

Differential Equations of Motion

- 1 A particle moves in a straight line. At time t seconds its acceleration is $(6t - 2) \text{ m s}^{-2}$. When $t = 0$ its velocity is 1 m s^{-1} .
Find the velocity of the particle when $t = 3$. [2]
- 2 A particle of mass 4 kg starts from rest and moves in a straight line under the action of a single force of magnitude $8t \text{ N}$, where t is the time in seconds.
Find the speed of the particle when $t = 2$. [3]
- 3 A particle moves in a straight line with acceleration $12e^{-2t} \text{ m s}^{-2}$, where t is the time in seconds. The particle starts from rest.
Find its speed when $t = \ln 2$. [3]
- 4 A particle moves in a straight line with speed $v \text{ m s}^{-1}$ at time t seconds. It satisfies $\frac{dv}{dt} = -2v$ and $v = 10$ when $t = 0$.
Find v in terms of t . [3]
- 5 A particle moves along the positive x -axis. When its displacement from the origin O is x metres, its acceleration is $2x \text{ m s}^{-2}$ and its speed is $v \text{ m s}^{-1}$. When $x = 0$, $v = 3$.
Show that $v^2 = 2x^2 + 9$. [3]
- 6 A particle moves along a straight line. At time t seconds its displacement from a fixed point O is x metres, where $\frac{d^2x}{dt^2} = 6t - 4$. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 2$.
Find the value of x when $t = 2$. [3]
- 7 A particle of mass 2 kg moves in a straight line. At time t seconds the resultant force on the particle is $\frac{12}{2t+1} \text{ N}$ in the direction of motion. When $t = 0$ the speed of the particle is 1 m s^{-1} .
Find the exact speed of the particle when $t = 4$. [4]
- 8 A particle P moves in a straight line. At time t seconds its acceleration is $4\cos 2t \text{ m s}^{-2}$. When $t = 0$, P is at the origin O with velocity zero.
 - (a) Find an expression for the velocity of P at time t . [2]
 - (b) Find an expression for the displacement of P from O at time t . [2]
 - (c) Find the greatest distance of P from O . [1]
- 9 A particle moves in a straight line with velocity $v \text{ m s}^{-1}$ at time t seconds. It starts from rest and satisfies $\frac{dv}{dt} = 6 - 2v$.
 - (a) Show that $v = 3(1 - e^{-2t})$. [4]
 - (b) State the limiting value of v as t becomes large. [1]
- 10 A particle P moves along the positive x -axis. When P is at displacement x metres from the origin O , its acceleration is $\frac{8}{(x+1)^2} \text{ m s}^{-2}$ directed away from O . P starts from rest at O .
Find the speed of P when $x = 3$. [5]
- 11 A particle of mass 0.5 kg moves in a straight line, starting from rest. At time t seconds the resultant force on the particle is $(t^2 - 5t + 6) \text{ N}$ in the direction of motion.
Find the speed of the particle when the acceleration is first zero. [5]

- 12** A particle moves along a straight line through a point O. At time t seconds its acceleration is $(4 - 2t) \text{ m s}^{-2}$ in the positive direction. When $t = 0$ the particle is at O and is at rest. Find the time at which the particle returns to O. [4]
- 13** A particle P moves along the positive x -axis. When P is at displacement x metres from the origin O its acceleration is $(12 - 3x) \text{ m s}^{-2}$ directed away from O. P starts from rest at O. Find the maximum speed of P. [5]
- 14** A particle P moves along the positive x -axis. When P is at displacement x metres from O its acceleration is $4e^{-x/2} \text{ m s}^{-2}$ directed away from O. P starts from rest at O.
 (a) Show that $v^2 = 16(1 - e^{-x/2})$, where $v \text{ m s}^{-1}$ is the speed of P. [3]
 (b) Find the limit of the speed of P as x increases. [1]
 (c) Find the exact value of x when $v = 2$. [2]
- 15** A particle moves along the positive x -axis. When its displacement from O is x metres its acceleration is $\frac{2}{1 + x^2} \text{ m s}^{-2}$ directed away from O. When $x = 0$ the speed of the particle is 1 m s^{-1} . Find an expression for the speed $v \text{ m s}^{-1}$ of the particle in terms of x , and find the limit of v as $x \rightarrow \infty$, giving your answer in exact form. [5]
- 16** A particle moves in a straight line, starting from rest. At time t seconds its acceleration is $t e^{-t} \text{ m s}^{-2}$.
 (a) Show that the velocity of the particle at time t is $[1 - (1 + t)e^{-t}] \text{ m s}^{-1}$. [4]
 (b) Find the limit of the velocity as $t \rightarrow \infty$. [1]
 (c) Explain why the speed of the particle is always less than this limit. [1]
- 17** A particle P moves in a straight line, with speed $v \text{ m s}^{-1}$ at time t seconds ($v > 0$). Its acceleration is $\frac{2}{v} \text{ m s}^{-2}$ and P starts from the point O with speed 1 m s^{-1} .
 (a) Show that $v^2 = 4t + 1$. [3]
 (b) Find the distance of P from O when $t = 2$. [3]
- 18** A particle P of mass 3 kg moves along the positive x -axis. When its displacement from O is x metres, its speed is $v = \frac{4}{1 + x} \text{ m s}^{-1}$, directed away from O.
 (a) Find the magnitude and direction of the resultant force on P when $x = 1$. [3]
 (b) Find the time taken for P to move from $x = 0$ to $x = 3$. [3]
- 19** A particle of mass 0.4 kg is at rest at time $t = 0$. It moves in a straight line under the action of a force of magnitude $t e^{-t/2} \text{ N}$, where t is the time in seconds. Find the speed of the particle when $t = 2$, giving your answer to 3 significant figures. [6]
- 20** A particle moves in a straight line. Its velocity $v \text{ m s}^{-1}$ at time t seconds satisfies $\frac{dv}{dt} = -(1 + v^2)$, and $v = 1$ when $t = 0$. Find v as a function of t , and find the time at which the particle first comes to rest. [5]
- 21** A particle moves in a straight line. At time t seconds, $t \geq 0$, its acceleration is $(12t - 3t^2) \text{ m s}^{-2}$. When $t = 0$ the particle is at the point O and is at rest.
 (a) Find expressions for the velocity and for the displacement from O at time t . [3]
 (b) Show that the particle next comes to rest when $t = 6$, and find its distance from O at this time. [2]
 (c) Find the greatest speed of the particle. [2]

