})$$

A1

$$\text{Answer: shown}$$

3 marks

(b) As $x \rightarrow \infty$, $v^2 \rightarrow 16$, so $v \rightarrow 4 \text{ m s}^{-1}$

Answer: 4 m s^{-1}

B1

1 mark

(c) $4 = 16(1 - e^{-x/2})$, so $e^{-x/2} = \frac{3}{4}$

M1

$$x = 2 \ln \frac{4}{3}$$

A1

Answer: $x = 2 \ln(\frac{4}{3}) \approx 0.575 \text{ m}$

2 marks

15 (5 marks)

$$v \frac{dv}{dx} = \frac{2}{1+x^2}$$

M1

$$\frac{1}{2}v^2 = 2 \tan^{-1} x + c$$

A1

$v = 1$ at $x = 0$ gives $c = \frac{1}{2}$, so $v^2 = 1 + 4 \tan^{-1} x$

M1 A1

As $x \rightarrow \infty$, $\tan^{-1} x \rightarrow \frac{\pi}{2}$, so $v \rightarrow \sqrt{1 + 2\pi} \approx 2.70 \text{ m s}^{-1}$

A1

Answer: $v = \sqrt{1 + 4 \tan^{-1} x}$; limit $\sqrt{1 + 2\pi} \approx 2.70 \text{ m s}^{-1}$

5 marks

16 (6 marks)

(a) $v = \int t e^{-t} dt = -t e^{-t} + \int e^{-t} dt$ (by parts)

M1 A1

$$= -t e^{-t} - e^{-t} + c$$

A1

$v = 0$ at $t = 0$ gives $c = 1$, so $v = 1 - (1 + t)e^{-t}$

A1

Answer: shown

4 marks

(b) $(1 + t)e^{-t} \rightarrow 0$, so $v \rightarrow 1 \text{ m s}^{-1}$

B1

Answer: 1 m s^{-1}

1 mark

(c) $(1 + t)e^{-t} > 0$ for all $t \geq 0$, so $v < 1$

E1

Answer: $(1 + t)e^{-t} > 0$

1 mark

17 (6 marks)

(a) $v \frac{dv}{dt} = 2$, so $\int v dv = \int 2 dt$

M1

$$\frac{1}{2}v^2 = 2t + c, \text{ and } v = 1 \text{ at } t = 0 \text{ gives } c = \frac{1}{2}$$

M1

$$v^2 = 4t + 1$$

A1

Answer: shown

3 marks

(b) $\frac{dx}{dt} = (4t + 1)^{1/2}$, so $x = \frac{1}{6}(4t + 1)^{3/2} + k$

M1 A1

$x = 0$ at $t = 0$ gives $k = -\frac{1}{6}$; at $t = 2$: $x = \frac{27 - 1}{6} = \frac{13}{3} \text{ m}$

A1

Answer: $\frac{13}{3} \text{ m} \approx 4.33 \text{ m}$

3 marks

18 (6 marks)

(a) $a = v \frac{dv}{dx} = \frac{4}{1+x} \times \left(-\frac{4}{(1+x)^2}\right) = -\frac{16}{(1+x)^3}$

M1 A1

At $x = 1$, $a = -2 \text{ m s}^{-2}$, so $F = 3 \times 2 = 6 \text{ N}$ directed towards O

A1

Answer: 6 N towards O

3 marks

(b) $\frac{dx}{dt} = \frac{4}{1+x}$, so $\int (1+x) dx = \int 4 dt$

M1

$$\left[x + \frac{x^2}{2}\right]_0^3 = 4t$$

A1

$$\frac{15}{2} = 4t, \text{ so } t = \frac{15}{8} \text{ s}$$

A1

Answer: $\frac{15}{8} \text{ s}$

3 marks

19 (6 marks)

$$a = \frac{t e^{-t/2}}{0.4} = \frac{5}{2} t e^{-t/2} \quad \text{M1}$$

$$v = \frac{5}{2} \int t e^{-t/2} dt = \frac{5}{2} [-2t e^{-t/2} + 2 \int e^{-t/2} dt] \quad \text{M1 A1}$$

$$= \frac{5}{2} [-2t e^{-t/2} - 4e^{-t/2}] + c = -5(t+2)e^{-t/2} + c \quad \text{A1}$$

$$v = 0 \text{ at } t = 0 \text{ gives } c = 10, \text{ so } v = 10 - 5(t+2)e^{-t/2} \quad \text{A1}$$

$$\text{When } t = 2, v = 10 - 20e^{-1} \approx 2.64 \text{ m s}^{-1} \quad \text{A1}$$

Answer: 2.64 m s⁻¹ 6 marks

20 (5 marks)

$$\int \frac{1}{1+v^2} dv = \int -1 dt \quad \text{M1}$$

$$\tan^{-1} v = -t + c \quad \text{A1}$$

$$v = 1 \text{ at } t = 0 \text{ gives } c = \frac{\pi}{4}, \text{ so } v = \tan\left(\frac{\pi}{4} - t\right) \quad \text{M1 A1}$$

$$v = 0 \text{ when } t = \frac{\pi}{4} \text{ s} \quad \text{A1}$$

Answer: $v = \tan\left(\frac{\pi}{4} - t\right)$; $t = \frac{\pi}{4} \text{ s}$ 5 marks

21 (7 marks)

$$\text{(a) } v = \int (12t - 3t^2) dt = 6t^2 - t^3 \text{ (} v = 0 \text{ at } t = 0) \quad \text{M1 A1}$$

$$x = \int (6t^2 - t^3) dt = 2t^3 - \frac{t^4}{4} \quad \text{A1}$$

Answer: $v = 6t^2 - t^3$; $x = 2t^3 - \frac{t^4}{4}$ 3 marks

$$\text{(b) } v = t^2(6 - t) = 0 \text{ gives } t = 6 \quad \text{M1}$$

$$x = 432 - 324 = 108 \text{ m} \quad \text{A1}$$

Answer: 108 m 2 marks

$$\text{(c) Greatest when } a = 0: 3t(4 - t) = 0, t = 4 \quad \text{M1}$$

$$v = 96 - 64 = 32 \text{ m s}^{-1} \quad \text{A1}$$

Answer: 32 m s⁻¹ 2 marks

22 (6 marks)

$$\text{(a) } x \text{ varies with } t, \text{ so it cannot be treated as a constant when integrating with respect to } t \quad \text{E1}$$

Answer: x is not constant 1 mark

$$\text{(b) } v \frac{dv}{dx} = 8 - 2x \text{ so } \frac{1}{2} v^2 = 8x - x^2 + c \quad \text{M1 A1}$$

$$v = 0 \text{ at } x = 0 \text{ gives } c = 0, \text{ so } v^2 = 16x - 2x^2 \quad \text{A1}$$

Answer: $v^2 = 16x - 2x^2$ 3 marks

$$\text{(c) Max speed when } a = 0, x = 4: v^2 = 64 - 32 = 32, v = 4\sqrt{2} \quad \text{M1}$$

$$v = 0 \text{ again when } x = 8, \text{ so greatest } x \text{ is } 8 \text{ m} \quad \text{A1}$$

Answer: max speed $4\sqrt{2} \approx 5.66 \text{ m s}^{-1}$; greatest $x = 8 \text{ m}$ 2 marks

23 (7 marks)

$$\text{(a) } a = \frac{12 \cos 3t}{2} = 6 \cos 3t \quad \text{M1}$$

$$v = 2 \sin 3t \text{ (} v = 0 \text{ at } t = 0) \quad \text{A1}$$

$$x = -\frac{2}{3} \cos 3t + k, x = 0 \text{ at } t = 0 \text{ gives } k = \frac{2}{3} \quad \text{M1}$$

$$x = \frac{2}{3}(1 - \cos 3t) \quad \text{A1}$$

Answer: $x = \frac{2}{3}(1 - \cos 3t)$ 4 marks

$$\text{(b) } 1 - \cos 3t \geq 0 \text{ so } x \geq 0; \text{ the greatest value is } \frac{4}{3} \text{ m} \quad \text{B1}$$

Answer: $x \geq 0$; greatest $x = \frac{4}{3} \text{ m}$

1 mark

(c) $2 \sin 3t = 1$, so $\sin 3t = \frac{1}{2}$

M1

$3t = \frac{\pi}{6}, t = \frac{\pi}{18} \approx 0.175 \text{ s}$

A1

Answer: $t = \frac{\pi}{18} \approx 0.175 \text{ s}$

2 marks

24 (7 marks)

(a) $\frac{1}{4-v^2} = \frac{1}{4} \left[\frac{1}{2-v} + \frac{1}{2+v} \right]$

M1 A1

$t = \int \frac{1}{4-v^2} dv = \frac{1}{4} [-\ln(2-v) + \ln(2+v)] + c$

M1

$v = 0$ at $t = 0$ gives $c = 0$, so $t = \frac{1}{4} \ln \frac{2+v}{2-v}$

A1

Answer: shown

4 marks

(b) $t = \frac{1}{4} \ln 3 \approx 0.275 \text{ s}$

B1

Answer: $\frac{1}{4} \ln 3 \approx 0.275 \text{ s}$

1 mark

(c) $e^{4t} = \frac{2+v}{2-v}$, so $v = \frac{2(e^{4t}-1)}{e^{4t}+1}$

M1

As $t \rightarrow \infty$, $v \rightarrow 2 \text{ m s}^{-1}$

A1

Answer: $v = \frac{2(e^{4t}-1)}{e^{4t}+1}$; limit 2 m s^{-1}

2 marks

25 (6 marks)

(a) $a = v \frac{dv}{dx} = \frac{1}{2} \frac{d}{dx}(v^2)$

M1

$= \frac{1}{2}(12-2x) = 6-x \text{ m s}^{-2}$

A1

Answer: $a = 6 - x$

2 marks

(b) $\frac{dx}{dt} = \sqrt{12x-x^2} = \sqrt{36-(x-6)^2}$

M1

$t = \int \frac{1}{\sqrt{36-(x-6)^2}} dx = \sin^{-1}\left(\frac{x-6}{6}\right)$

M1 A1

$[\sin^{-1}\left(\frac{x-6}{6}\right)]_3^6 = \sin^{-1} 0 - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6}$

A1

Answer: shown

4 marks

26 (7 marks)

(a) $\frac{1}{v(4-v)} = \frac{1}{4} \left[\frac{1}{v} + \frac{1}{4-v} \right]$

M1 A1

$\frac{1}{4} [\ln v - \ln(4-v)] = t + c$

M1

$v = 1$ at $t = 0$: $c = -\frac{1}{4} \ln 3$, so $\frac{v}{4-v} = \frac{1}{3} e^{4t}$

A1

$3v = (4-v)e^{4t}$, so $v(3+e^{4t}) = 4e^{4t}$ and $v = \frac{4}{1+3e^{-4t}}$

A1

Answer: shown

5 marks

(b) $2 = \frac{4}{1+3e^{-4t}}$ gives $e^{-4t} = \frac{1}{3}$

M1

$t = \frac{1}{4} \ln 3 \approx 0.275 \text{ s}$

A1

Answer: $\frac{1}{4} \ln 3 \approx 0.275 \text{ s}$

2 marks

27 (8 marks)

$$(a) v \frac{dv}{dx} = -(2x + 4x^3) \quad \text{M1}$$

$$\frac{1}{2}v^2 = -x^2 - x^4 + c \quad \text{A1}$$

$$v = 0 \text{ at } x = 1: c = 2, \text{ so } v^2 = 4 - 2x^2 - 2x^4 \quad \text{A1}$$

Answer: shown 3 marks

$$(b) 4 - 2x^2 - 2x^4 = -2(x^4 + x^2 - 2) = -2(x^2 - 1)(x^2 + 2) \quad \text{M1 A1}$$

$$v = 0 \text{ when } x^2 = 1 \ (x^2 + 2 > 0), \text{ so the other rest position is } x = -1 \quad \text{A1}$$

Answer: $x = -1$ 3 marks

$$(c) \text{ At } x = 0, v^2 = 4 \text{ so } v = 2 \text{ m s}^{-1} \quad \text{B1}$$

The force always acts towards O, so P is pulled back each time it reaches ± 1 : it oscillates between $x = 1$ and $x = -1$ E1

Answer: 2 m s^{-1} 2 marks

28 (9 marks)

$$(a) \text{ Integrating factor } e^{\int 1 dt} = e^t \quad \text{B1}$$

$$\frac{d}{dt}(v e^t) = 2e^{-t} e^t = 2 \quad \text{M1}$$

$$v e^t = 2t + c, \text{ and } v = 0 \text{ when } t = 0 \text{ so } c = 0 \quad \text{M1}$$

$$v = 2t e^{-t} \quad \text{A1}$$

Answer: shown 4 marks

$$(b) \frac{dv}{dt} = 2e^{-t}(1 - t) = 0 \text{ gives } t = 1 \quad \text{M1 A1}$$

$$v = \frac{2}{e} \approx 0.736 \text{ m s}^{-1} \text{ at } t = 1 \text{ s (a maximum since } v \text{ is increasing then decreasing)} \quad \text{A1}$$

Answer: max speed $\frac{2}{e} \text{ m s}^{-1}$ at $t = 1 \text{ s}$ 3 marks

$$(c) \int_0^\infty 2t e^{-t} dt = [-2t e^{-t} - 2e^{-t}]_0^\infty \quad \text{M1}$$

$$= 0 - (-2) = 2 \text{ m} \quad \text{A1}$$

Answer: 2 m 2 marks

29 (8 marks)

$$(a) a = 2 \text{ for } 0 \leq x \leq 2, \text{ so } v^2 = 2 \times 2 \times x = 4x \ (v^2 = 2ax) \quad \text{M1 A1}$$

$$x = t^2 \text{ (from rest with } a = 2), \text{ so } 2 = t^2 \text{ and } t = \sqrt{2} \approx 1.41 \text{ s} \quad \text{A1}$$