- 22** A particle moves along the positive x -axis with acceleration $(8 - 2x) \text{ m s}^{-2}$ when its displacement from O is x metres. It starts from rest at O. A student writes:
 $"a = \frac{dv}{dt} = 8 - 2x, \text{ so } v = 8t - 2xt + c."$
- (a) Explain the error in the student's working. [1]
 (b) Find the correct expression for v^2 in terms of x . [3]
 (c) Find the maximum speed of the particle and the greatest value of x reached. [2]
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- 23** A particle P of mass 2 kg moves along the x -axis. At time t seconds it is acted on by a force $12\cos 3t$ N in the positive x -direction. At $t = 0$, P is at the origin O at rest.
- (a) Find the displacement of P from O at time t . [4]
 (b) Show that P never moves to the left of O, and state the greatest value of x . [1]
 (c) Find the first time at which the speed of P is 1 m s^{-1} . [2]
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- 24** A particle moves in a straight line with velocity $v \text{ m s}^{-1}$ at time t seconds. It starts from rest and $\frac{dv}{dt} = 4 - v^2$, with $0 \leq v < 2$.
- (a) Use partial fractions to show that $t = \frac{1}{4} \ln \frac{2+v}{2-v}$. [4]
 (b) Find the time at which $v = 1$. [1]
 (c) Make v the subject of the formula in part (a) and hence state the limiting value of v . [2]
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- 25** A particle P moves along the x -axis. When its displacement from the origin is x metres ($0 \leq x \leq 12$), its speed $v \text{ m s}^{-1}$ satisfies $v^2 = 12x - x^2$. When $x = 3$, P is moving in the direction of x increasing.
- (a) Find the acceleration of P when its displacement is x . [2]
 (b) Show that the time taken for P to move from $x = 3$ to $x = 6$ is $\frac{\pi}{6}$ s. [4]
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- 26** A particle moves in a straight line with speed $v \text{ m s}^{-1}$ at time t seconds, where $\frac{dv}{dt} = v(4 - v)$, $0 < v < 4$. When $t = 0$, $v = 1$.
- (a) Show that $v = \frac{4}{1 + 3e^{-4t}}$. [5]
 (b) Find the time at which the speed is 2 m s^{-1} . [2]
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- 27** A particle P of mass 1 kg moves along the x -axis. When its displacement from O is x metres, the resultant force on P is $-(2x + 4x^3)$ N, directed along the x -axis. P is released from rest at $x = 1$.
- (a) Show that $v^2 = 4 - 2x^2 - 2x^4$, where $v \text{ m s}^{-1}$ is the speed of P. [3]
 (b) Show that $v^2 = 2(1 - x^2)(2 + x^2)$ and hence find the other value of x at which P is instantaneously at rest. [3]
 (c) Find the speed of P as it passes through O, and explain why the motion is oscillatory. [2]
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- 28** A particle moves in a straight line. At time t seconds its velocity $v \text{ m s}^{-1}$ satisfies $\frac{dv}{dt} + v = 2e^{-t}$, and $v = 0$ when $t = 0$.
- (a) Use an integrating factor to show that $v = 2te^{-t}$. [4]
 (b) Show that the greatest speed is $\frac{2}{e} \text{ m s}^{-1}$, and state when it occurs. [3]
 (c) Find the total distance travelled by the particle as $t \rightarrow \infty$. [2]

- 29** A particle P of mass 3 kg moves along the positive x-axis, starting from rest at the origin O at time $t = 0$. The resultant force on P is 6 N in the positive x-direction while $0 \leq x \leq 2$, and $\frac{12}{x}$ N in the positive x-direction when $x > 2$.
- (a) Find the speed of P when $x = 2$ and the time taken to reach this point. [3]
- (b) Show that for $x > 2$, $v^2 = 8 + 8 \ln \frac{x}{2}$. [3]
- (c) Find the value of x when $v = 6$. [2]
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- 30** A particle P of mass 2 kg moves in a straight line. It starts from rest at the point O at time $t = 0$. For $0 \leq t \leq 4$ the resultant force on P in the direction of motion is $t(4 - t)$ N, and for $t > 4$ the resultant force is zero.
- (a) Show that for $0 \leq t \leq 4$ the speed of P is $v = t^2 - \frac{t^3}{6}$. [3]
- (b) Find the speed of P and its distance from O when $t = 4$. [3]
- (c) Find the time at which P is 40 m from O. [2]
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Motion with Resistance

- 1** A particle of mass 0.5 kg moves in a straight line on a smooth horizontal surface. The only horizontal force on the particle is a resistance of magnitude $2v$ N, where $v \text{ m s}^{-1}$ is its speed. At time $t = 0$ the speed is 8 m s^{-1} . Show that $v = 8e^{-4t}$, and find the speed when $t = \ln 2$. [4]
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- 2** A parcel of mass 2 kg falls vertically through air. The air resistance has magnitude $4v$ N when the speed of the parcel is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$. Find the terminal velocity of the parcel. [2]
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- 3** A particle of mass 3 kg is moving vertically upwards through a liquid. The resistance has magnitude $0.6v$ N when the speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Show that $\frac{dv}{dt} = -9.8 - 0.2v$. [2]
- (b) Find the magnitude of the deceleration when $v = 14$. [1]
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- 4** A particle moves in a straight line with deceleration $0.4v^2 \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its speed. At $t = 0$ the speed is 5 m s^{-1} . Show that $v = \frac{5}{1 + 2t}$. [3]
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- 5** A particle moves in a straight line with deceleration $0.2v \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its speed. Its initial speed is 10 m s^{-1} . Find the distance the particle travels while its speed falls from 10 m s^{-1} to 5 m s^{-1} . [3]
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- 6** A boat of mass 200 kg moves through still water. With its engine switched off, the only horizontal force on the boat is a resistance of magnitude $50v$ N, where $v \text{ m s}^{-1}$ is the speed. When the engine is switched off the speed is 4 m s^{-1} . Find the speed of the boat 8 seconds later. [3]
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- 7** A stone of mass m kg falls from rest through water. The resistance to its motion has magnitude $0.5mv$ N when its speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Show that $v = 19.6(1 - e^{-0.5t})$, where t is the time in seconds. [4]
- (b) Find the speed of the stone when $t = 2$. [1]
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- 8** A ball of mass m kg is projected vertically upwards with speed 20 m s^{-1} through air. The air resistance has magnitude $0.2mv \text{ N}$ when the speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$. Find the time taken to reach the highest point, giving your answer to 3 significant figures. [4]
- 9** A particle moves in a straight line with deceleration $0.05v^2 \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its speed. The initial speed is 20 m s^{-1} .
 (a) Show that $v = 20e^{-0.05x}$, where x metres is the distance travelled. [3]
 (b) Find the distance travelled while the speed halves. [2]
- 10** A car of mass 800 kg moves along a straight horizontal road. The engine produces a constant driving force of 1200 N and the resistance to motion has magnitude $3v^2 \text{ N}$ when the speed is $v \text{ m s}^{-1}$.
 (a) Find the acceleration of the car when $v = 10$. [2]
 (b) Find the maximum speed of the car. [2]
- 11** A sky-diver falls vertically. Her acceleration is $(9.8 - 0.02v^2) \text{ m s}^{-2}$ when her speed is $v \text{ m s}^{-1}$.
 (a) Find her terminal speed, correct to 3 significant figures. [2]
 (b) Find her speed when her acceleration is 4.9 m s^{-2} . [2]
- 12** A particle of mass 2 kg moves in a straight line on a smooth horizontal table. The only horizontal force is a resistance of magnitude $0.6v \text{ N}$, where $v \text{ m s}^{-1}$ is the speed. The initial speed is 12 m s^{-1} . Show that the particle cannot travel further than 40 m . [4]
- 13** A block of mass m kg slides down a smooth plane inclined at 30° to the horizontal. In addition to its weight and the normal reaction, the block is subject to a resistance of magnitude $0.4mv \text{ N}$, where $v \text{ m s}^{-1}$ is its speed, acting up the plane. The block starts from rest. Take $g = 9.8 \text{ m s}^{-2}$.
 (a) Show that the speed approaches 12.25 m s^{-1} . [3]
 (b) Find the time taken to reach a speed of 6 m s^{-1} . [2]
- 14** A small sphere of mass m kg is released from rest in a liquid. The resistance has magnitude $0.5mv \text{ N}$ when the speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$.
 (a) Show that the speed after t seconds is $19.6(1 - e^{-0.5t}) \text{ m s}^{-1}$. [3]
 (b) Find the distance fallen in the first 4 seconds. [3]
- 15** A particle of mass m kg is projected vertically upwards with speed 30 m s^{-1} through a medium which exerts a resistance of magnitude $0.4mv \text{ N}$ when the speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$. Find the greatest height reached above the point of projection, to 3 significant figures. [6]
- 16** A ball of mass m kg is thrown vertically upwards with speed 25 m s^{-1} . Air resistance has magnitude $0.01mv^2 \text{ N}$ when the speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$. Find the greatest height reached, giving your answer to 3 significant figures. [5]
- 17** A particle of mass m kg falls from rest through air. The air resistance has magnitude $0.02mv^2 \text{ N}$ when its speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$.
 (a) Show that $v^2 = 490(1 - e^{-0.04x})$, where x metres is the distance fallen. [4]
 (b) Find the speed after the particle has fallen 30 m . [2]
- 18** A boat moves on a lake with its engine off. The only horizontal force on the boat is a resistance which gives it a deceleration of $0.05v^2 \text{ m s}^{-2}$ when its speed is $v \text{ m s}^{-1}$. The initial speed is 20 m s^{-1} .
 (a) Show that $v = \frac{20}{1+t}$ at time t seconds. [3]
 (b) Find the distance travelled in the first 10 seconds. [3]