Answer: $v = 2\sqrt{2} \text{ m s}^{-1}$; $t = \sqrt{2} \text{ s}$ 3 marks

$$(b) a = \frac{4}{x}, \text{ so } v \frac{dv}{dx} = \frac{4}{x} \quad \text{M1}$$

$$\frac{1}{2}v^2 = 4 \ln x + c \quad \text{A1}$$

$$v^2 = 8 \text{ at } x = 2 \text{ gives } c = 4 - 4 \ln 2, \text{ so } v^2 = 8 + 8 \ln \frac{x}{2} \quad \text{A1}$$

Answer: shown 3 marks

$$(c) 36 = 8 + 8 \ln \frac{x}{2}, \text{ so } \ln \frac{x}{2} = \frac{7}{2} \quad \text{M1}$$

$$x = 2e^{7/2} \approx 66.2 \text{ m} \quad \text{A1}$$

Answer: $x = 2e^{7/2} \approx 66.2 \text{ m}$ 2 marks

30 (8 marks)

$$(a) a = \frac{t(4-t)}{2} = 2t - \frac{t^2}{2} \quad \text{M1}$$

$$v = t^2 - \frac{t^3}{6} + c \quad \text{A1}$$

$$v = 0 \text{ at } t = 0 \text{ gives } c = 0, \text{ so } v = t^2 - \frac{t^3}{6} \quad \text{A1}$$

Answer: shown 3 marks

$$(b) v = 16 - \frac{64}{6} = \frac{16}{3} \text{ m s}^{-1} \quad \text{B1}$$

$$x = \int_0^4 \left(t^2 - \frac{t^3}{6}\right) dt = \left[\frac{t^3}{3} - \frac{t^4}{24}\right]_0^4 \quad \text{M1}$$

$$= \frac{64}{3} - \frac{32}{3} = \frac{32}{3} \text{ m} \quad \text{A1}$$

Answer: $v = \frac{16}{3} \text{ m s}^{-1}$, $x = \frac{32}{3} \text{ m}$ 3 marks

(c) After $t = 4$ the speed is constant at $\frac{16}{3}$; need a further $40 - \frac{32}{3} = \frac{88}{3} \text{ m}$ M1

Extra time $= \frac{88}{3} \div \frac{16}{3} = 5.5 \text{ s}$, so $t = 9.5 \text{ s}$ A1

Answer: $t = 9.5 \text{ s}$ 2 marks

Motion with Resistance

1 (4 marks)

$$0.5 \frac{dv}{dt} = -2v, \text{ so } \frac{dv}{dt} = -4v \quad \text{M1}$$

$$\int \frac{1}{v} dv = \int -4 dt, \text{ so } \ln v = -4t + c; v = 8 \text{ at } t = 0 \text{ gives } v = 8e^{-4t} \quad \text{M1 A1}$$

$$t = \ln 2: v = 8e^{-4 \ln 2} = 8 \times \frac{1}{16} = 0.5 \text{ m s}^{-1} \quad \text{A1}$$

Answer: $v = 8e^{-4t}$; 0.5 m s^{-1} 4 marks

2 (2 marks)

At terminal velocity the acceleration is zero: $2g = 4v$ M1

$$v = \frac{2 \times 9.8}{4} = 4.9 \text{ m s}^{-1} \quad \text{A1}$$

Answer: 4.9 m s^{-1} 2 marks

3 (3 marks)

(a) N2L upwards: $3 \frac{dv}{dt} = -3g - 0.6v$ M1

$$\frac{dv}{dt} = -9.8 - 0.2v \quad \text{A1}$$

Answer: shown 2 marks

(b) $9.8 + 0.2 \times 14 = 12.6 \text{ m s}^{-2}$ B1

Answer: 12.6 m s^{-2} 1 mark

4 (3 marks)

$$\frac{dv}{dt} = -0.4v^2, \text{ so } \int \frac{1}{v^2} dv = \int -0.4 dt \quad \text{M1}$$

$$-\frac{1}{v} = -0.4t + c; v = 5 \text{ at } t = 0 \text{ gives } c = -\frac{1}{5} \quad \text{A1}$$

$$\frac{1}{v} = 0.4t + 0.2 = \frac{1 + 2t}{5}, \text{ so } v = \frac{5}{1 + 2t} \quad \text{A1}$$

Answer: shown 3 marks

5 (3 marks)

$$v \frac{dv}{dx} = -0.2v, \text{ so } \frac{dv}{dx} = -0.2 \quad \text{M1}$$

$$v = -0.2x + 10 \text{ (} v = 10 \text{ when } x = 0\text{)} \quad \text{A1}$$

$$v = 5 \text{ when } x = 25 \text{ m} \quad \text{A1}$$

Answer: 25 m 3 marks

6 (3 marks)

$$200 \frac{dv}{dt} = -50v, \text{ so } \frac{dv}{dt} = -\frac{1}{4}v \text{ and } v = 4e^{-t/4}$$

M1 A1

$$t = 8: v = 4e^{-2} \approx 0.541 \text{ m s}^{-1}$$

A1

Answer: 0.541 m s⁻¹

3 marks

7 (5 marks)

$$(a) m \frac{dv}{dt} = mg - 0.5mv, \text{ so } \frac{dv}{dt} = 9.8 - 0.5v$$

M1

$$\int \frac{1}{9.8 - 0.5v} dv = \int 1 dt, \text{ so } -2 \ln(9.8 - 0.5v) = t + c$$

M1 A1

$$v = 0 \text{ at } t = 0: 9.8 - 0.5v = 9.8e^{-0.5t}, \text{ so } v = 19.6(1 - e^{-0.5t})$$

A1

Answer: shown

4 marks

$$(b) v = 19.6(1 - e^{-1}) \approx 12.4 \text{ m s}^{-1}$$

B1

Answer: 12.4 m s⁻¹

1 mark

8 (4 marks)

$$\frac{dv}{dt} = -(9.8 + 0.2v)$$

B1

$$t = \int_0^{20} \frac{1}{9.8 + 0.2v} dv = 5[\ln(9.8 + 0.2v)]_0^{20}$$

M1 A1

$$= 5 \ln \frac{13.8}{9.8} \approx 1.71 \text{ s}$$

A1

Answer: 1.71 s

4 marks

9 (5 marks)

$$(a) v \frac{dv}{dx} = -0.05v^2, \text{ so } \int \frac{1}{v} dv = \int -0.05 dx$$

M1

$$\ln v = -0.05x + c; v = 20 \text{ at } x = 0 \text{ gives } c = \ln 20$$

A1

$$v = 20e^{-0.05x}$$

A1

Answer: shown

3 marks

$$(b) 10 = 20e^{-0.05x}, \text{ so } e^{-0.05x} = \frac{1}{2}$$

M1

$$x = 20 \ln 2 \approx 13.9 \text{ m}$$

A1

Answer: 20 ln 2 ≈ 13.9 m

2 marks

10 (4 marks)

$$(a) 800a = 1200 - 3 \times 10^2$$

M1

$$a = \frac{900}{800} = 1.125 \text{ m s}^{-2}$$

A1

Answer: 1.125 m s⁻²

2 marks

$$(b) \text{ At maximum speed } a = 0: 3v^2 = 1200$$

M1

$$v = 20 \text{ m s}^{-1}$$

A1

Answer: 20 m s⁻¹

2 marks

11 (4 marks)

$$(a) \text{ Terminal speed when acceleration is zero: } v^2 = \frac{9.8}{0.02} = 490$$

M1

$$v = \sqrt{490} \approx 22.1 \text{ m s}^{-1}$$

A1

Answer: 22.1 m s⁻¹

2 marks

$$(b) 9.8 - 0.02v^2 = 4.9, \text{ so } v^2 = 245$$

M1

$$v = \sqrt{245} \approx 15.7 \text{ m s}^{-1}$$

A1

Answer: 15.7 m s⁻¹

2 marks

12 (4 marks)

$$2v \frac{dv}{dx} = -0.6v, \text{ so } \frac{dv}{dx} = -0.3$$

M1 A1

$$v = 12 - 0.3x$$

A1

$v > 0$ requires $x < 40$, so $v = 0$ only when $x = 40$ and the particle travels at most 40 m

E1

Answer: shown

4 marks

13 (5 marks)

$$(a) m \frac{dv}{dt} = mg \sin 30^\circ - 0.4mv, \text{ so } \frac{dv}{dt} = 4.9 - 0.4v$$

M1 A1

$$\text{Limiting speed when } \frac{dv}{dt} = 0: v = \frac{4.9}{0.4} = 12.25 \text{ m s}^{-1}$$

A1

Answer: shown

3 marks

$$(b) v = 12.25(1 - e^{-0.4t}); \text{ set equal to } 6$$

M1

$$t = -\frac{1}{0.4} \ln(1 - \frac{6}{12.25}) \approx 1.68 \text{ s}$$

A1

Answer: 1.68 s

2 marks

14 (6 marks)

$$(a) \frac{dv}{dt} = 9.8 - 0.5v$$

B1

Separating variables and integrating: $-2 \ln(9.8 - 0.5v) = t + c$

M1

$$v = 0 \text{ at } t = 0 \text{ gives } v = 19.6(1 - e^{-0.5t})$$

A1

Answer: shown

3 marks

$$(b) x = \int_0^4 19.6(1 - e^{-0.5t}) dt = 19.6[t + 2e^{-0.5t}]_0^4$$

M1 A1

$$= 19.6(4 + 2e^{-2} - 2) = 19.6(2 + 2e^{-2}) \approx 44.5 \text{ m}$$

A1

Answer: 44.5 m

3 marks

15 (6 marks)

$$v \frac{dv}{dx} = -(9.8 + 0.4v)$$

M1 A1

$$x = \int_0^{30} \frac{v}{9.8 + 0.4v} dv$$

M1

$$\frac{v}{9.8 + 0.4v} = \frac{1}{0.4} [1 - \frac{9.8}{9.8 + 0.4v}]$$

M1

$$x = 2.5[v - 24.5 \ln(9.8 + 0.4v)]_0^{30} = 2.5[30 - 24.5 \ln \frac{21.8}{9.8}]$$

A1

$$= 26.0 \text{ m}$$

A1

Answer: 26.0 m

6 marks

16 (5 marks)

$$v \frac{dv}{dx} = -(9.8 + 0.01v^2)$$

M1 A1

$$x = \int_0^{25} \frac{v}{9.8 + 0.01v^2} dv$$

M1

$$= [50 \ln(9.8 + 0.01v^2)]_0^{25}$$

A1

$$= 50 \ln \frac{16.05}{9.8} \approx 24.7 \text{ m}$$

A1

Answer: 24.7 m

5 marks

17 (6 marks)

$$(a) v \frac{dv}{dx} = 9.8 - 0.02v^2$$

M1

$$\int \frac{v}{9.8 - 0.02v^2} dv = \int 1 dx, \text{ so } -25 \ln(9.8 - 0.02v^2) = x + c$$

M1 A1

$$v = 0 \text{ at } x = 0: 9.8 - 0.02v^2 = 9.8e^{-0.04x}, \text{ so } v^2 = 490(1 - e^{-0.04x})$$

A1

Answer: shown

4 marks

$$(b) v^2 = 490(1 - e^{-1.2})$$

M1

$$v \approx 18.5 \text{ m s}^{-1}$$

A1

Answer: 18.5 m s^{-1} **2 marks****18 (6 marks)**

(a) $\frac{dv}{dt} = -0.05v^2$, so $\int \frac{1}{v^2} dv = -0.05 \int 1 dt$

M1

$$-\frac{1}{v} = -0.05t + c; v = 20 \text{ at } t = 0 \text{ gives } \frac{1}{v} = 0.05t + 0.05$$

A1

$$v = \frac{20}{1+t}$$

A1**Answer: shown****3 marks**

(b) $x = \int_0^{10} \frac{20}{1+t} dt$

M1

$$= 20[\ln(1+t)]_0^{10}$$

A1

$$= 20 \ln 11 \approx 48.0 \text{ m}$$

A1**Answer: 48.0 m****3 marks****19 (6 marks)**

(a) $500 \frac{dv}{dt} = 600 - 40v$, so $\frac{dv}{dt} = 1.2 - 0.08v$

M1

$$\int \frac{1}{1.2 - 0.08v} dv = \int 1 dt, \text{ so } -12.5 \ln(1.2 - 0.08v) = t + c$$

M1 A1

$$v = 0 \text{ at } t = 0 \text{ gives } v = 15(1 - e^{-0.08t})$$

A1**Answer: shown****4 marks**

(b) $10 = 15(1 - e^{-0.08t})$, so $e^{-0.08t} = \frac{1}{3}$

M1

$$t = \frac{\ln 3}{0.08} \approx 13.7 \text{ s}$$

A1**Answer: 13.7 s****2 marks****20 (6 marks)**

(a) $\frac{dv}{dt} = -2v^{1/2}$, so $\int v^{-1/2} dv = \int -2 dt$

M1

$$2v^{1/2} = -2t + c; v = 16 \text{ at } t = 0 \text{ gives } c = 8, \text{ so } \sqrt{v} = 4 - t$$

A1

$$v = 0 \text{ when } t = 4 \text{ s}$$

A1**Answer: shown****3 marks**

(b) $v \frac{dv}{dx} = -2v^{1/2}$, so $\int v^{1/2} dv = \int -2 dx$

M1

$$\frac{2}{3}v^{3/2} = -2x + c; v = 16 \text{ at } x = 0: c = \frac{2}{3} \times 64$$

A1

$$v = 0: x = \frac{64}{3} \approx 21.3 \text{ m}$$

A1

Answer: $\frac{64}{3} \approx 21.3 \text{ m}$

3 marks**21 (8 marks)**

(a) Going up: $v \frac{dv}{dx} = -(9.8 + 0.01v^2)$

M1

$$x = \int_0^{20} \frac{v}{9.8 + 0.01v^2} dv = [50 \ln(9.8 + 0.01v^2)]_0^{20}$$

M1 A1

$$H = 50 \ln \frac{13.8}{9.8} \approx 17.1 \text{ m}$$

A1**Answer: 17.1 m****4 marks**

(b) Coming down: $v \frac{dv}{dx} = 9.8 - 0.01v^2$

M1

$$v^2 = 980(1 - e^{-0.02x}) \quad (v = 0 \text{ at the top})$$

M1 A1

$$\text{At } x = H: e^{-0.02H} = \frac{9.8}{13.8}, \text{ so } v^2 = 980 \times \frac{4}{13.8} \approx 284, v \approx 16.9 \text{ m s}^{-1}$$

A1**Answer: 16.9 m s^{-1}** **4 marks**

22 (10 marks)**(a)** Terminal speed: $9.8 = 0.04V^2$, so $V^2 = 245$ **M1**

$$V = \sqrt{245} = 7\sqrt{5} \approx 15.7 \text{ m s}^{-1}$$

A1**Answer: $V = 7\sqrt{5} \approx 15.7 \text{ m s}^{-1}$** **2 marks**

(b) $\frac{dv}{dt} = 9.8 - 0.04v^2 = 0.04(245 - v^2)$; $t = \int \frac{1}{0.04(245 - v^2)} dv$

B1**Answer: shown****1 mark**

(c) $\frac{1}{245 - v^2} = \frac{1}{2V} \left[\frac{1}{V - v} + \frac{1}{V + v} \right]$

M1 A1

$$0.04t = \frac{1}{2V} \ln \frac{V+v}{V-v} + c$$

M1 A1

$$v = 0 \text{ at } t = 0, c = 0; \frac{V+v}{V-v} = e^{0.08Vt}, v = V \tanh(0.04Vt) \text{ where } 0.04V \approx 0.626$$

A1**Answer: shown****5 marks**

(d) $\tanh(0.626t) = \frac{1}{2}$, so $0.626t = \frac{1}{2} \ln 3$

M1

$$t \approx 0.877 \text{ s}$$

A1**Answer: 0.877 s****2 marks****23 (8 marks)****(a)** v^2 is a function of t , so it cannot be treated as a constant (the equation must be separated first)**E1**The claimed final result does not follow and is the solution of a different equation, $\frac{dv}{dt} = -0.1v$ **E1****Answer: variable v^2 treated as constant; wrong model****2 marks**

(b) $\int \frac{1}{v^2} dv = \int -0.1 dt$, so $-\frac{1}{v} = -0.1t + c$

M1

$$v = 10 \text{ at } t = 0 \text{ gives } c = -0.1$$

A1

$$\frac{1}{v} = 0.1t + 0.1, \text{ so } v = \frac{10}{1+t}$$

A1

Answer: $v = \frac{10}{1+t}$

3 marks

(c) $v \frac{dv}{dx} = -0.1v^2$, so $\int \frac{1}{v} dv = \int -0.1 dx$, giving $v = 10e^{-0.1x}$

M1 A1 $v > 0$ for every finite x , so the particle never stops ($v \rightarrow 0$ only as $x \rightarrow \infty$)**E1****Answer: $v = 10e^{-0.1x}$; never stops****3 marks****24 (9 marks)**

(a) $\frac{dv}{dt} = 9.8 - 0.8v$; terminal speed $V = \frac{9.8}{0.8} = 12.25 \text{ m s}^{-1}$

B1

$$\int \frac{1}{9.8 - 0.8v} dv = \int 1 dt, \text{ so } -\frac{1}{0.8} \ln(9.8 - 0.8v) = t + c$$

M1 A1

$$v = 0 \text{ at } t = 0: v = 12.25(1 - e^{-0.8t})$$

A1**Answer: $V = 12.25 \text{ m s}^{-1}$** **4 marks**

(b) $e^{-0.8t} = 0.1$

M1

$$t = \frac{\ln 10}{0.8} \approx 2.88 \text{ s}$$

A1**Answer: 2.88 s****2 marks**

(c) $x = \int_0^{t_1} 12.25(1 - e^{-0.8t}) dt = 12.25 \left[t + \frac{1}{0.8} e^{-0.8t} \right]_0^{t_1}$

M1 A1

$$= 12.25(t_1 - 1.25 \times 0.9) \text{ with } t_1 = 2.88, \text{ giving } x \approx 21.5 \text{ m}$$

A1**Answer: 21.5 m****3 marks****25 (7 marks)**

(a) Driving force = $\frac{24000}{v}$; N2L: $1000 \frac{dv}{dt} = \frac{24000}{v} - 3v^2$

M1

Maximum speed when $\frac{dv}{dt} = 0$: $v^3 = 8000$, so $v = 20 \text{ m s}^{-1}$

A1**Answer: shown****2 marks**

$$(b) 1000v \frac{dv}{dx} = \frac{24000}{v} - 3v^2, \text{ so } \frac{dv}{dx} = \frac{24000 - 3v^3}{1000v^2}$$

M1 A1

$$x = \int_0^{10} \frac{1000v^2}{24000 - 3v^3} dv$$

M1

$$= \left[-\frac{1000}{9} \ln(24000 - 3v^3) \right]_0^{10}$$

A1

$$= \frac{1000}{9} \ln \frac{24000}{21000} \approx 14.8 \text{ m}$$

A1

Answer: 14.8 m

5 marks

26 (8 marks)

$$(a) m \frac{dv}{dt} = mg - mkv; \text{ terminal speed when } \frac{dv}{dt} = 0: V = \frac{g}{k}$$

B1

$$\int \frac{1}{g - kv} dv = \int 1 dt, \text{ so } -\frac{1}{k} \ln(g - kv) = t + c$$

M1 A1

$$v = 0 \text{ at } t = 0: v = \frac{g}{k}(1 - e^{-kt}) = V(1 - e^{-kt})$$

A1

Answer: shown

4 marks

$$(b) \frac{1}{2} = 1 - e^{-kt} \text{ gives } e^{-kt} = \frac{1}{2}, \text{ so } t = \frac{\ln 2}{k}$$

B1

Answer: shown

1 mark

$$(c) x = \int_0^T \frac{g}{k}(1 - e^{-kt}) dt = \frac{g}{k} \left[t + \frac{1}{k} e^{-kt} \right]_0^T$$

M1 A1

$$T = \frac{\ln 2}{k}, e^{-kT} = \frac{1}{2}: x = \frac{g}{k} \left(\frac{\ln 2}{k} + \frac{1}{2k} - \frac{1}{k} \right) = \frac{g}{k^2} \left(\ln 2 - \frac{1}{2} \right)$$

A1

Answer: shown

3 marks

27 (10 marks)

$$(a) \text{ Rising: } v \frac{dv}{dx} = -(g + kv^2)$$

M1

$$H = \int_0^U \frac{v}{g + kv^2} dv$$

M1

$$= \frac{1}{2k} [\ln(g + kv^2)]_0^U$$

A1

$$= \frac{1}{2k} \ln \frac{g + kU^2}{g} = \frac{1}{2k} \ln \left(1 + \frac{kU^2}{g} \right)$$

A1

Answer: shown

4 marks

$$(b) \text{ Falling: } v \frac{dv}{dx} = g - kv^2$$

M1

$$\int \frac{v}{g - kv^2} dv = \int 1 dx \text{ gives } -\frac{1}{2k} \ln(g - kv^2) = x + c; v = 0 \text{ at } x = 0$$

M1

$$\text{At } x = H: g - kV^2 = g e^{-2kH}, \text{ so } V^2 = \frac{g}{k}(1 - e^{-2kH})$$

A1

Answer: shown

3 marks

$$(c) e^{-2kH} = \frac{1}{1 + kU^2/g} = \frac{g}{g + kU^2}$$

M1

$$V^2 = \frac{g}{k} \times \frac{kU^2}{g + kU^2} = \frac{gU^2}{g + kU^2}, \text{ so } \frac{1}{V^2} = \frac{1}{U^2} + \frac{k}{g}$$

A1

$$k/g > 0 \text{ so } \frac{1}{V^2} > \frac{1}{U^2}, \text{ hence } V < U$$

E1

Answer: shown

3 marks

28 (10 marks)

$$(a) \frac{dv}{dt} = -(9.8 + 0.04v^2) = -0.04(245 + v^2) \quad \text{M1}$$

$$t = \int_0^{15} \frac{1}{0.04(245 + v^2)} dv \quad \text{M1}$$

$$= \frac{1}{0.04\sqrt{245}} [\tan^{-1} \frac{v}{\sqrt{245}}]_0^{15} \quad \text{A1}$$

$$= \frac{\tan^{-1}(0.958)}{0.626} \approx 1.22 \text{ s} \quad \text{A1}$$

Answer: 1.22 s

4 marks

$$(b) \text{ Falling: } \frac{dv}{dt} = 9.8 - 0.04v^2 = 0.04(245 - v^2) \quad \text{M1}$$

$$\frac{1}{245 - v^2} = \frac{1}{2\sqrt{245}} \left[\frac{1}{\sqrt{245} - v} + \frac{1}{\sqrt{245} + v} \right] \quad \text{M1 A1}$$

$$0.04t = \frac{1}{2\sqrt{245}} \ln \frac{\sqrt{245} + v}{\sqrt{245} - v}; \text{ solve for } v \text{ with } v = 0 \text{ at } t = 0: v = \sqrt{245} \tanh(0.04\sqrt{245}t) \text{ with} \quad \text{A1}$$

$$0.04\sqrt{245} \approx 0.626$$

Answer: shown

4 marks

$$(c) \tanh(0.626t) = \frac{1}{2}, \text{ so } t = \frac{1}{0.626} \tanh^{-1}\left(\frac{1}{2}\right) = \frac{\ln 3}{1.252} \quad \text{M1}$$

$$t \approx 0.877 \text{ s} \quad \text{A1}$$

Answer: 0.877 s

2 marks

29 (10 marks)

$$(a) 1200 \frac{dv}{dt} = 3000 - 300v, \text{ so } \frac{dv}{dt} = 2.5 - 0.25v \quad \text{M1}$$

$$\int \frac{1}{2.5 - 0.25v} dv = \int 1 dt, \text{ so } -4 \ln(2.5 - 0.25v) = t + c \quad \text{M1 A1}$$

$$v = 0 \text{ at } t = 0: v = 10(1 - e^{-t/4}) \quad \text{A1}$$

Answer: shown

4 marks

$$(b) x = \int_0^8 10(1 - e^{-t/4}) dt = 10[t + 4e^{-t/4}]_0^8 \quad \text{M1 A1}$$

$$= 10(8 + 4e^{-2} - 4) \approx 45.4 \text{ m} \quad \text{A1}$$

Answer: 45.4 m

3 marks

$$(c) \text{ Speed at } t = 8: u = 10(1 - e^{-2}) \approx 8.65 \text{ m s}^{-1} \quad \text{B1}$$

$$1200v \frac{dv}{dx} = -300v, \text{ so } \frac{dv}{dx} = -0.25 \text{ and } v = u - 0.25x \quad \text{M1}$$

$$x = \frac{u}{0.25} \approx 34.6 \text{ m. The distance is finite: the resistance (linear in } v) \text{ brings the boat to rest only at this limiting point} \quad \text{A1}$$

Answer: 34.6 m

3 marks

30 (10 marks)

$$(a) \frac{dv}{dt} = 9.8 - 0.98v; \text{ terminal speed } V = \frac{9.8}{0.98} = 10 \text{ m s}^{-1} \quad \text{B1}$$

$$\int \frac{1}{9.8 - 0.98v} dv = \int 1 dt \quad \text{M1}$$

$$-\frac{1}{0.98} \ln|9.8 - 0.98v| = t + c \quad \text{A1}$$

$$v = 25 \text{ at } t = 0: 9.8 - 0.98v = -14.7e^{-0.98t} \quad \text{M1}$$

$$v = 10 + 15e^{-0.98t} \quad \text{A1}$$

Answer: shown

5 marks

(b) The speed decreases from 25 towards 10 because the initial speed exceeds the terminal speed (resistance exceeds weight)

E1

Answer: speed decreases towards 10 m s⁻¹

1 mark

$$(c) 15 = 10 + 15e^{-0.98t}, \text{ so } e^{-0.98t} = \frac{1}{3} \quad \text{M1}$$

$$t = \frac{\ln 3}{0.98} \approx 1.12 \text{ s} \quad \text{A1}$$

Answer: 1.12 s**2 marks**

$$(d) x = \int_0^{t_1} (10 + 15e^{-0.98t}) dt = 10t_1 + \frac{15}{0.98}(1 - e^{-0.98t_1})$$

M1

$$= 10 \times 1.12 + 15.31 \times \frac{2}{3} \approx 21.4 \text{ m}$$

A1**Answer: 21.4 m****2 marks**

Variable Mass and Force Models

1 (2 marks)

$$F = \frac{d}{dt}(\text{momentum}) = 6t + 2$$

M1

$$t = 2: F = 14 \text{ N}$$

A1**Answer: 14 N****2 marks**

2 (2 marks)

$$\text{Force} \propto \frac{1}{x^2}: \text{at } 3R \text{ the force is } \frac{1}{9} \text{ of its surface value}$$

M1

$$\text{Weight} = \frac{80}{9} \approx 8.89 \text{ N}$$

A1

$$\text{Answer: } \frac{80}{9} \approx 8.89 \text{ N}$$

2 marks

3 (2 marks)

$$\text{Thrust} = (\text{rate of mass ejection}) \times (\text{relative speed of ejection})$$

M1

$$= 5 \times 800 = 4000 \text{ N}$$

A1**Answer: 4000 N****2 marks**

4 (2 marks)

$$\text{Sand gains horizontal speed } 2 \text{ m s}^{-1}: \text{rate of change of momentum} = 3 \times 2$$