- 19** A boat of mass 500 kg is propelled by an engine which gives a constant forward force of 600 N. The water exerts a resistance of magnitude $40v$ N when the speed of the boat is v m s⁻¹. The boat starts from rest.
- (a) Show that $v = 15(1 - e^{-0.08t})$. [4]
- (b) Find the time taken to reach a speed of 10 m s⁻¹. [2]
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- 20** A particle moves in a straight line with deceleration $2\sqrt{v}$ m s⁻² when its speed is v m s⁻¹. The initial speed is 16 m s⁻¹.
- (a) Show that the particle comes to rest after exactly 4 seconds. [3]
- (b) Find the total distance travelled before the particle comes to rest. [3]
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- 21** A ball of mass m kg is thrown vertically upwards from ground level with speed 20 m s⁻¹. Air resistance has magnitude $0.01mv^2$ N when the speed is v m s⁻¹. Take $g = 9.8$ m s⁻².
- (a) Find the greatest height H reached by the ball. [4]
- (b) Find the speed with which the ball returns to the ground. [4]
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- 22** A particle of mass m kg is released from rest and falls vertically. The air resistance has magnitude $0.04mv^2$ N when its speed is v m s⁻¹. Take $g = 9.8$ m s⁻².
- (a) Find the terminal speed V , giving its exact value and its value to 3 significant figures. [2]
- (b) Show that $\frac{dv}{dt} = 0.04(245 - v^2)$ and hence find t as an integral in terms of v . [1]
- (c) Use partial fractions to show that $v = V \tanh(0.626t)$, where V is as in part (a), and the constant 0.626 is correct to 3 significant figures. [5]
- (d) Find the time taken to reach half of the terminal speed. [2]
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- 23** A particle moves in a straight line with deceleration $0.1v^2$ m s⁻² when its speed is v m s⁻¹. Its initial speed is 10 m s⁻¹. A student writes:
- “ $\frac{dv}{dt} = -0.1v^2$, so integrating, $v = -0.1v^2t + 10$, which gives $v = 10e^{-0.1t}$.”
- (a) Identify the errors in the student's working. [2]
- (b) Find v in terms of t . [3]
- (c) Find v in terms of the distance x travelled, and say whether the particle ever stops. [3]
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- 24** A parachutist of mass m kg falls vertically from rest. The air resistance has magnitude $0.8mv$ N when the speed is v m s⁻¹. Take $g = 9.8$ m s⁻².
- (a) Find the terminal speed V , and show that $v = V(1 - e^{-0.8t})$. [4]
- (b) Find the time at which the speed reaches 90% of V . [2]
- (c) Find the distance fallen in this time. [3]
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- 25** A car of mass 1000 kg travels along a straight horizontal road with its engine working at a constant power of 24 000 W. The resistance to motion has magnitude $3v^2$ N when the speed is v m s⁻¹.
- (a) Show that the car's maximum speed is 20 m s⁻¹. [2]
- (b) The car accelerates from rest. Find the distance it travels in reaching a speed of 10 m s⁻¹, correct to 3 significant figures. [5]
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- 26** A particle of mass m falls from rest through a medium which exerts a resistance of magnitude mkv , where v is the speed and k is a positive constant. The terminal speed is V .
- (a) Show that $V = \frac{g}{k}$ and that $v = V(1 - e^{-kt})$. [4]
- (b) Show that the time taken to reach a speed of $\frac{1}{2}V$ is $\frac{\ln 2}{k}$. [1]
- (c) Show that the distance fallen in this time is $\frac{g}{k^2}(\ln 2 - \frac{1}{2})$. [3]

- 27** A particle of mass m is projected vertically upwards with speed U in a medium which exerts a resistance of magnitude mkv^2 , where v is the speed and k is a positive constant. The particle rises to a greatest height H and then falls back to its starting point with speed V .
- (a) Show that $H = \frac{1}{2k} \ln\left(1 + \frac{kU^2}{g}\right)$. [4]
- (b) Show that $V^2 = \frac{g}{k}(1 - e^{-2kH})$. [3]
- (c) Hence show that $\frac{1}{V^2} = \frac{1}{U^2} + \frac{k}{g}$, and deduce that $V < U$. [3]
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- 28** A particle of mass m kg is projected vertically upwards with speed 15 m s^{-1} through a medium which exerts a resistance of magnitude $0.04mv^2 \text{ N}$ when the speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Find the time taken to reach the highest point, to 3 significant figures. [4]
- (b) After reaching the highest point the particle falls. Show that the speed t seconds after leaving the highest point is $v = \sqrt{245} \tanh(0.626t)$. [4]
- (c) Find the time after leaving the highest point at which the particle's speed is half of its terminal speed. [2]
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- 29** A boat of mass 1200 kg is driven by an engine producing a constant forward force of 3000 N . The water exerts a resistance of magnitude $300v \text{ N}$ when the speed of the boat is $v \text{ m s}^{-1}$. The boat starts from rest.
- (a) Show that $v = 10(1 - e^{-t/4})$, where t is the time in seconds. [4]
- (b) Find the distance travelled in the first 8 seconds. [3]
- (c) When $t = 8$ the engine is switched off. Find how much further the boat travels before it comes to rest, if the resistance is unchanged. Comment on the answer. [3]
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- 30** A stone of mass m kg is thrown vertically downwards with speed 25 m s^{-1} in a medium which exerts a resistance of magnitude $0.98mv \text{ N}$ when the speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Show that the terminal speed is $V = 10 \text{ m s}^{-1}$, and that $v = 10 + 15e^{-0.98t}$. [5]
- (b) Explain what the result shows about the speed of the stone as it falls. [1]
- (c) Find the time at which the speed is 15 m s^{-1} . [2]
- (d) Find the distance travelled in this time. [2]