M1

$$F = 6 \text{ N}$$

A1**Answer: 6 N****2 marks**

5 (2 marks)

$$\text{Distance from centre is } x = 2R$$

M1

$$\frac{gR^2}{(2R)^2} = \frac{g}{4} = 2.45 \text{ m s}^{-2}$$

A1**Answer: 2.45 m s⁻²****2 marks**

6 (3 marks)

$$\text{Final mass} = 800 - 300 = 500 \text{ kg}$$

B1

$$v = 2000 \ln \frac{800}{500}$$

M1

$$= 940 \text{ m s}^{-1}$$

A1**Answer: 940 m s⁻¹****3 marks**

7 (4 marks)

$$v \frac{dv}{dx} = -\frac{gR^2}{x^2}, \text{ so } \frac{1}{2}v^2 = \frac{gR^2}{x} + c$$

M1 A1

$$v = u \text{ at } x = R: v^2 = u^2 - 2gR + \frac{2gR^2}{x}$$

M1

$$v^2 > 0 \text{ as } x \rightarrow \infty \text{ requires } u^2 \geq 2gR, \text{ i.e. } u \geq \sqrt{2gR}; \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \approx 1.12 \times 10^4 \text{ m s}^{-1}$$

A1**Answer: 1.12 × 10⁴ m s⁻¹****4 marks**

8 (5 marks)

$$v \frac{dv}{dx} = -\frac{gR^2}{x^2} \quad \text{M1}$$

$$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right) \quad \text{M1 A1}$$

$$v = 0: \frac{1}{x} = \frac{1}{R} - \frac{u^2}{2gR^2}, \text{ giving } x = 7.99 \times 10^6 \text{ m} \quad \text{M1}$$

$$\text{Height} = x - R \approx 1.59 \times 10^6 \text{ m} \quad \text{A1}$$

Answer: $1.59 \times 10^6 \text{ m}$ 5 marks

9 (4 marks)

$$v \frac{dv}{dx} = -\frac{gR^2}{x^2} \text{ (x decreasing); } v = 0 \text{ at } x = 2R \quad \text{M1}$$

$$\frac{1}{2}v^2 = \int_{x=2R}^x \frac{gR^2}{s^2} ds = gR^2\left(\frac{1}{x} - \frac{1}{2R}\right) \quad \text{M1 A1}$$

$$\frac{1}{2}v^2 = gR^2\left(\frac{1}{x} - \frac{1}{2R}\right); \text{ at } x = R, v^2 = 2gR^2 \times \frac{1}{2R} = gR, \text{ so } v = \sqrt{gR} \approx 7.92 \times 10^3 \text{ m s}^{-1} \quad \text{A1}$$

Answer: $7.92 \times 10^3 \text{ m s}^{-1}$ 4 marks

10 (5 marks)

$$\text{(a) Mass at time } t: m = 1500 - 10t \quad \text{B1}$$

$$m \frac{dv}{dt} = \text{thrust} = 10 \times 1800 = 18000 \quad \text{M1}$$

$$\frac{dv}{dt} = \frac{18000}{1500 - 10t} \quad \text{A1}$$

$$v = -1800 \ln(1500 - 10t) + c; v = 0 \text{ at } t = 0 \text{ gives } v = 1800 \ln \frac{1500}{1500 - 10t} \quad \text{A1}$$

Answer: shown 4 marks

$$\text{(b) } v = 1800 \ln \frac{1500}{900} \approx 919 \text{ m s}^{-1} \quad \text{B1}$$

Answer: 919 m s^{-1} 1 mark

11 (4 marks)

Horizontal momentum is conserved: no external horizontal force E1

Mass after 10 s: $20 + 2 \times 10 = 40 \text{ kg}$ B1

$$20 \times 6 = 40v \quad \text{M1}$$

$$v = 3 \text{ m s}^{-1} \quad \text{A1}$$

Answer: 3 m s^{-1} 4 marks

12 (4 marks)

$$0.5a = \frac{4}{x^2}, \text{ so } v \frac{dv}{dx} = -\frac{8}{x^2} \quad \text{M1}$$

$$\frac{1}{2}v^2 = \frac{8}{x} + c; v = 0 \text{ at } x = 4 \text{ gives } c = -2 \quad \text{M1 A1}$$

$$x = 1: \frac{1}{2}v^2 = 6, \text{ so } v = 2\sqrt{3} \approx 3.46 \text{ m s}^{-1} \quad \text{A1}$$

Answer: $2\sqrt{3} \approx 3.46 \text{ m s}^{-1}$ 4 marks

13 (5 marks)

$$\text{(a) Thrust} = 20 \times 2500 = 50\,000 \text{ N; } N2L: 5000a = 50\,000 - 5000 \times 9.8 \quad \text{M1}$$

$$a = 0.2 \text{ m s}^{-2} \quad \text{A1}$$

Answer: 0.2 m s^{-2} 2 marks

$$\text{(b) Mass} = 5000 - 20 \times 10 = 4800 \text{ kg} \quad \text{B1}$$

$$4800a = 50\,000 - 4800 \times 9.8 \quad \text{M1}$$

$$a = 0.617 \text{ m s}^{-2} \quad \text{A1}$$

Answer: 0.617 m s^{-2} 3 marks

14 (7 marks)

$$(a) m = 4000 - 25t; \text{thrust} = 25 \times 2200 = 55\,000 \text{ N}$$

B1

$$N2L: m \frac{dv}{dt} = 55\,000 - mg$$

M1

$$\frac{dv}{dt} = \frac{55\,000}{4000 - 25t} - 9.8$$

A1

$$v = -2200 \ln(4000 - 25t) - 9.8t + c$$

M1

$$v = 0 \text{ at } t = 0 \text{ gives } v = 2200 \ln \frac{4000}{4000 - 25t} - 9.8t$$

A1

Answer: shown

5 marks

$$(b) m = 4000 - 750 = 3250; v = 2200 \ln \frac{4000}{3250} - 294$$

M1

$$\approx 163 \text{ m s}^{-1}$$

A1

Answer: 163 m s⁻¹

2 marks

15 (6 marks)

$$(a) \frac{d}{dt}(mv) = mg$$

B1

$$m \frac{dv}{dt} + v \frac{dm}{dt} = mg$$

M1

$$m \frac{dv}{dt} + 0.2mv = mg, \text{ so } \frac{dv}{dt} + 0.2v = 9.8$$

A1

Answer: shown

3 marks

$$(b) \frac{dv}{dt} = 9.8 - 0.2v \text{ gives } v = 49(1 - e^{-0.2t})$$

M1 A1

$$t = 5: v = 49(1 - e^{-1}) \approx 31.0 \text{ m s}^{-1}$$

A1

Answer: 31.0 m s⁻¹

3 marks

16 (5 marks)

$$(a) m = 1500 - 5t$$

B1

$$\text{Sand leaves with the lorry's velocity, so there is no thrust from it: } m \frac{dv}{dt} = 600$$

M1 A1

Answer: shown

3 marks

$$(b) v = \int_0^{100} \frac{600}{1500 - 5t} dt = [-120 \ln(1500 - 5t)]_0^{100}$$

M1

$$= 120 \ln \frac{1500}{1000} \approx 48.7 \text{ m s}^{-1}$$

A1

Answer: 48.7 m s⁻¹

2 marks

17 (5 marks)

$$W = \int_R^{R+h} \frac{mgR^2}{x^2} dx$$

M1

$$= mgR^2 \left[-\frac{1}{x} \right]_R^{R+h} = mgR^2 \left(\frac{1}{R} - \frac{1}{R+h} \right)$$

M1 A1

$$= mgR^2 \times \frac{h}{R(R+h)} = \frac{mgRh}{R+h}$$

A1

$$h = R, m = 1000: W = \frac{1000 \times 9.8 \times 6.4 \times 10^6}{2} = 3.14 \times 10^{10} \text{ J}$$

A1

Answer: 3.14 × 10¹⁰ J

5 marks

18 (6 marks)

$$\text{After stage 1 the mass is } 6000 - 2500 = 3500 \text{ kg}$$

B1

$$\text{Speed gained in stage 1} = 2000 \ln \frac{6000}{3500} \approx 1080 \text{ m s}^{-1}$$

M1 A1

$$\text{After dropping the casing the mass is } 3000 \text{ kg, and after stage 2 it is } 3000 - 1800 = 1200 \text{ kg}$$

B1

$$\text{Speed gained in stage 2} = 2400 \ln \frac{3000}{1200} \approx 2200 \text{ m s}^{-1}$$

M1

$$\text{Final speed} \approx 1080 + 2200 = 3280 \text{ m s}^{-1}$$

A1

Answer: 3280 m s⁻¹

6 marks

19 (6 marks)**(a)** Rate of change of momentum = 40×1.5

M1

$$F = 60 \text{ N}$$

A1

Answer: 60 N

2 marks

(b) Power = $Fv = 60 \times 1.5 = 90 \text{ W}$

B1

Rate of gain of KE = $\frac{1}{2} \times 40 \times 1.5^2$

M1

$$= 45 \text{ W}$$

A1

Answer: 90 W; 45 W

3 marks

(c) The other 45 W is lost as heat and sound through friction as the sand accelerates to the belt's speed

E1

Answer: dissipated as heat (friction)

1 mark

20 (6 marks)**(a)** Each part of the chain still in the air falls freely from rest through distance x

E1

$$v^2 = 2gx, \text{ so } v = \sqrt{2gx}$$

A1

Answer: shown

2 marks

(b) Mass on the table = λx , whose weight is λgx

B1

Mass landing per second = λv ; rate of loss of momentum = $\lambda v \times v = \lambda v^2 = 2\lambda gx$

M1 A1

Total force = $\lambda gx + 2\lambda gx = 3\lambda gx = 3 \times 2 \times 9.8 \times 0.5 = 29.4 \text{ N}$

A1

Answer: 29.4 N

4 marks

21 (9 marks)**(a)** $m \frac{dv}{dt} = 20 \times 2000 - mg$ with $m = 3000 - 20t$

M1

$$\frac{dv}{dt} = \frac{40\,000}{3000 - 20t} - 9.8$$

A1

$$v = 2000 \ln \frac{3000}{3000 - 20t} - 9.8t \quad (v = 0 \text{ at } t = 0)$$

A1

$$t = 50: v = 2000 \ln \frac{3000}{2000} - 490 \approx 321 \text{ m s}^{-1}$$

A1

Answer: 321 m s⁻¹

4 marks

(b) Use $\int \ln \frac{m_0}{m_0 - kt} dt = \frac{m_0}{k} [(1 - s) \ln(1 - s) + s]$, $s = \frac{kt}{m_0}$

M1

$$x = 2000 \times 150 \times \left[\frac{2}{3} \ln \frac{2}{3} + \frac{1}{3} \right] - 4.9 \times 2500$$

A1

$$\approx 6660 \text{ m}$$

A1

Answer: 6660 m

3 marks

(c) Further rise = $\frac{v^2}{2g} \approx 5250 \text{ m}$

M1

Greatest height $\approx 6660 + 5250 \approx 11900 \text{ m}$

A1

Answer: 11900 m

2 marks

22 (9 marks)**(a)** Mass per unit length $\lambda = 2 \text{ kg m}^{-1}$; mass raised = $2x$

B1

$$\frac{d}{dt}(mv) = F - mg \text{ with } v = 2: v \frac{dm}{dt} = F - mg$$

M1

$$\frac{dm}{dt} = \lambda \frac{dx}{dt} = 4, \text{ so } 2 \times 4 = F - 2x \times 9.8$$

A1

$$F = 19.6x + 8$$

A1

Answer: shown

4 marks

(b) $W = \int_0^3 (19.6x + 8) dx = [9.8x^2 + 8x]_0^3$

M1

$$= 88.2 + 24 = 112.2 \text{ J}$$

A1

Answer: 112.2 J

2 marks

(c) PE gain = $6 \times 9.8 \times 1.5 = 88.2 \text{ J}$ (centre of mass at height 1.5 m)

B1

KE = $\frac{1}{2} \times 6 \times 2^2 = 12 \text{ J}$, so energy lost = $112.2 - 88.2 - 12 = 12 \text{ J}$

M1 A1

Answer: 12 J lost (heat and sound as links jerk into motion)**3 marks****23 (8 marks)**

$$(a) \frac{d}{dt}(mv) = mg, \text{ so } m \frac{dv}{dt} + v \frac{dm}{dt} = mg$$

M1

$$\frac{dm}{dt} = \frac{dm}{dx} \times v = 0.02mv, \text{ and } \frac{dv}{dt} = v \frac{dv}{dx}$$

A1

$$\text{So } v \frac{dv}{dx} + 0.02v^2 = 9.8$$

A1**Answer: shown****3 marks**

$$(b) \text{ With } w = v^2: \frac{1}{2} \frac{dw}{dx} + 0.02w = 9.8, \text{ i.e. } \frac{dw}{dx} + 0.04w = 19.6$$

M1

$$\text{Integrating factor } e^{0.04x}: w e^{0.04x} = 490e^{0.04x} + c$$

A1

$$w = 0 \text{ at } x = 0 \text{ gives } c = -490, \text{ so } v^2 = 490(1 - e^{-0.04x})$$

A1**Answer: shown****3 marks**

$$(c) v^2 = 490(1 - e^{-4}), \text{ so } v \approx 21.9 \text{ m s}^{-1}$$

B1

$$\text{Limiting speed } \sqrt{490} \approx 22.1 \text{ m s}^{-1}$$

B1

$$\text{Answer: } 21.9 \text{ m s}^{-1}; 22.1 \text{ m s}^{-1}$$

2 marks**24 (8 marks)**

$$(a) m = 5000 - 4t; \text{ thrust } 4 \times 3000 = 12\,000 \text{ N upwards}$$

B1

$$\text{Taking downwards as positive: } m \frac{dv}{dt} = mg_M - 12\,000$$

M1

$$\frac{dv}{dt} = 1.6 - \frac{12\,000}{5000 - 4t}$$

A1

$$v = 1.6t + 3000 \ln(5000 - 4t) + c$$

M1

$$v = 100 \text{ at } t = 0: v = 100 + 1.6t - 3000 \ln \frac{5000}{5000 - 4t}$$

A1**Answer: shown****5 marks**

$$(b) v = 100 + 96 - 3000 \ln \frac{5000}{4760} \approx 48.4 \text{ m s}^{-1}$$

M1 A1

$$\text{Acceleration (downwards)} = 1.6 - \frac{12\,000}{4760} = -0.921, \text{ so the deceleration is } 0.921 \text{ m s}^{-2}$$