Variable Mass and Force Models

- 1** The momentum of a particle of mass 5 kg moving in a straight line is $(3t^2 + 2t) \text{ kg m s}^{-1}$ at time t seconds.
Find the magnitude of the resultant force on the particle when $t = 2$. [2]
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- 2** A particle has weight 80 N on the surface of the Earth, whose radius is R . The gravitational force on the particle is inversely proportional to the square of its distance from the Earth's centre.
Find the weight of the particle when it is at a distance $3R$ from the centre of the Earth. [2]
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- 3** A rocket in deep space ejects burnt fuel at a constant rate of 5 kg s^{-1} . The fuel leaves the rocket at a speed of 800 m s^{-1} relative to the rocket.
Find the thrust on the rocket. [2]
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- 4** Sand falls vertically onto a horizontal conveyor belt at a constant rate of 3 kg s^{-1} . The belt moves horizontally at a constant speed of 2 m s^{-1} .
Find the horizontal driving force needed on the belt, ignoring friction in the machinery, to keep it at this constant speed. [2]

- 5** The acceleration due to gravity at a distance x from the centre of the Earth ($x \geq R$) is $\frac{gR^2}{x^2}$, where R is the radius of the Earth and $g = 9.8 \text{ m s}^{-2}$ is the value at the surface. Find the acceleration due to gravity at a height R above the surface of the Earth. [2]
- 6** A rocket in deep space has initial mass 800 kg (including fuel). It ejects 300 kg of fuel at a constant speed of 2000 m s^{-1} relative to the rocket. The rocket starts from rest. It can be shown that the final speed is $v = c \ln \frac{m_0}{m}$, where c is the relative speed of ejection, m_0 the initial mass and m the final mass. Calculate the final speed of the rocket. [3]
- 7** The acceleration due to gravity at a distance x from the centre of the Earth ($x \geq R$) is $\frac{gR^2}{x^2}$, where $R = 6.4 \times 10^6 \text{ m}$ is the radius of the Earth and $g = 9.8 \text{ m s}^{-2}$. Ignore air resistance. A particle is projected vertically upwards from the surface with speed u . By considering $v \frac{dv}{dx}$, show that the particle escapes from the Earth (never returns) if $u \geq \sqrt{2gR}$, and calculate this speed in m s^{-1} to 3 significant figures. [4]
- 8** Take the Earth's radius as $R = 6.4 \times 10^6 \text{ m}$, $g = 9.8 \text{ m s}^{-2}$, and the acceleration due to gravity at distance x from the centre as $\frac{gR^2}{x^2}$. A particle is projected vertically upwards from the surface of the Earth with speed 5000 m s^{-1} . Ignore air resistance. Find the greatest height above the surface reached by the particle, to 3 significant figures. [5]
- 9** Use $g = 9.8 \text{ m s}^{-2}$ and $R = 6.4 \times 10^6 \text{ m}$, with the acceleration due to gravity at distance x from the Earth's centre equal to $\frac{gR^2}{x^2}$. A small meteor falls from rest at a height R above the Earth's surface. Ignore air resistance. Show that its speed on reaching the surface is $\sqrt{gR}$, and calculate this speed. [4]
- 10** A rocket in deep space ejects fuel at a constant rate of 10 kg s^{-1} with speed 1800 m s^{-1} relative to the rocket. The initial mass of the rocket and fuel is 1500 kg and the rocket starts from rest. Assume the thrust is the only force on the rocket.
(a) Show that the speed of the rocket after t seconds is $v = 1800 \ln \frac{1500}{1500 - 10t}$. [4]
(b) Find the speed after 60 seconds. [1]
- 11** A trolley of mass 20 kg moves on smooth horizontal rails with initial speed 6 m s^{-1} . Sand falls vertically into the trolley at a constant rate of 2 kg s^{-1} . No horizontal external force acts on the trolley. Find the speed of the trolley after 10 seconds. [4]
- 12** A particle P of mass 0.5 kg is attracted towards a fixed point O by a force of magnitude $\frac{4}{x^2} \text{ N}$, where x metres is its distance from O. P is released from rest at $x = 4$. Find the speed of P when $x = 1$. [4]
- 13** A rocket of initial mass 5000 kg (including fuel) is launched vertically upwards from rest. It ejects burnt fuel at a constant rate of 20 kg s^{-1} with speed 2500 m s^{-1} relative to the rocket. Take $g = 9.8 \text{ m s}^{-2}$ and assume g is constant and there is no air resistance.
(a) Find the initial acceleration of the rocket. [2]
(b) Find the acceleration after 10 seconds. [3]

- 14** A rocket is launched vertically upwards from rest. Its initial mass is 4000 kg and it ejects burnt fuel at a constant rate of 25 kg s^{-1} with speed 2200 m s^{-1} relative to the rocket. Take $g = 9.8 \text{ m s}^{-2}$, assume g is constant and ignore air resistance.
- (a) Show that the speed of the rocket t seconds after launch is $v = 2200 \ln \frac{4000}{4000 - 25t} - 9.8t$. [5]
- (b) Find the speed after 30 seconds, given that fuel remains. [2]
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- 15** A raindrop falls vertically from rest through still, saturated air and gains mass by condensation, so that its mass m at time t satisfies $\frac{dm}{dt} = 0.2m$ (t in seconds). The condensing water is initially at rest. Take $g = 9.8 \text{ m s}^{-2}$ and ignore air resistance.
- (a) Using the equation $\frac{d}{dt}(mv) = mg$, show that $\frac{dv}{dt} + 0.2v = 9.8$, where $v \text{ m s}^{-1}$ is the speed of the drop. [3]
- (b) Hence find v in terms of t , and the speed when $t = 5$. [3]
-
- 16** A lorry of initial mass 1500 kg, including a load of sand, travels along a straight horizontal road. Sand leaks steadily from the lorry at 5 kg s^{-1} , leaving the lorry with the same velocity as the lorry at that instant. The engine provides a constant driving force of 600 N and there is no other horizontal force. The lorry starts from rest.
- (a) Show that the mass of the lorry at time t is $1500 - 5t$, and that $\frac{dv}{dt} = \frac{600}{1500 - 5t}$. [3]
- (b) Find the speed of the lorry after 100 seconds. [2]
-
- 17** Use $g = 9.8 \text{ m s}^{-2}$, $R = 6.4 \times 10^6 \text{ m}$, and the gravitational force on a body of mass m at distance x from the Earth's centre as $\frac{mgR^2}{x^2}$.
- Show that the work done against gravity in raising a satellite of mass m from the surface of the Earth to a height h above the surface is $\frac{mgRh}{R+h}$, and calculate this work for a satellite of mass 1000 kg raised to a height R . [5]
-
- 18** A two-stage rocket in deep space starts from rest with total mass 6000 kg. In the first stage it burns 2500 kg of fuel, ejecting it at 2000 m s^{-1} relative to the rocket. The empty first-stage casing of mass 500 kg is then detached (with no change in the velocity of the rest of the rocket). In the second stage the remaining rocket burns 1800 kg of fuel, ejecting it at 2400 m s^{-1} relative to the rocket. The speed gained when a rocket of mass m_0 burns fuel to mass m with relative ejection speed c is $c \ln \frac{m_0}{m}$.
- Find the final speed of the rocket. [6]
-
- 19** Sand is dropped vertically onto a horizontal conveyor belt at a constant rate of 40 kg s^{-1} . The belt moves at a constant speed of 1.5 m s^{-1} . Assume the sand is initially at rest and ends up moving with the belt.
- (a) Find the extra driving force the motor must provide to maintain the belt's speed. [2]
- (b) Find the rate at which the motor does work, and the rate at which kinetic energy is given to the sand. [3]
- (c) Explain what happens to the remainder of the energy. [1]
-
- 20** A uniform chain of mass 4 kg and length 2 m is held vertically with its lower end just touching a horizontal table. The chain is released from rest and falls onto the table, the links coming to rest as they land. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) When the top of the chain has fallen a distance x , show that the part of the chain still in the air has speed $\sqrt{2gx}$. [2]
- (b) Show that the force exerted by the table on the chain is $3\lambda gx$, where λ is the mass per unit length, and calculate it when $x = 0.5$. [4]