A1

$$\text{Answer: speed } 48.4 \text{ m s}^{-1}; \text{ deceleration } 0.921 \text{ m s}^{-2}$$

3 marks**25 (7 marks)**

$$(a) v^2 = u^2 - 2gR^2 \left(\frac{1}{R} - \frac{1}{x} \right)$$

M1 A1

$$v = 0: \frac{2gR^2}{x} = 2gR - u^2, \text{ so } x = \frac{2gR^2}{2gR - u^2}$$

A1**Answer: shown****3 marks**

$$(b) x = \frac{2gR^2}{2gR - gR} = 2R$$

B1**Answer: shown****1 mark**

$$(c) \frac{2gR^2}{2gR - u^2} = 60R \text{ gives } 2gR - u^2 = \frac{gR}{30}$$

M1

$$u^2 = 2gR - \frac{gR}{30} = \frac{59gR}{30}, u = \sqrt{59gR/30} \approx 1.11 \times 10^4 \text{ m s}^{-1}$$

A1

$$u/\sqrt{2gR} = \sqrt{59/60} \approx 99.2\%$$

A1

$$\text{Answer: } 1.11 \times 10^4 \text{ m s}^{-1}; 99.2\%$$

3 marks**26 (7 marks)**

$$(a) \text{ Mass at time } t: 800 + 2t; \text{ rain has no horizontal momentum}$$

B1

$$\text{Rate of change of horizontal momentum} = \text{driving force: } \frac{d}{dt}[(800 + 2t)v] = 200$$

M1

$$(800 + 2t)v = 200t + c; v = 5 \text{ at } t = 0 \text{ gives } c = 4000$$

M1

$$v = \frac{4000 + 200t}{800 + 2t}$$

A1**Answer: shown****4 marks**

$$(b) v(100) = \frac{24\,000}{1000} = 24 \text{ m s}^{-1}$$

B1

$$\frac{dv}{dt} = \frac{200 - 2v}{800 + 2t} \text{ (from the equation in (a))}$$

M1

$$t = 100: \frac{200 - 48}{1000} = 0.152 \text{ m s}^{-2}$$

A1

Answer: 24 m s⁻¹; 0.152 m s⁻²

3 marks

27 (10 marks)

$$(a) v \frac{dv}{dx} = -\frac{2}{x^2}, \text{ so } \frac{1}{2}v^2 = \frac{2}{x} + c$$

M1

$$v = 0 \text{ at } x = 4: c = -\frac{1}{2}, \text{ so } v^2 = \frac{4}{x} - 1 = 4\left(\frac{1}{x} - \frac{1}{4}\right)$$

A1

Answer: shown

2 marks

$$(b) \frac{dx}{dt} = -v = -\sqrt{\frac{4}{x} - 1} = -\sqrt{\frac{4-x}{x}}, \text{ so } t = \int_2^4 \sqrt{\frac{x}{4-x}} dx$$

M1 A1

$$x = 4 \sin^2 \theta, dx = 8 \sin \theta \cos \theta d\theta; \sqrt{\frac{x}{4-x}} = \tan \theta$$

M1

$$\text{Integrand becomes } 8 \sin^2 \theta = 4(1 - \cos 2\theta)$$

A1

$$\text{Limits: } x = 2 \text{ gives } \theta = \frac{\pi}{4}; x = 4 \text{ gives } \theta = \frac{\pi}{2}$$

B1

$$t = [4\theta - 2 \sin 2\theta]_{\pi/4}^{\pi/2} = 2\pi - (\pi - 2) = \pi + 2$$

A1

Answer: shown

6 marks

$$(c) \theta \text{ runs from } 0 \text{ to } \frac{\pi}{2}: t = [4\theta - 2 \sin 2\theta]_0^{\pi/2}$$

M1

$$t = 2\pi \text{ s} \approx 6.28 \text{ s}$$

A1

Answer: 2π s

2 marks

28 (10 marks)

$$(a) \frac{d}{dt}(mv) = mg - 0.3mv$$

M1

$$m \frac{dv}{dt} + v \frac{dm}{dt} = mg - 0.3mv \text{ with } \frac{dm}{dt} = 0.2m$$

A1

$$\frac{dv}{dt} + 0.2v = 9.8 - 0.3v, \text{ so } \frac{dv}{dt} + 0.5v = 9.8$$

A1

Answer: shown

3 marks

$$(b) \text{Integrating factor } e^{0.5t}: v e^{0.5t} = 19.6e^{0.5t} + c$$

M1 A1

$$v = 0 \text{ at } t = 0: v = 19.6(1 - e^{-0.5t}); \text{ terminal speed } 19.6 \text{ m s}^{-1}$$

A1

$$\text{Answer: } v = 19.6(1 - e^{-0.5t}); 19.6 \text{ m s}^{-1}$$

3 marks

$$(c) m = m_0 e^{0.2t} = 2m_0 \text{ when } t = 5 \ln 2 \approx 3.47 \text{ s}$$

M1

$$e^{-0.5t} = 2^{-5/2}: v = 19.6(1 - 2^{-5/2}) \approx 16.1 \text{ m s}^{-1}$$

A1

$$x = \int_0^t 19.6(1 - e^{-0.5t}) dt = 19.6[t + 2e^{-0.5t}]_0^t$$

M1

$$= 19.6(5 \ln 2 + 2 \times 2^{-5/2} - 2) \approx 35.7 \text{ m}$$

A1

Answer: 16.1 m s⁻¹; 35.7 m

4 marks

29 (10 marks)

(a) Mass per unit length = 2 kg m⁻¹; the only external force along the chain is the weight of the hanging part, 2xg

B1

$$\text{Whole chain accelerates together: } 4 \frac{d^2x}{dt^2} = 2xg$$

M1

$$\frac{d^2x}{dt^2} = 4.9x$$

A1

Answer: shown

3 marks

$$(b) v \frac{dv}{dx} = 4.9x$$

M1

$$\frac{1}{2}v^2 = 2.45x^2 + c$$

A1

$$v = 0 \text{ at } x = 0.5: v^2 = 4.9(x^2 - 0.25)$$

Answer: shown

A1
3 marks

$$(c) x = 2: v^2 = 4.9 \times 3.75, v \approx 4.29 \text{ m s}^{-1}$$

M1 A1

$$t = \int_{0.5}^2 \frac{dx}{\sqrt{4.9(x^2 - 0.25)}} = \frac{1}{\sqrt{4.9}} [\cosh^{-1}(2x)]_{0.5}^2$$

M1

$$= \frac{\cosh^{-1} 4}{2.214} \approx 0.932 \text{ s}$$

A1

Answer: 4.29 m s⁻¹; 0.932 s

4 marks

30 (11 marks)

(a) The only forces are internal and equal and opposite, so total momentum stays zero and the centre of mass stays at rest

E1

Acceleration of A: $\frac{16}{3r^2}$ towards B

B1

Acceleration of B: $\frac{16}{6r^2} = \frac{8}{3r^2}$ towards A

B1

Answer: a_A = $\frac{16}{3r^2}$, a_B = $\frac{8}{3r^2}$

3 marks

(b) Relative acceleration = sum of the two accelerations (they approach each other)

M1

$$\frac{16}{3r^2} + \frac{8}{3r^2} = \frac{8}{r^2} \text{ so } \frac{d^2r}{dt^2} = -\frac{8}{r^2}$$

A1

Answer: shown

2 marks

$$(c) \frac{1}{2} v_{\text{rel}}^2 = \int_1^4 \frac{8}{r^2} dr = 8(1 - \frac{1}{4}) = 6, \text{ so } v_{\text{rel}}^2 = 12$$

M1 A1

Momentum zero: $3v_A = 6v_B$, and $v_A + v_B = 2\sqrt{3}$

M1

$$v_A = \frac{2}{3}(2\sqrt{3}) = \frac{4\sqrt{3}}{3} \approx 2.31 \text{ m s}^{-1}; v_B = \frac{2\sqrt{3}}{3} \approx 1.15 \text{ m s}^{-1}$$

A1

Answer: A: $\frac{4\sqrt{3}}{3} \approx 2.31 \text{ m s}^{-1}$; B: $\frac{2\sqrt{3}}{3} \approx 1.15 \text{ m s}^{-1}$

4 marks

(d) Using $x = 4 \sin^2\theta$ as for a single particle attracted with $K = 8$ from rest at $r = 4$

M1

$$t = \frac{\pi}{2} \sqrt{\frac{4^3}{2 \times 8}} = \pi \text{ s} \approx 3.14 \text{ s}$$

A1

Answer: π s

2 marks

Simple Harmonic Motion

1 (3 marks)

Amplitude = 3 m; $\omega = 2$ so period = $\frac{2\pi}{2} = \pi \text{ s} \approx 3.14 \text{ s}$

B1 B1

Maximum speed = $a\omega = 3 \times 2 = 6 \text{ m s}^{-1}$

B1

Answer: 3 m; π s; 6 m s⁻¹

3 marks

2 (2 marks)

$$T = \frac{2\pi}{4} = \frac{\pi}{2} \text{ s} \approx 1.57 \text{ s}$$

B1

Maximum acceleration = $a\omega^2 = 0.5 \times 16 = 8 \text{ m s}^{-2}$

B1

Answer: $\frac{\pi}{2}$ s; 8 m s⁻²

2 marks

3 (3 marks)

Compare with $v^2 = \omega^2(a^2 - x^2)$: $\omega = 4$ and $a = 3$

M1 A1

Maximum speed $a\omega = 12 \text{ m s}^{-1}$

A1

Answer: amplitude 3 m; 12 m s⁻¹

3 marks

4 (2 marks)

$$\omega = \frac{2\pi}{2} = \pi \text{ rad s}^{-1}$$

M1

$$v_{\max} = a\omega = 0.4\pi \approx 1.26 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 0.4\pi \approx 1.26 \text{ m s}^{-1}$$

2 marks

5 (2 marks)

$$\omega^2 = \frac{k}{m} = \frac{8}{0.5} = 16, \omega = 4$$

M1

$$T = \frac{2\pi}{4} = \frac{\pi}{2} \approx 1.57 \text{ s}$$

A1

$$\text{Answer: } \frac{\pi}{2} \text{ s} \approx 1.57 \text{ s}$$

2 marks

6 (3 marks)

$$\frac{dx}{dt} = -6 \sin 3t, \frac{d^2x}{dt^2} = -18 \cos 3t$$

M1 A1

$$= -9(2 \cos 3t) = -9x; \text{ period} = \frac{2\pi}{3} \text{ s}$$

B1

$$\text{Answer: shown; period } \frac{2\pi}{3} \text{ s}$$

3 marks

7 (3 marks)

$$v^2 = \omega^2(a^2 - x^2)$$

M1

$$= 4(25 - 9) = 64$$

A1

$$v = 8 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 8 \text{ m s}^{-1}$$

3 marks

8 (4 marks)

$$\omega = \frac{2\pi}{6} = \frac{\pi}{3}, x = 0.8 \sin\left(\frac{\pi}{3}t\right)$$

M1 A1

$$0.4 = 0.8 \sin\left(\frac{\pi}{3}t\right) \text{ gives } \sin\left(\frac{\pi}{3}t\right) = \frac{1}{2}$$

M1

$$\frac{\pi}{3}t = \frac{\pi}{6}, \text{ so } t = 0.5 \text{ s}$$

A1

$$\text{Answer: } 0.5 \text{ s}$$

4 marks

9 (4 marks)

$$\text{(a) Tension} = \frac{\lambda e}{l} = \frac{24e}{0.6} = 40e = 0.4 \times 9.8$$

M1

$$e = 0.098 \text{ m}$$

A1

$$\text{Answer: } 0.098 \text{ m}$$

2 marks

$$\text{(b) } \omega^2 = \frac{k}{m} = \frac{40}{0.4} = 100, \omega = 10$$

M1

$$T = \frac{2\pi}{10} = \frac{\pi}{5} \approx 0.628 \text{ s}$$

A1

$$\text{Answer: } \frac{\pi}{5} \approx 0.628 \text{ s}$$

2 marks

10 (4 marks)

$$\text{Tension} = \frac{\lambda x}{l}$$

B1

$$\text{Equation of motion: } m \frac{d^2x}{dt^2} = -\frac{\lambda x}{l}$$

M1

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{ml}x, \text{ which is of the form } \ddot{x} = -\omega^2x \text{ with } \omega^2 = \frac{\lambda}{ml}, \text{ so the motion is SHM}$$

A1

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{\lambda}}$$

A1

$$\text{Answer: } T = 2\pi\sqrt{(ml/\lambda)}$$

4 marks

11 (5 marks)

$$(a) t = 0: x = 4 \cos \frac{\pi}{6} = 2\sqrt{3} \approx 3.46 \text{ m}$$

B1

$$\frac{dx}{dt} = -12 \sin(3t + \frac{\pi}{6})$$

M1

$$t = 0: \text{velocity} = -12 \sin \frac{\pi}{6} = -6 \text{ m s}^{-1}$$

A1

$$\text{Answer: } x = 2\sqrt{3} \text{ m; } v = -6 \text{ m s}^{-1}$$

3 marks

$$(b) \frac{d^2x}{dt^2} = -36 \cos(3t + \frac{\pi}{6}) = -9x$$

M1

$$\text{Maximum magnitude} = 36 \text{ m s}^{-2}$$

A1

$$\text{Answer: } 36 \text{ m s}^{-2}$$

2 marks

12 (4 marks)

$$\text{Amplitude } a = 0.4 \text{ m}$$

B1

$$v_{\max} = a\omega \text{ gives } \omega = \frac{1.2}{0.4} = 3, T = \frac{2\pi}{3} \approx 2.09 \text{ s}$$