- 21** A rocket of initial mass 3000 kg is launched vertically upwards from rest. It burns fuel at a constant rate of 20 kg s^{-1} , ejecting it at 2000 m s^{-1} relative to the rocket, until burn-out at $t = 50 \text{ s}$. Take $g = 9.8 \text{ m s}^{-2}$, assume g is constant and ignore air resistance.
- (a) Show that for $0 \leq t \leq 50$ the speed of the rocket is $v = 2000 \ln \frac{3000}{3000 - 20t} - 9.8t$, and find its speed at burn-out. [4]
- (b) Show that the height of the rocket at burn-out is $300\,000[(1 - s) \ln(1 - s) + s] - 4.9 \times 50^2$, where $s = \frac{1}{3}$, and evaluate it. [3]
- (c) After burn-out the rocket moves freely under gravity. Find the greatest height reached above the launch point. [2]
-
- 22** A uniform chain of mass 6 kg and length 3 m is initially in a heap on a horizontal floor. One end is raised vertically upwards at a constant speed of 2 m s^{-1} by a force F newtons. Take $g = 9.8 \text{ m s}^{-2}$. When the raised end is at height x metres, the part of the chain off the floor is vertical.
- (a) Show that $F = 19.6x + 8$. [4]
- (b) Find the work done by F while the whole chain is lifted clear of the floor (x from 0 to 3). [2]
- (c) Calculate the gain in potential energy and the kinetic energy of the chain when it has just left the floor, and hence find the energy lost in the process. Suggest where it goes. [3]
-
- 23** A raindrop falls vertically from rest through a still cloud of fine mist. It gains mass from the mist, which is initially at rest, so that its mass m satisfies $\frac{dm}{dx} = 0.02m$, where x metres is the distance fallen. Let $v \text{ m s}^{-1}$ be its speed. Take $g = 9.8 \text{ m s}^{-2}$; ignore air resistance.
- (a) Show that $v \frac{dv}{dx} + 0.02v^2 = 9.8$. [3]
- (b) By writing $w = v^2$ or otherwise, show that $v^2 = 490(1 - e^{-0.04x})$. [3]
- (c) Find the speed after the drop has fallen 100 m, and the limiting speed. [2]
-
- 24** A spacecraft of initial mass 5000 kg descends vertically towards the Moon's surface. At the instant $t = 0$ its speed is 100 m s^{-1} downwards. Its engine, pointing downwards, ejects burnt fuel at a constant rate of 4 kg s^{-1} with speed 3000 m s^{-1} relative to the spacecraft. Assume that the Moon's gravitational acceleration is a constant 1.6 m s^{-2} and ignore any other forces.
- (a) Show that the speed $v \text{ m s}^{-1}$ of the spacecraft at time t seconds (while fuel remains) is $v = 100 + 1.6t - 3000 \ln \frac{5000}{5000 - 4t}$. [5]
- (b) Find the speed when $t = 60$, and the deceleration at that instant. [3]
-
- 25** Use $g = 9.8 \text{ m s}^{-2}$ and $R = 6.4 \times 10^6 \text{ m}$ for the Earth, with gravitational acceleration $\frac{gR^2}{x^2}$ at distance x from the Earth's centre. A projectile is fired vertically upwards from the surface of the Earth with speed u . Ignore air resistance and the gravitational effect of the Moon.
- (a) Show that the greatest distance from the Earth's centre reached is $x = \frac{2gR^2}{2gR - u^2}$ (for $u^2 < 2gR$). [3]
- (b) Show that if $u = \sqrt{gR}$ the projectile reaches a distance $2R$ from the Earth's centre. [1]
- (c) The Moon's centre is about $60R$ from the Earth's centre. Find the minimum launch speed needed to reach this distance, to 3 significant figures, and express it as a percentage of the escape speed $\sqrt{2gR}$. [3]

- 26** An open truck of mass 800 kg (when empty) travels along a straight horizontal track. Rain falls vertically into the truck at a constant rate of 2 kg s^{-1} . The engine provides a constant horizontal driving force of 200 N and there is no resistance to motion. When $t = 0$, the truck is empty and moving at 5 m s^{-1} .
- (a) Show that the speed $v \text{ m s}^{-1}$ at time t seconds satisfies $\frac{d}{dt}[(800 + 2t)v] = 200$, and
hence that $v = \frac{4000 + 200t}{800 + 2t}$. [4]
- (b) Find the speed of the truck when $t = 100$, and its acceleration at that instant. [3]
-
- 27** A particle P of mass 1 kg is attracted towards a fixed point O by a force of magnitude $\frac{2}{x^2}$ N, where x metres is the distance OP. P is released from rest at $x = 4$.
- (a) Show that the speed of P at distance x from O is given by $v^2 = 4(\frac{1}{x} - \frac{1}{4})$. [2]
- (b) Using the substitution $x = 4 \sin^2\theta$, show that the time taken for P to move from $x = 4$ to $x = 2$ is $(\pi + 2)$ seconds. [6]
- (c) Find the total time for P to reach O. [2]
-
- 28** A raindrop falls vertically from rest through a still, saturated cloud and gains mass so that its mass m at time t satisfies $\frac{dm}{dt} = 0.2m$. The condensing water is at rest. In addition to its weight, the drop experiences a resistance of magnitude $0.3mv$ N, where $v \text{ m s}^{-1}$ is its speed. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Show that $\frac{dv}{dt} + 0.5v = 9.8$. [3]
- (b) Find v in terms of t , and state the terminal speed. [3]
- (c) Find the speed and the distance fallen at the instant the mass of the drop has doubled. [4]
-
- 29** A uniform chain of length 2 m and mass 4 kg lies on a smooth horizontal table, with a length 0.5 m hanging vertically over the straight edge of the table. The chain is released from rest. Let x metres be the length hanging over the edge at time t seconds. Assume the chain stays straight and in contact with the table and its edge. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Show that $\frac{d^2x}{dt^2} = 4.9x$. [3]
- (b) Show that the speed v of the chain satisfies $v^2 = 4.9(x^2 - 0.25)$. [3]
- (c) Find the speed with which the end of the chain leaves the table, and the time taken for the chain to leave the table. [4]
-
- 30** Two particles A and B, of masses 3 kg and 6 kg, are alone in deep space. They attract each other with a force of magnitude $\frac{16}{r^2}$ N, where r metres is their separation. They are released from rest when $r = 4$.
- (a) Explain why the centre of mass of the particles remains at rest, and write down the accelerations of A and B in terms of r . [3]
- (b) Show that the rate of change of r satisfies $\frac{d^2r}{dt^2} = -\frac{8}{r^2}$. [2]
- (c) Find the speed of each particle when $r = 1$. [4]
- (d) Treating the particles as points, find the time until they collide. [2]

Simple Harmonic Motion

- 1** A particle moves in a straight line so that its displacement from O at time t seconds is $x = 3 \sin 2t$ metres.
Write down the amplitude and the period of the motion, and find the maximum speed of the particle. [3]

- 2** A particle moves with simple harmonic motion of amplitude 0.5 m and angular frequency 4 rad s^{-1} .
Find the period and the maximum acceleration. [2]
-
- 3** A particle P moves in a straight line with simple harmonic motion about O. When P is at displacement x metres its speed $v \text{ m s}^{-1}$ satisfies $v^2 = 16(9 - x^2)$.
Find the amplitude of the motion and the maximum speed of P. [3]
-
- 4** A particle moves with simple harmonic motion of period 2 s and amplitude 0.4 m.
Calculate the maximum speed of the particle. [2]
-
- 5** A particle of mass 0.5 kg lies on a smooth horizontal table attached to one end of a light spring of stiffness 8 N m^{-1} . The other end of the spring is fixed. The particle oscillates in a straight line along the spring, which stays taut. Find the period of the oscillations. [2]
-
- 6** Show that $x = 2\cos 3t$ satisfies the equation $\frac{d^2x}{dt^2} = -9x$. State the period of the motion. [3]
-
- 7** A particle moves with simple harmonic motion with amplitude 5 m and angular frequency 2 rad s^{-1} .
Find its speed when it is 3 m from the centre of the motion. [3]
-
- 8** A particle P moves with simple harmonic motion with period 6 s and amplitude 0.8 m about a point O. At time $t = 0$, P is at O moving in the positive direction.
Find the time taken for P to travel from O to a point 0.4 m from O. [4]
-
- 9** A particle of mass 0.4 kg hangs in equilibrium from the lower end of a light elastic spring of natural length 0.6 m and modulus of elasticity 24 N. The upper end is fixed.
(a) Find the extension of the spring in equilibrium. [2]
(b) The particle is pulled down a small distance and released. Find the period of the resulting oscillations. [2]
-
- 10** A particle P of mass m lies on a smooth horizontal table, attached to one end of a light elastic string of natural length l and modulus of elasticity λ . The other end of the string is fixed to a point A on the table. P is released from rest with the string stretched, and moves along the line through A. At time t the extension of the string is x .
Show that, while the string is taut, P moves with simple harmonic motion, and find the period in terms of m , l and λ . [4]
-
- 11** A particle moves on a straight line with displacement $x = 4\cos(3t + \frac{\pi}{6})$ metres from O at time t seconds.
(a) Find the displacement and velocity when $t = 0$. [3]
(b) Find the maximum acceleration. [2]
-
- 12** A particle moves with simple harmonic motion between two points A and B, 0.8 m apart. The maximum speed is 1.2 m s^{-1} .
Find the period of the motion and the magnitude of the acceleration at A. [4]
-
- 13** A particle P moves with simple harmonic motion along a straight line about a centre O. When P is 2 m from O its speed is 6 m s^{-1} , and when P is 3 m from O its speed is 4 m s^{-1} .
Find the amplitude and the period of the motion. [5]
-

- 14** A particle of mass 0.5 kg hangs in equilibrium at the lower end of a light elastic string of natural length 1.2 m and modulus of elasticity 30 N. The upper end of the string is fixed. The particle is pulled down 0.1 m from the equilibrium position and released from rest. Take $g = 9.8 \text{ m s}^{-2}$.
- Show that the extension in equilibrium is 0.196 m. [2]
 - Show that while the string is taut the particle performs simple harmonic motion, and find the period and maximum speed. [4]
 - Explain why the string remains taut throughout the motion. [1]
-
- 15** A particle P moves with simple harmonic motion along a straight line with centre O, amplitude 0.5 m and period 4 s. P passes through O at time $t = 0$ moving in the positive direction.
Find the time taken for P to travel directly from the point 0.2 m from O to the point 0.4 m from O, to 3 significant figures. [5]
-
- 16** A straight smooth narrow tunnel is bored through the Earth's centre. Assume the Earth is a uniform sphere of radius $R = 6.4 \times 10^6 \text{ m}$, so that the gravitational force on a particle of mass m at distance x from the centre ($x \leq R$) is $\frac{mgx}{R}$, directed towards the centre, where $g = 9.8 \text{ m s}^{-2}$. A particle is released from rest at one end of the tunnel.
- Show that the particle moves with simple harmonic motion, and find the period in minutes, to 3 significant figures. [4]
 - Find the speed of the particle as it passes through the centre of the Earth. [2]
-
- 17** A uniform cylinder of mass 0.2 kg and cross-sectional area 20 cm^2 floats in water with its axis vertical. The density of water is 1000 kg m^{-3} . The cylinder is pushed down slightly from its equilibrium position and released. Take $g = 9.8 \text{ m s}^{-2}$. Assume the cylinder remains partly out of the water and ignore water resistance.
Show that the cylinder moves with simple harmonic motion, and find the period of the oscillations. [6]
-
- 18** A particle P of mass 0.8 kg lies on a smooth horizontal table between two fixed points A and B. P is attached to A by a light spring of stiffness 12 N m^{-1} and to B by a light spring of stiffness 20 N m^{-1} . Both springs are in line and are stretched (taut) when P is in its equilibrium position. P is displaced a small distance along the line AB and released.
Show that P performs simple harmonic motion and find its period. [6]
-
- 19** A simple pendulum consists of a small bob of mass m attached to a light inextensible string of length 0.98 m. It oscillates in a vertical plane. The angle θ radians that the string makes with the downward vertical satisfies $L \frac{d^2\theta}{dt^2} = -g \sin \theta$. Take $g = 9.8 \text{ m s}^{-2}$.
- For small oscillations $\sin \theta \approx \theta$. Show that the motion is approximately simple harmonic and find its period. [3]
 - The bob is released from rest with $\theta = 0.1$. Find its greatest speed. [3]
-
- 20** A particle P moves with simple harmonic motion of amplitude 0.5 m and period 2 s about a centre O.
Find the fraction of each period for which P is within 0.25 m of O. [5]
-
- 21** A particle of mass 0.5 kg hangs in equilibrium at the end of a light elastic string of natural length 1.2 m and modulus of elasticity 30 N, the other end of the string being fixed. The particle is pulled down 0.4 m from the equilibrium position and released from rest. Take $g = 9.8 \text{ m s}^{-2}$.
- Show that the string becomes slack during the motion, and find the speed of the particle at the instant this happens. [5]
 - Find the greatest height of the particle above the equilibrium position. [3]