M1 A1

$$\text{Acceleration at } A = a\omega^2 = 0.4 \times 9 = 3.6 \text{ m s}^{-2}$$

A1

$$\text{Answer: } \frac{2\pi}{3} \approx 2.09 \text{ s; } 3.6 \text{ m s}^{-2}$$

4 marks

13 (5 marks)

$$36 = \omega^2(a^2 - 4) \text{ and } 16 = \omega^2(a^2 - 9)$$

M1 A1

$$\text{Subtracting: } 20 = 5\omega^2, \text{ so } \omega = 2$$

M1

$$a^2 = 4 + \frac{36}{4} = 13, a = \sqrt{13} \approx 3.61 \text{ m}$$

A1

$$T = \frac{2\pi}{\omega} = \pi \approx 3.14 \text{ s}$$

A1

$$\text{Answer: } a = \sqrt{13} \approx 3.61 \text{ m; } T = \pi \text{ s}$$

5 marks

14 (7 marks)

$$(a) \frac{\lambda e}{1} = mg: \frac{30e}{1.2} = 0.5 \times 9.8$$

M1

$$e = \frac{4.9 \times 1.2}{30} = 0.196 \text{ m}$$

A1

$$\text{Answer: shown}$$

2 marks

$$(b) \text{ At displacement } x \text{ below equilibrium: } 0.5 \frac{d^2x}{dt^2} = mg - \frac{30(e+x)}{1.2} = -25x$$

M1 A1

$$\ddot{x} = -50x \text{ so SHM with } \omega = \sqrt{50} \approx 7.07, T = \frac{2\pi}{\sqrt{50}} \approx 0.889 \text{ s}$$

A1

$$v_{\max} = a\omega = 0.1\sqrt{50} \approx 0.707 \text{ m s}^{-1}$$

A1

$$\text{Answer: } T \approx 0.889 \text{ s; } v_{\max} \approx 0.707 \text{ m s}^{-1}$$

4 marks

$$(c) \text{ The amplitude } 0.1 \text{ m is less than the equilibrium extension } 0.196 \text{ m, so the extension never falls to zero}$$

E1

$$\text{Answer: } a < e$$

1 mark

15 (5 marks)

$$\omega = \frac{2\pi}{4} = \frac{\pi}{2}, x = 0.5 \sin(\frac{\pi}{2}t)$$

M1 A1

$$x = 0.2: \sin(\frac{\pi}{2}t) = 0.4, t_1 = \frac{2}{\pi} \sin^{-1} 0.4$$

M1

$$x = 0.4: t_2 = \frac{2}{\pi} \sin^{-1} 0.8$$

A1

$$t_2 - t_1 \approx 0.328 \text{ s}$$

A1

$$\text{Answer: } 0.328 \text{ s}$$

5 marks

16 (6 marks)

(a) $m \frac{d^2x}{dt^2} = -\frac{mgx}{R}$, so $\frac{d^2x}{dt^2} = -\frac{g}{R}x$, which is SHM

M1 A1

$$\omega = \sqrt{\frac{g}{R}} \text{ so } T = 2\pi\sqrt{\frac{R}{g}}$$

M1

$$= 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}} \approx 5080 \text{ s} \approx 84.6 \text{ minutes}$$

A1

Answer: $T \approx 84.6$ minutes

4 marks

(b) $v_{\max} = R\omega = R\sqrt{\frac{g}{R}} = \sqrt{gR}$

M1

$$= \sqrt{9.8 \times 6.4 \times 10^6} \approx 7920 \text{ m s}^{-1}$$

A1

Answer: 7920 m s^{-1}

2 marks

17 (6 marks)

$$\text{Area} = 20 \times 10^{-4} = 2 \times 10^{-3} \text{ m}^2$$

B1

$$\text{Extra depth } x \text{ below equilibrium gives extra upthrust} = 1000 \times 2 \times 10^{-3}x \times g = 19.6x \text{ N}$$

M1 A1

$$\text{N2L: } 0.2 \frac{d^2x}{dt^2} = -19.6x$$

M1

$$\ddot{x} = -98x, \text{ so SHM with } \omega = \sqrt{98}$$

A1

$$T = \frac{2\pi}{\sqrt{98}} \approx 0.635 \text{ s}$$

A1

Answer: $T \approx 0.635$ s

6 marks

18 (6 marks)

Let P be displaced x towards B from equilibrium. Tension in spring AP increases by $12x$; tension in spring BP decreases by $20x$

M1 A1

$$\text{Resultant force towards equilibrium} = 12x + 20x = 32x$$

M1

$$0.8\ddot{x} = -32x, \text{ so } \ddot{x} = -40x$$

A1

$$\text{SHM with } \omega = \sqrt{40} = 2\sqrt{10}$$

A1

$$T = \frac{2\pi}{2\sqrt{10}} = \frac{\pi}{\sqrt{10}} \approx 0.993 \text{ s}$$

A1

Answer: $T = \pi/\sqrt{10} \approx 0.993$ s

6 marks

19 (6 marks)

(a) $\frac{d^2\theta}{dt^2} \approx -\frac{g}{L}\theta = -10\theta$

M1 A1

$$\omega = \sqrt{10}, T = \frac{2\pi}{\sqrt{10}} \approx 1.99 \text{ s}$$

A1

Answer: $T \approx 1.99$ s

3 marks

(b) Amplitude of θ is 0.1, so $\theta = 0.1 \cos(\sqrt{10}t)$

M1

$$\text{Greatest angular speed} = 0.1\sqrt{10}$$

A1

$$\text{Greatest speed} = L\omega_{\max} = 0.98 \times 0.1\sqrt{10} \approx 0.310 \text{ m s}^{-1}$$

A1

Answer: 0.310 m s^{-1}

3 marks

20 (5 marks)

$$x = 0.5 \sin \pi t, \omega = \pi$$

B1

$$|x| \leq 0.25 \text{ when } |\sin \pi t| \leq \frac{1}{2}$$

M1

$$\text{In one quarter period: } \pi t \leq \frac{\pi}{6}, \text{ i.e. } t \leq \frac{1}{6} \text{ s}$$

A1

$$\text{Time in one period} = 4 \times \frac{1}{6} = \frac{2}{3} \text{ s}$$

M1

$$\text{Fraction} = \frac{2}{3} \div 2 = \frac{1}{3}$$

A1

Answer: $\frac{1}{3}$

5 marks

21 (8 marks)

(a) Equilibrium extension $e = 0.196 \text{ m} < 0.4 \text{ m}$, so the string goes slack (the particle rises above the natural length point) E1

SHM about equilibrium with $\omega^2 = \frac{\lambda}{ml} = 50$ B1

String slack when the particle is $e = 0.196 \text{ m}$ above equilibrium M1

$$v^2 = \omega^2(a^2 - e^2) = 50(0.16 - 0.038416)$$
 M1

$$= 6.08, v \approx 2.47 \text{ m s}^{-1}$$
 A1

Answer: $v \approx 2.47 \text{ m s}^{-1}$ 5 marks

(b) After the string goes slack the particle moves freely: further rise $= \frac{v^2}{2g} = 0.310 \text{ m}$ M1 A1

Height above equilibrium $= 0.196 + 0.310 \approx 0.506 \text{ m}$ A1

Answer: 0.506 m 3 marks

22 (8 marks)

(a) With extension x : $0.5\ddot{x} = -\frac{10x}{0.4}$, so $\ddot{x} = -50x$: SHM with $\omega = \sqrt{50}$ M1 A1

Amplitude 0.3 m , so speed at N (centre) $= 0.3\sqrt{50}$ M1

$$\approx 2.12 \text{ m s}^{-1}$$
 A1

Answer: 2.12 m s⁻¹ 4 marks

(b) Time from release to N = quarter period $= \frac{\pi}{2\sqrt{50}}$, and the same on the return: total $\frac{\pi}{\sqrt{50}} \approx$ M1 A1

$$0.444 \text{ s}$$

Time slack (to wall and back) $= \frac{2 \times 0.15}{2.12} \approx 0.141 \text{ s}$ M1

$$\text{Total} \approx 0.586 \text{ s}$$
 A1

Answer: 0.586 s 4 marks

23 (9 marks)

(a) The student has used the period as ω ; in fact $\omega = \frac{2\pi}{T}$ E1

$$\omega = \pi$$
 M1

$$v_{\max} = a\omega = 0.5\pi \approx 1.57 \text{ m s}^{-1}$$
 A1

Answer: $\omega = \pi$; 1.57 m s⁻¹ 3 marks

(b) $x = 0.5 \sin \pi t$, $v = 0.5\pi \cos \pi t$ M1 A1

Moving towards O means $v < 0$; speed 1 needs $0.5\pi \cos \pi t = -1$, $\cos \pi t = -\frac{2}{\pi}$ M1

$$\pi t = \cos^{-1}\left(-\frac{2}{\pi}\right) \approx 2.26 \text{ rad}, t \approx 0.720 \text{ s}$$
 A1

Answer: 0.720 s 4 marks

(c) $x = 0.5 \sin(\pi t)$ with $\cos \pi t = -\frac{2}{\pi}$, so $\sin \pi t = +\sqrt{1 - \frac{4}{\pi^2}}$ M1

$$x = 0.5\sqrt{1 - \frac{4}{\pi^2}} \approx 0.386 \text{ m}$$
 A1

Answer: 0.386 m 2 marks

24 (8 marks)

Speed of B before impact: $u = \sqrt{2 \times 9.8 \times 0.1} = 1.4 \text{ m s}^{-1}$ M1 A1

Conservation of momentum: $0.5v = 0.3 \times 1.4$, $v = 0.84 \text{ m s}^{-1}$ M1 A1

Old equilibrium extension $\frac{0.2g}{50} = 0.0392 \text{ m}$; new equilibrium extension $\frac{0.5g}{50} = 0.098 \text{ m}$, so after impact the particle is 0.0588 m above the new equilibrium M1 A1

$$\omega^2 = \frac{50}{0.5} = 100$$
 B1

$$a^2 = x^2 + \frac{v^2}{\omega^2} = 0.0588^2 + 0.007056, \text{ so } a \approx 0.103 \text{ m}$$
 A1

Answer: 0.103 m 8 marks

25 (7 marks)

(a) $x = A \cos 2t + B \sin 2t$; $x(0) = 1$ gives $A = 1$ M1 A1

$\frac{dx}{dt}(0) = 2B = -2\sqrt{3}$, so $B = -\sqrt{3}$ A1

$x = \cos 2t - \sqrt{3} \sin 2t = 2 \cos(2t + \frac{\pi}{3})$, $R = 2$, $\alpha = \frac{\pi}{3}$ A1

Answer: $x = 2\cos(2t + \frac{\pi}{3})$ 4 marks

(b) Amplitude 2 m; maximum speed $2 \times 2 = 4 \text{ m s}^{-1}$ B1

$\cos(2t + \frac{\pi}{3}) = 0$ first when $2t + \frac{\pi}{3} = \frac{\pi}{2}$ M1

$t = \frac{\pi}{12} \approx 0.262 \text{ s}$ A1

Answer: 2 m; 4 m s⁻¹; $\frac{\pi}{12} \text{ s}$ 3 marks

26 (9 marks)

(a) $20 = \frac{1}{2} \times 9.8t^2$, $t = \sqrt{\frac{40}{9.8}} \approx 2.02 \text{ s}$ M1 A1

$v = gt = \sqrt{2 \times 9.8 \times 20} \approx 19.8 \text{ m s}^{-1}$ A1

Answer: $t \approx 2.02 \text{ s}$; $v \approx 19.8 \text{ m s}^{-1}$ 3 marks

(b) Take x downwards from equilibrium: $70\ddot{x} = 70g - \frac{1400}{20}(9.8 + x) = -70x$ M1

So $\ddot{x} = -x$: SHM with $\omega = 1$ A1

Equilibrium: $70x = 70 \times 9.8$, extension 9.8 m A1

Answer: $\omega = 1$; extension 9.8 m 3 marks

(c) At the instant the rope becomes taut she is 9.8 m above equilibrium with speed 19.8: $a^2 = 9.8^2 + 19.8^2$ ($\omega = 1$) M1 A1

$a \approx 22.1 \text{ m}$; greatest depth $= 20 + 9.8 + a \approx 51.9 \text{ m}$ A1

Answer: 51.9 m below the platform 3 marks

27 (11 marks)

(a) Friction $F = 0.2 \times 0.5 \times 9.8 = 0.98 \text{ N}$ opposes motion B1

$0.5\ddot{x} = -50x + 0.98$ where x is the extension, i.e. $\ddot{x} = -100(x - 0.0196)$ M1 A1

So SHM about $x = 0.0196$ with $\omega = 10$, period $\frac{2\pi}{10} = \frac{\pi}{5} \text{ s}$ A1

Answer: shown 4 marks

(b) First half: amplitude about 0.0196 is $0.2 - 0.0196 = 0.1804$, so it stops at $0.0196 - 0.1804 = -0.1608$ M1 A1

Next half (friction reverses): centre -0.0196 , amplitude $0.1608 - 0.0196 = 0.1412$, so it ends at $0.1412 - 0.0196 = 0.1216$; again a drop of 0.0392 A1

Answer: shown 3 marks

(c) Stops when $|x| \leq 0.0196$ (spring force $\leq$ limiting friction): $|x_n| = 0.2 - 0.0392n$ M1

$n = 5$ gives $|x| = 0.004 < 0.0196$, so there are 5 half-oscillations A1

Time $= 5 \times \frac{\pi}{10} = \frac{\pi}{2} \approx 1.57 \text{ s}$ A1

Swings have lengths 0.2 + 0.1608, 0.1608 + 0.1216, 0.1216 + 0.0824, 0.0824 + 0.0432, 0.0432 + 0.004; total $\approx 1.02 \text{ m}$ A1

Answer: $t = \frac{\pi}{2} \text{ s} \approx 1.57 \text{ s}$; distance $\approx 1.02 \text{ m}$ 4 marks

28 (10 marks)

(a) The string has length $2\sqrt{0.64 + y^2}$, so extension $2\sqrt{0.64 + y^2} - 1$ B1

$$\text{Tension} = \frac{\lambda x}{l} = 20(2\sqrt{0.64 + y^2} - 1)$$
 M1

$$\text{Each half makes angle } \phi \text{ with AB where } \sin \phi = \frac{y}{\sqrt{0.64 + y^2}}$$
 M1

$$\text{Force towards M} = 2T \sin \phi = \frac{2Ty}{\sqrt{0.64 + y^2}}$$
 A1

Answer: shown 4 marks

(b) For small y , $\sqrt{0.64 + y^2} \approx 0.8$, so $T \approx 20(1.6 - 1) = 12 \text{ N}$ M1 A1

$$\text{Force} \approx \frac{2 \times 12y}{0.8} = 30y$$
 M1

$$0.4\ddot{y} = -30y, \ddot{y} = -75y: \text{SHM with } \omega = \sqrt{75}$$
 A1

$$T = \frac{2\pi}{\sqrt{75}} = \frac{2\pi}{5\sqrt{3}} \approx 0.726 \text{ s}$$
 A1

Answer: $T \approx 0.726 \text{ s}$ 5 marks

(c) Because $\sqrt{0.64 + y^2} \approx 0.8$ requires $y^2 \ll 0.64$; for larger y the tension and angle change significantly and the force is no longer proportional to y E1

Answer: force not proportional to y for large y 1 mark

29 (10 marks)

$$(a) \omega^2 = \frac{k}{(m_A + m_B)} = \frac{60}{2} = 30$$
 M1

$$T = \frac{2\pi}{\sqrt{30}} \approx 1.15 \text{ s}$$
 A1

Answer: $T \approx 1.15 \text{ s}$ 2 marks

(b) B moves with the same acceleration as A: $|a| = \omega^2 x = 30x$ M1

$$\text{Friction on B} = m_B |a| = 0.5 \times 30x = 15x \text{ N}$$
 A1

Answer: shown 2 marks

(c) B does not slip if $15x \leq \mu R = 0.4 \times 0.5 \times 9.8 = 1.96$ M1 A1

$$\text{Greatest amplitude: } x = \frac{1.96}{15} \approx 0.131 \text{ m}$$
 A1

Answer: 0.131 m 3 marks

(d) Slipping begins at $|x| = 0.131 \text{ m}$ B1

$$v^2 = \omega^2(a^2 - x^2) = 30(0.04 - 0.131^2)$$
 M1

$$v \approx 0.829 \text{ m s}^{-1}$$
 A1

Answer: 0.829 m s⁻¹ 3 marks

30 (11 marks)

$$(a) \text{Equilibrium extension: } \frac{30e}{1.2} = 0.5g, e = 0.196 \text{ m}$$
 B1

$$\text{Displacement } x \text{ below equilibrium: } 0.5\ddot{x} = 0.5g - \frac{30(0.196 + x)}{1.2} = -25x$$
 M1

$$\ddot{x} = -50x, \text{ so } \omega = \sqrt{50}$$
 A1

Answer: $e = 0.196 \text{ m}$ 3 marks

(b) $x = 0.4 \cos(\sqrt{50}t)$; the string becomes slack when $x = -0.196$ M1

$$\cos(\sqrt{50}t) = -0.49, \text{ so } \sqrt{50}t = \cos^{-1}(-0.49) \approx 2.08, t \approx 0.295 \text{ s}$$
 M1 A1

$$v^2 = 50(0.4^2 - 0.196^2), v \approx 2.47 \text{ m s}^{-1}$$
 A1

Answer: $t \approx 0.295 \text{ s}$; $v \approx 2.47 \text{ m s}^{-1}$ 4 marks

(c) Time taut going up = 0.295 s, the same time again on the way down B1

$$\text{Time slack: projected upwards at } 2.47 \text{ m s}^{-1} \text{ under gravity, time } \frac{2v}{g} = 0.503 \text{ s}$$
 M1 A1

$$\text{Period} = 2 \times 0.295 + 0.503 \approx 1.09 \text{ s}$$
 A1

Answer: 1.09 s 4 marks

Damped and Forced Motion

1 (3 marks)

Auxiliary equation $m^2 + 5m + 6 = 0$, so $(m + 2)(m + 3) = 0$ and $m = -2$ or -3 M1 A1

$x = Ae^{-2t} + Be^{-3t}$; two distinct real roots so overdamped B1

Answer: $x = Ae^{-2t} + Be^{-3t}$; overdamped 3 marks

2 (2 marks)

Forces along the line: spring $-8x$ and resistance $-4\frac{dx}{dt}$ B1

$$2\frac{d^2x}{dt^2} = -8x - 4\frac{dx}{dt}, \text{ so } \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = 0$$

M1

Answer: shown 2 marks

3 (3 marks)

$m^2 + 6m + 9 = (m + 3)^2 = 0$ has a repeated root $m = -3$ (discriminant $36 - 36 = 0$), so critically damped M1 A1

$$x = (A + Bt)e^{-3t}$$

B1

Answer: $x = (A + Bt)e^{-3t}$ 3 marks

4 (3 marks)

$$m^2 + 2m + 5 = 0 \text{ gives } m = -1 \pm 2i$$

M1

$$x = e^{-t}(A \cos 2t + B \sin 2t)$$

A1

$$\text{Period} = \frac{2\pi}{2} = \pi$$

B1

Answer: $x = e^{-t}(A \cos 2t + B \sin 2t)$; period π 3 marks

5 (2 marks)

Critical damping: discriminant $c^2 - 4 \times 16 = 0$ M1

$$c = 8$$

A1

Answer: $c = 8$ 2 marks

6 (3 marks)

$$x = 2: \frac{d^2x}{dt^2} = 0, \text{ so } 0 + 4 \times 2 = 8$$

B1

Complementary function: $A \cos 2t + B \sin 2t$ B1

$$x = A \cos 2t + B \sin 2t + 2$$

B1

Answer: $x = A \cos 2t + B \sin 2t + 2$ 3 marks

7 (5 marks)

$$m^2 + 7m + 12 = 0, m = -3, -4$$

M1 A1

$$x = Ae^{-3t} + Be^{-4t}, \text{ and } A + B = 1$$

B1

$$\frac{dx}{dt} = -3Ae^{-3t} - 4Be^{-4t} = 0 \text{ at } t = 0: 3A + 4B = 0$$

M1

$$A = 4, B = -3, \text{ so } x = 4e^{-3t} - 3e^{-4t}$$

A1

Answer: $x = 4e^{-3t} - 3e^{-4t}$ 5 marks

8 (5 marks)

(a) Repeated root $m = -2$, so $x = (A + Bt)e^{-2t}$ M1

$$x(0) = 3 \text{ gives } A = 3$$

A1

$$\frac{dx}{dt} = (B - 2A - 2Bt)e^{-2t}, \text{ and at } t = 0: B - 6 = -2$$

M1

$$B = 4, \text{ so } x = (3 + 4t)e^{-2t}$$

A1

Answer: $x = (3 + 4t)e^{-2t}$ 4 marks

(b) $3 + 4t > 0$ and $e^{-2t} > 0$ for $t > 0$, so $x > 0$ E1

Answer: $x > 0$ for all t **1 mark****9 (5 marks)**

$$m = -1 \pm 2i, \text{ so } x = e^{-t}(A \cos 2t + B \sin 2t)$$

M1 A1

$$x(0) = 2 \text{ gives } A = 2$$

B1

$$\frac{dx}{dt} = e^{-t}[(-A + 2B) \cos 2t + (-B - 2A) \sin 2t]; \text{ at } t = 0: -A + 2B = 0$$

M1

$$B = 1, \text{ so } x = e^{-t}(2 \cos 2t + \sin 2t)$$

A1

Answer: $x = e^{-t}(2 \cos 2t + \sin 2t)$

5 marks**10 (5 marks)**

$$\text{Try } x = p \cos t + q \sin t: \frac{dx}{dt} = -p \sin t + q \cos t, \frac{d^2x}{dt^2} = -p \cos t - q \sin t$$

M1

$$\text{Coefficients of } \cos t: -p + 3q + 2p = 0, \text{ i.e. } p + 3q = 0$$

A1

$$\text{Coefficients of } \sin t: -q - 3p + 2q = 10, \text{ i.e. } q - 3p = 10$$

A1

$$p = -3, q = 1$$

A1

$$x = Ae^{-t} + Be^{-2t} + \sin t - 3 \cos t$$

A1

Answer: $x = Ae^{-t} + Be^{-2t} + \sin t - 3 \cos t$

5 marks**11 (5 marks)**

$$0.5\ddot{x} = -18x - 3\dot{x}, \text{ so } \ddot{x} + 6\dot{x} + 36x = 0$$

M1 A1

$$\text{Discriminant } 36 - 144 < 0, \text{ so underdamped}$$

B1

$$m = -3 \pm i\sqrt{27} = -3 \pm 3\sqrt{3}i$$

M1

$$x = e^{-3t}(A \cos 3\sqrt{3}t + B \sin 3\sqrt{3}t)$$

A1

Answer: underdamped; $x = e^{-3t}(A \cos 3\sqrt{3}t + B \sin 3\sqrt{3}t)$

5 marks**12 (4 marks)**

Maxima of x occur approximately at the maxima of $\cos 3t$, spaced $\frac{2\pi}{3}$ apart (the factor $e^{-0.2t}$ only scales each value)

E1

$$\text{Maximum values at } t \text{ and } t + \frac{2\pi}{3} \text{ have ratio } \frac{e^{-0.2(t + 2\pi/3)}}{e^{-0.2t}}$$

M1

$$= e^{-0.4\pi/3} = e^{-2\pi/15}$$

A1

$$\approx 0.658$$

A1

Answer: 0.658

4 marks**13 (6 marks)**

$$\text{Complementary function: } A \cos 3t + B \sin 3t$$

B1

$$\text{Try } x = a \cos 2t: -4a + 9a = 18$$

M1 A1

$$a = \frac{18}{5}, \text{ so } x = A \cos 3t + B \sin 3t + \frac{18}{5} \cos 2t$$

A1

$$x(0) = 0 \text{ gives } A = -\frac{18}{5}; \frac{dx}{dt}(0) = 0 \text{ gives } B = 0$$

M1

$$x = \frac{18}{5}(\cos 2t - \cos 3t)$$

A1

Answer: $x = \frac{18}{5}(\cos 2t - \cos 3t)$

6 marks**14 (7 marks)**

$$(a) \ddot{x} + \frac{c}{1000}\dot{x} + 40x = 0; \text{ critical when } \left(\frac{c}{1000}\right)^2 = 4 \times 40$$

M1 A1

$$c = 1000\sqrt{160} \approx 12600 \text{ N s m}^{-1}$$

A1

Answer: 12600 N s m⁻¹

3 marks

$$(b) \ddot{x} + 10\dot{x} + 40x = 0; \text{ discriminant } 100 - 160 < 0 \text{ so underdamped}$$

M1 A1

$$m = -5 \pm i\sqrt{15}$$

M1

$$\text{Period} = \frac{2\pi}{\sqrt{15}} \approx 1.62 \text{ s}$$

A1

Answer: $T \approx 1.62$ s**4 marks****15 (7 marks)****(a)** Complementary function: $m^2 + 4m + 5 = 0$, $m = -2 \pm i$, so $e^{-2t}(A \cos t + B \sin t)$ **M1 A1**Try $x = p \cos t + q \sin t$: $\cos t$ terms $4p + 4q = 10$ **M1 A1** $\sin t$ terms: $4q - 4p = 0$, so $p = q = \frac{5}{4}$ **A1****Answer: $x = e^{-2t}(A \cos t + B \sin t) + \frac{5}{4}(\cos t + \sin t)$** **5 marks****(b)** The e^{-2t} term decays to zero, so $x \rightarrow \frac{5}{4}(\cos t + \sin t)$ **B1**Amplitude = $\frac{5}{4}\sqrt{2} \approx 1.77$ **B1****Answer: steady state amplitude $\frac{5\sqrt{2}}{4} \approx 1.77$** **2 marks****16 (7 marks)****(a)** These terms are part of the complementary function, so they would give 0 on the left-hand side**E1****Answer: part of the complementary function****1 mark****(b)** Try $x = at \cos 2t$: $\frac{dx}{dt} = a \cos 2t - 2at \sin 2t$, $\frac{d^2x}{dt^2} = -4a \sin 2t - 4at \cos 2t$ **M1 A1**Substituting: $-4a \sin 2t = 8 \sin 2t$, $a = -2$ **A1** $x = A \cos 2t + B \sin 2t - 2t \cos 2t$; $x(0) = 0$ gives $A = 0$ **M1** $\frac{dx}{dt}(0) = 0$ gives $2B - 2 = 0$, $B = 1$, so $x = \sin 2t - 2t \cos 2t$ **A1****Answer: $x = \sin 2t - 2t \cos 2t$** **5 marks****(c)** The amplitude grows without limit (resonance), approximately $2t$ **E1****Answer: grows without limit****1 mark****17 (7 marks)****(a)** $2\ddot{x} = -8x - 4\dot{x} + 10 \cos t$ **M1** $\ddot{x} + 2\dot{x} + 4x = 5 \cos t$ **A1****Answer: shown****2 marks****(b)** Try $x = a \cos t + b \sin t$ **M1** $\cos t$: $-a + 2b + 4a = 5$, i.e. $3a + 2b = 5$; $\sin t$: $-b - 2a + 4b = 0$, i.e. $3b = 2a$ **M1 A1** $a = \frac{15}{13}$, $b = \frac{10}{13}$ **A1**Amplitude = $\sqrt{a^2 + b^2} = \frac{5}{\sqrt{13}} \approx 1.39$ m**A1****Answer: $x = \frac{15}{13}\cos t + \frac{10}{13}\sin t$; amplitude $\frac{5}{\sqrt{13}} \approx 1.39$ m****5 marks****18 (6 marks)****(a)** $m = -1 \pm 3i$, $x = e^{-t}(A \cos 3t + B \sin 3t)$, $A = 3$ **M1 A1** $\frac{dx}{dt} = e^{-t}[(-A + 3B) \cos 3t + (-B - 3A) \sin 3t]$ **M1** $-A + 3B = 0$ gives $B = 1$ **A1****Answer: shown****4 marks****(b)** $3 \cos 3t + \sin 3t = 0$, $\tan 3t = -3$, $3t = \pi - \tan^{-1} 3$ **M1** $t = \frac{\pi - \tan^{-1} 3}{3} \approx 0.631$ s**A1****Answer: $t \approx 0.631$ s****2 marks****19 (7 marks)****(a)** $x = e^{-t}(A \cos 2t + B \sin 2t)$, $x(0) = 0$ so $A = 0$ **M1 A1** $\frac{dx}{dt}(0) = 2B = 4$, so $B = 2$ **A1****Answer: shown****3 marks**

$$(b) \frac{dx}{dt} = 2e^{-t}(2 \cos 2t - \sin 2t) = 0, \tan 2t = 2$$

M1 A1

$$t = \frac{1}{2} \tan^{-1} 2 \approx 0.554 \text{ s}$$

A1

$$x \approx 1.03 \text{ m}$$

A1

Answer: $t \approx 0.554 \text{ s}$; $x \approx 1.03$

4 marks

20 (6 marks)