- 22** A particle P of mass 0.5 kg lies on a smooth horizontal table. It is attached to one end of a light elastic string of natural length 0.4 m and modulus of elasticity 10 N, whose other end is fixed to a point O on the table. A smooth vertical wall on the table is perpendicular to the line OP, at distance 0.15 m on the O side of the point N where $OP = 0.4$ m, so that P can reach it when the string is slack. P is held with the string stretched to a total length of 0.7 m and released from rest. The collision with the wall is perfectly elastic.
- (a) Show that P moves with simple harmonic motion while the string is taut and find the speed of P at N. [4]
- (b) Find the time for P to return to its starting position for the first time. [4]
-
- 23** A particle moves with simple harmonic motion of period 2 s and amplitude 0.5 m. A student writes: "The angular frequency is 2, so the maximum speed is $0.5 \times 2 = 1 \text{ m s}^{-1}$."
- (a) Identify the error and give the correct maximum speed. [3]
- (b) The particle is at the centre O at $t = 0$ moving in the positive direction. Find the first time at which it is moving towards O with speed exactly 1 m s^{-1} . [4]
- (c) Find the displacement of the particle at the time found in part (b). [2]
-
- 24** A particle A of mass 0.2 kg is attached to the lower end of a light vertical spring of stiffness 50 N m^{-1} , whose upper end is fixed, and rests in equilibrium. A particle B of mass 0.3 kg is released from rest 0.1 m above A and falls onto A. B sticks to A on impact. Take $g = 9.8 \text{ m s}^{-2}$.
- Find the amplitude of the subsequent simple harmonic motion, to 3 significant figures. [8]
-
- 25** A particle moves on a straight line so that its displacement x metres from O at time t seconds satisfies $\frac{d^2x}{dt^2} + 4x = 0$. When $t = 0$, $x = 1$ and $\frac{dx}{dt} = -2\sqrt{3}$.
- (a) Find x in terms of t in the form $x = R \cos(2t + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. [4]
- (b) State the amplitude and maximum speed, and find the first time the particle is at O. [3]
-
- 26** A bungee jumper of mass 70 kg steps off a platform attached to an elastic rope of natural length 20 m and modulus of elasticity 1400 N, fixed at the platform. She falls vertically from rest. Model her as a particle, the rope as light and ignore air resistance. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Show that the rope becomes taut after about 2.02 s and find her speed at that instant. [3]
- (b) Show that, once the rope is taut, the jumper moves with simple harmonic motion with $\omega = 1$, and find the extension of the rope in equilibrium. [3]
- (c) Find the amplitude of the SHM and the greatest distance she falls below the platform. [3]
-
- 27** A particle P of mass 0.5 kg is attached to one end of a light spring of stiffness 50 N m^{-1} , the other end being fixed, and lies on a rough horizontal table with coefficient of friction 0.2. The spring is in line with the motion. P is held with the spring extended by 0.2 m, released from rest, and moves along the line of the spring. Take $g = 9.8 \text{ m s}^{-2}$. P stays at rest if the spring force does not exceed the limiting friction.
- (a) Show that while P is moving towards the natural-length position, it performs SHM about a point 0.0196 m from that position, with period $\frac{\pi}{5}$ s. [4]
- (b) Show that the greatest displacement from the natural-length position decreases by 0.0392 m in each successive half-oscillation. [3]
- (c) Find the total time for which P moves and the total distance it travels before coming to rest permanently. [4]

- 28** A particle P of mass 0.4 kg lies on a smooth horizontal table. It is attached to the midpoint of a light elastic string of natural length 1.0 m and modulus of elasticity 20 N. The ends of the string are fixed to points A and B on the table, where $AB = 1.6$ m, and P rests at the midpoint M of AB. P is given a small displacement y metres perpendicular to AB, and the string stays taut.
- (a) Show that when P is at distance y from M the tension in the string is $T = 20(2\sqrt{0.64 + y^2} - 1)$ N, and that the magnitude of the force on P directed towards M is $\frac{2Ty}{\sqrt{0.64 + y^2}}$. [4]
- (b) Hence show that, for small y , P performs approximate simple harmonic motion, and find the period. [5]
- (c) Explain why the approximation is valid only for y small compared with 0.8 m. [1]
-
- 29** A block A of mass 1.5 kg lies on a smooth horizontal table, attached to one end of a light spring of stiffness 60 N m^{-1} , the other end being fixed. A block B of mass 0.5 kg rests on top of A. The coefficient of friction between A and B is 0.4. The blocks move together along the line of the spring with simple harmonic motion. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Find the period of oscillations when the blocks move together. [2]
- (b) Show that the frictional force on B when the displacement from the centre is x is $15x$ in magnitude. [2]
- (c) Find the greatest amplitude of the motion for which B does not slip on A. [3]
- (d) The blocks are released from rest with the spring extended by 0.2 m. Find the speed at which B starts to slip. [3]
-
- 30** A particle of mass 0.5 kg hangs in equilibrium at the end of a light elastic string of natural length 1.2 m and modulus of elasticity 30 N, the other end of the string being fixed. The particle is pulled down 0.4 m from the equilibrium position and released from rest at time $t = 0$. Take $g = 9.8 \text{ m s}^{-2}$ and ignore air resistance. The motion is periodic.
- (a) Find the equilibrium extension and show that, while the string is taut, the motion is simple harmonic with $\omega = \sqrt{50}$. [3]
- (b) Show that the particle reaches the natural-length position after approximately 0.294 s, and find the speed there. [4]
- (c) Hence find the period of the complete motion, to 3 significant figures. [4]

Damped and Forced Motion

- 1** Find the general solution of the differential equation $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, and state whether the system is underdamped, critically damped or overdamped. [3]
-
- 2** A particle of mass 2 kg is attached to a spring of stiffness 8 N m^{-1} and moves in a medium which exerts a resistance of magnitude $4v$ N, where $v \text{ m s}^{-1}$ is its speed. Show that the displacement x metres from the equilibrium position satisfies $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = 0$. [2]
-
- 3** Show that the motion described by $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$ is critically damped, and write down its general solution. [3]
-
- 4** Find the general solution of $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$ and state the period of the oscillations in the solution. [3]