(a) Oscillatory when $k^2 - 36 < 0$

M1

$$0 < k < 6$$

A1

Critical damping: $k = 6$

A1

Answer: $0 < k < 6$; $k = 6$

3 marks

$$(b) m^2 + 2m + 9 = 0, m = -1 \pm 2\sqrt{2}i; \omega = 2\sqrt{2}$$

M1

$$\text{Period} = \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}} \approx 2.22 \text{ s}$$

A1

$$\text{Factor } e^{-pT} = e^{-\pi/\sqrt{2}} \approx 0.108$$

A1

Answer: $T = \pi/\sqrt{2} \approx 2.22 \text{ s}$; factor 0.108

3 marks

21 (9 marks)

(a) Try $x = a \cos \omega t + b \sin \omega t$

M1

$$\cos: (25 - \omega^2)a + 2\omega b = 20; \sin: (25 - \omega^2)b - 2\omega a = 0$$

M1 A1

$$\text{Multiply and add: } a^2 + b^2 = \frac{400}{(25 - \omega^2)^2 + 4\omega^2}$$

M1

$$A = \sqrt{a^2 + b^2} = \frac{20}{\sqrt{(25 - \omega^2)^2 + 4\omega^2}}$$

A1

Answer: shown

5 marks

(b) Minimise $g = (25 - \omega^2)^2 + 4\omega^2$ with $u = \omega^2$: $g = (25 - u)^2 + 4u$

M1

$$\frac{dg}{du} = -2(25 - u) + 4 = 0 \text{ gives } u = 23$$

A1

$$g(23) = 4 + 92 = 96 \text{ (a minimum, since } g \text{ is a positive quadratic in } u)$$

M1

$$A_{\max} = \frac{20}{\sqrt{96}} \approx 2.04$$

A1

Answer: $\omega^2 = 23$; $A_{\max} = \frac{20}{\sqrt{96}} \approx 2.04$

4 marks

22 (7 marks)

Complementary function: $m^2 + 3m + 2 = 0$, $m = -1, -2$, so $Ae^{-t} + Be^{-2t}$

M1 A1

$$\text{Try } x = ae^{-3t}: 9a - 9a + 2a = 4$$

M1 A1

$$a = 2, \text{ so } x = Ae^{-t} + Be^{-2t} + 2e^{-3t}$$

A1

$$x(0) = 1: A + B = -1; \frac{dx}{dt}(0) = 0: -A - 2B - 6 = 0$$

M1

$$A = 4, B = -5: x = 4e^{-t} - 5e^{-2t} + 2e^{-3t}$$

A1

Answer: $x = 4e^{-t} - 5e^{-2t} + 2e^{-3t}$

7 marks

23 (7 marks)

(a) $Ae^{-2t} + Be^{-2t} = (A + B)e^{-2t}$ has only one arbitrary constant, so it cannot satisfy two initial conditions; the general solution is $(A + Bt)e^{-2t}$

E1

Answer: only one arbitrary constant

1 mark

(b) $x = (A + Bt)e^{-2t}$, $x(0) = 0$ so $A = 0$

M1 A1

$$\frac{dx}{dt} = B(1 - 2t)e^{-2t}, \text{ at } t = 0: B = 3, \text{ so } x = 3te^{-2t}$$

A1

Answer: $x = 3te^{-2t}$

3 marks

$$(c) \frac{dx}{dt} = 3(1 - 2t)e^{-2t} = 0 \text{ gives } t = \frac{1}{2} \text{ s}$$

M1 A1

$$x = \frac{3}{2}e^{-1} \approx 0.552 \text{ m}$$

A1

Answer: $t = 0.5 \text{ s}$; $x \approx 0.552 \text{ m}$

3 marks

24 (8 marks)

(a) $m = -1, -4; x = Ae^{-t} + Be^{-4t}$

M1

$x(0) = 0: A = -B; \frac{dx}{dt}(0) = -A - 4B = 6, \text{ so } 3A = 6$

M1

$x = 2(e^{-t} - e^{-4t})$

A1

Answer: $x = 2(e^{-t} - e^{-4t})$

3 marks

(b) $\frac{dx}{dt} = 2(-e^{-t} + 4e^{-4t}) = 0$

M1

$e^{-3t} = \frac{1}{4}, t = \frac{1}{3} \ln 4 \approx 0.462 \text{ s}$

A1

$x = 2(4^{-1/3} - 4^{-4/3}) = 2 \times 4^{-1/3}(1 - \frac{1}{4})$

M1

$\approx 0.945 \text{ m}$

A1

Answer: $t = \frac{1}{3} \ln 4; x \approx 0.945$

4 marks

(c) $e^{-t} > e^{-4t}$ for $t > 0$, so $x > 0$; $x \rightarrow 0$ as $t \rightarrow \infty$

E1

Answer: $x > 0$; limit 0

1 mark

25 (8 marks)

(a) Try $x = a \cos 2t + b \sin 2t$

M1

$\cos 2t: (13 - 4)a + 12b = 26; \sin 2t: (13 - 4)b - 12a = 0$

M1 A1

$9b = 12a \text{ so } b = \frac{4a}{3}; 9a + 16a = 26$

M1

$a = \frac{26}{25}, b = \frac{104}{75}$

A1

Answer: $x = \frac{26}{25} \cos 2t + \frac{104}{75} \sin 2t$

5 marks

(b) $R = \sqrt{a^2 + b^2} = \frac{26}{25} \sqrt{1 + \frac{16}{9}} = \frac{26}{25} \times \frac{5}{3} = \frac{26}{15}$

M1 A1

$\tan \varphi = \frac{b}{a} = \frac{4}{3}, \varphi \approx 0.927 \text{ rad}$

A1

Answer: $R = \frac{26}{15}, \varphi \approx 0.927$

3 marks

26 (8 marks)

(a) $4\ddot{x} = -100x - 16\dot{x}$ (the weight is balanced by the equilibrium tension), so $\ddot{x} + 4\dot{x} + 25x = 0$

M1 A1

$m = -2 \pm i\sqrt{21}; x = e^{-2t}(A \cos \sqrt{21}t + B \sin \sqrt{21}t)$

M1 A1

$x(0) = 0.2 \text{ gives } A = 0.2; \frac{dx}{dt}(0) = -2A + \sqrt{21}B = 0$

M1

$B = \frac{0.4}{\sqrt{21}}, \text{ so } x = e^{-2t}(0.2 \cos \sqrt{21}t + \frac{0.4}{\sqrt{21}} \sin \sqrt{21}t)$

A1

Answer: $x = e^{-2t}(0.2 \cos \sqrt{21}t + (0.4/\sqrt{21}) \sin \sqrt{21}t)$

6 marks

(b) $0.2 \cos \sqrt{21}t + \frac{0.4}{\sqrt{21}} \sin \sqrt{21}t = 0, \tan \sqrt{21}t = -\frac{\sqrt{21}}{2}$

M1

$\sqrt{21}t = \pi - \tan^{-1} \frac{\sqrt{21}}{2}, t \approx 0.433 \text{ s}$

A1

Answer: $t \approx 0.433 \text{ s}$

2 marks

27 (10 marks)

(a) $\cos 3t$ and $\sin 3t$ appear in the complementary function (resonance), so multiply by t

E1

$\frac{dx}{dt} = a \cos 3t + b \sin 3t + t(-3a \sin 3t + 3b \cos 3t)$

M1

$\frac{d^2x}{dt^2} = -6a \sin 3t + 6b \cos 3t - 9x$

M1 A1

Hence $-6a \sin 3t + 6b \cos 3t = 6 \cos 3t, a = 0, b = 1: x_p = t \sin 3t$

A1

Answer: $x_p = t \sin 3t$

5 marks

(b) $x = A \cos 3t + B \sin 3t + t \sin 3t; x(0) = 0 \text{ gives } A = 0$

M1

$$\frac{dx}{dt}(0) = 3B = 0, \text{ so } x = t \sin 3t$$

A1

Answer: $x = t \sin 3t$

2 marks

(c) $x = t \sin 3t$, so the oscillations are bounded by $\pm t$ and the amplitude increases linearly

E1

Displacement is largest in magnitude when $|\sin 3t| = 1$, i.e. $3t = \frac{\pi}{2} + n\pi$; the first of these with $t \geq 10$ has $n = 10$

M1

$$t = \frac{(\pi/2) + 10\pi/3}{3} \approx 11.0 \text{ s}$$

A1

Answer: $t \approx 11.0 \text{ s}$

3 marks

28 (10 marks)(a) Newton's second law: $\ddot{x} = -kx - c\dot{x} + 10 \cos 2t$

B1

Answer: shown

1 mark

(b) $\dot{x} = -4 \cos \varphi \sin 2t + 4 \sin \varphi \cos 2t$, $\ddot{x} = -8 \cos \varphi \cos 2t - 8 \sin \varphi \sin 2t$

M1 A1

cos 2t terms: $-8 \cos \varphi + 4c \sin \varphi + 2k \cos \varphi = 10$

M1

i.e. $2(k - 4) \cos \varphi + 4c \sin \varphi = 10 \rightarrow (k - 4) \cos \varphi + 2c \sin \varphi = 5$; sin 2t terms give the second equation

A1

Answer: shown

4 marks

(c) From the second equation $\tan \varphi = \frac{2c}{k - 4} = \frac{3}{4}$, so $k - 4 = \frac{8c}{3}$

M1 A1

$$\cos \varphi = \frac{4}{5}, \sin \varphi = \frac{3}{5}$$

B1

$$\frac{8c}{3} \times \frac{4}{5} + 2c \times \frac{3}{5} = 5, \text{ so } \frac{32c}{15} + \frac{18c}{15} = 5$$

M1

$$c = 1.5, k = 4 + 4 = 8$$

A1

Answer: $c = 1.5, k = 8$

5 marks

29 (9 marks)(a) $x = (A + Bt)e^{-pt}$, $A = a$

M1 A1

$$\frac{dx}{dt} = (B - pA - pBt)e^{-pt}; \text{ at } t = 0: B - pa = -v, \text{ so } B = pa - v$$

A1

Answer: shown

3 marks

(b) $x = 0$ requires $a + (pa - v)t = 0$ (since $e^{-pt} > 0$)

M1

$$t = \frac{a}{v - pa} \text{ is positive only if } v > pa$$

A1

(if $v \leq pa$ the factor $a + (pa - v)t > 0$ for all $t > 0$, so no zero)

E1

Answer: $t = \frac{a}{(v - pa)}$ when $v > pa$

3 marks

$$(c) t = \frac{0.5}{4 - 1.5} = 0.2 \text{ s}$$

B1

$$x = (0.5 - 2.5t)e^{-3t}, \frac{dx}{dt} = (-2.5 - 1.5 + 7.5t)e^{-3t} = 0 \text{ gives } t = \frac{8}{15}$$

M1

$$x = (0.5 - \frac{4}{3})e^{-1.6} \approx -0.168 \text{ m (i.e. 0.168 m on the far side)}$$

A1

Answer: $t = 0.2 \text{ s}; |x| \approx 0.168 \text{ m}$

3 marks

30 (9 marks)(a) Complementary function $A \cos 5t + B \sin 5t$; try $x = a \cos 4t$; $(25 - 16)a = 10$

M1 A1

$$a = \frac{10}{9}; x(0) = 0 \text{ gives } A = -\frac{10}{9}$$

M1

$$\frac{dx}{dt}(0) = 5B = 0, \text{ so } x = \frac{10}{9}(\cos 4t - \cos 5t)$$

A1

Answer: shown

4 marks

$$(b) \cos 4t - \cos 5t = 2 \sin \frac{9t}{2} \sin \frac{t}{2}$$

M1

$$x = \frac{10}{9} \times 2 \sin \frac{9t}{2} \sin \frac{t}{2} = \frac{20}{9} \sin(\frac{1}{2}t) \sin(\frac{9}{2}t)$$

A1

Answer: shown

2 marks

(c) Rapid oscillation $\sin(\frac{9}{2}t)$ with slowly varying amplitude $\frac{20}{9}|\sin(\frac{1}{2}t)|$

E1

Maximum amplitude $\frac{20}{9} \approx 2.22$

B1

Envelope is zero when $\sin(\frac{1}{2}t) = 0$, i.e. every $2\pi \approx 6.28$ s

B1

Answer: beats; max amplitude $\frac{20}{9}$; envelope zeros 2π s apart

3 marks