- 5** Find the value of the constant c for which the equation $\frac{d^2x}{dt^2} + c\frac{dx}{dt} + 16x = 0$ ($c > 0$) represents critical damping. [2]
-
- 6** Show that $x = 2$ is a particular integral of $\frac{d^2x}{dt^2} + 4x = 8$, and hence write down the general solution. [3]
-
- 7** Solve $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 12x = 0$, given that $x = 1$ and $\frac{dx}{dt} = 0$ when $t = 0$. [5]
-
- 8** A critically damped system satisfies $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$. When $t = 0$, $x = 3$ and $\frac{dx}{dt} = -2$.
 (a) Find x in terms of t . [4]
 (b) Explain why the system never passes through the equilibrium position for $t > 0$. [1]
-
- 9** Solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$ given that $x = 2$ and $\frac{dx}{dt} = 0$ when $t = 0$. [5]
-
- 10** Find a particular integral of $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 10 \sin t$, and hence write down the general solution. [5]
-
- 11** A particle of mass 0.5 kg moves on the x -axis attached to a spring of stiffness 18 N m^{-1} , and is subject to a resistance of magnitude $3v \text{ N}$ where $v \text{ m s}^{-1}$ is the speed. Show that the equation of motion is $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 36x = 0$, and decide whether the motion is underdamped, critically damped or overdamped. Find the general solution. [5]
-
- 12** A damped oscillation is modelled by $x = e^{-0.2t} \cos 3t$.
 Show that successive maximum displacements are separated in time by $\frac{2\pi}{3} \text{ s}$, and that the ratio of each maximum to the previous one is $e^{-2\pi/15}$. Evaluate this ratio. [4]
-
- 13** Solve $\frac{d^2x}{dt^2} + 9x = 18 \cos 2t$, given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$. [6]
-
- 14** The suspension of a car of mass 1000 kg is modelled by a spring of stiffness $40\,000 \text{ N m}^{-1}$ and a damper which exerts a resistance of magnitude $cv \text{ N}$ when the vertical speed is $v \text{ m s}^{-1}$. The car's vertical displacement x metres from equilibrium satisfies $1000\ddot{x} + c\dot{x} + 40\,000x = 0$.
 (a) Find the value of c for which the suspension is critically damped. [3]
 (b) When $c = 10\,000$ show that the motion is underdamped and find the period of the damped oscillations. [4]
-
- 15** The motion of a damped, forced system is modelled by $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 10 \cos t$.
 (a) Find the general solution. [5]
 (b) Describe the long-term behaviour of the system, giving the amplitude of the steady state oscillation. [2]
-
- 16** An undamped system is modelled by $\frac{d^2x}{dt^2} + 4x = 8 \sin 2t$, with $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.
 (a) Explain why a trial particular integral of the form $x = a \cos 2t + b \sin 2t$ is not suitable. [1]
 (b) Find the particular solution. [5]
 (c) Describe how the amplitude of the oscillation behaves as t increases. [1]
-

- 17** A particle of mass 2 kg is attached to a spring of stiffness 8 N m^{-1} and moves on a horizontal table, subject to a resistance of magnitude $4v \text{ N}$. A periodic force $10 \cos t \text{ N}$ acts on the particle along the line of the spring. Let x metres be the extension.
- (a) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = 5 \cos t$. [2]
- (b) Find the steady state solution, and its amplitude. [5]
-
- 18** An underdamped system satisfies $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0$, with $x = 3$ and $\frac{dx}{dt} = 0$ when $t = 0$.
- (a) Show that $x = e^{-t}(3 \cos 3t + \sin 3t)$. [4]
- (b) Find the first time at which the particle passes through the equilibrium position. [2]
-
- 19** A damped system satisfies $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$, with $x = 0$ and $\frac{dx}{dt} = 4$ when $t = 0$.
- (a) Show that $x = 2e^{-t} \sin 2t$. [3]
- (b) Find the time and the value of the first maximum displacement. [4]
-
- 20** The displacement x of a system satisfies $\frac{d^2x}{dt^2} + k\frac{dx}{dt} + 9x = 0$, where $k > 0$ is a constant.
- (a) Find the range of values of k for which the motion is oscillatory, and the value of k for which it is critically damped. [3]
- (b) When $k = 2$, find the period of the oscillations and the factor by which the amplitude of the oscillation decreases during one period. [3]
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- 21** A damped system is driven by a periodic force and satisfies $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 25x = 20 \cos \omega t$, where $\omega > 0$ is a constant.
- (a) Show that the steady state solution has amplitude $A(\omega) = \frac{20}{\sqrt{(25 - \omega^2)^2 + 4\omega^2}}$. [5]
- (b) Show that A is greatest when $\omega^2 = 23$, and find the greatest value of the amplitude. [4]
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- 22** Solve $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 4e^{-3t}$, given that $x = 1$ and $\frac{dx}{dt} = 0$ when $t = 0$. [7]
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- 23** A student solves $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$ and writes: "The auxiliary equation has repeated root $m = -2$, so $x = Ae^{-2t} + Be^{-2t}$."
- (a) Explain why this is not a general solution. [1]
- (b) Given that $x = 0$ and $\frac{dx}{dt} = 3$ when $t = 0$, find x in terms of t . [3]
- (c) Find the greatest displacement and the time at which it occurs. [3]
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- 24** An overdamped system satisfies $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0$, with $x = 0$ and $\frac{dx}{dt} = 6$ when $t = 0$.
- (a) Find x in terms of t . [3]
- (b) Show that the greatest displacement occurs when $t = \frac{1}{3} \ln 4$, and find this displacement. [4]
- (c) Explain why $x > 0$ for all $t > 0$, and state the limit of x as $t \rightarrow \infty$. [1]
-
- 25** A forced damped system satisfies $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 13x = 26 \cos 2t$.
- (a) Find the steady state solution in the form $x = a \cos 2t + b \sin 2t$. [5]
- (b) Write the steady state solution in the form $R \cos(2t - \phi)$, giving R exactly and ϕ to 3 significant figures. [3]

- 26** A mass of 4 kg hangs in equilibrium from a vertical spring of stiffness 100 N m^{-1} , and moves in a fluid which exerts a resistance of magnitude $16v \text{ N}$ when its speed is $v \text{ m s}^{-1}$. The mass is pulled 0.2 m below the equilibrium position and released from rest. Let x metres be the displacement below equilibrium at time t seconds.

- (a) Show that $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 25x = 0$, and find x in terms of t . [6]
 (b) Find the time at which the mass first passes through the equilibrium position. [2]

- 27** A bridge is subject to a periodic force and its vertical displacement x metres at time t seconds satisfies $\frac{d^2x}{dt^2} + 9x = 6 \cos 3t$. Damping is negligible. Initially $x = 0$ and $\frac{dx}{dt} = 0$.

- (a) Explain why the particular integral must be of the form $x = t(a \cos 3t + b \sin 3t)$, and find it. [5]
 (b) Find the solution satisfying the initial conditions. [2]
 (c) Explain why the amplitude of the oscillation grows linearly, and find the first time at which the displacement has magnitude 10 m , giving the time to 3 significant figures. [3]

- 28** A particle of mass 1 kg is attached to a spring of stiffness $k \text{ N m}^{-1}$ and moves in a medium which exerts a resistance of magnitude $cv \text{ N}$ when its speed is $v \text{ m s}^{-1}$. A driving force $10 \cos 2t \text{ N}$ is applied along the line of motion. After a long time the displacement is observed to be $x = 2 \cos(2t - \varphi)$ metres, where $\tan \varphi = \frac{3}{4}$ and $0 < \varphi < \frac{\pi}{2}$.

- (a) Show that the equation of motion is $\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 10 \cos 2t$. [1]
 (b) Using $x = 2 \cos \varphi \cos 2t + 2 \sin \varphi \sin 2t$, show that $(k - 4) \cos \varphi + 2c \sin \varphi = 5$ and $(k - 4) \sin \varphi - 2c \cos \varphi = 0$. [4]
 (c) Hence find the values of c and k . [5]

- 29** A critically damped system satisfies $\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + p^2x = 0$ where $p > 0$ is constant. At $t = 0$ the displacement is $a > 0$ and the velocity is $-v$, where $v > 0$.

- (a) Show that $x = [a + (pa - v)t]e^{-pt}$. [3]
 (b) Show that the system passes through the equilibrium position for some $t > 0$ if and only if $v > pa$, and find that time. [3]
 (c) For $p = 3$, $a = 0.5$ and $v = 4$, find the time when the system passes through equilibrium, and the greatest displacement on the other side of equilibrium. [3]

- 30** An undamped system satisfies $\frac{d^2x}{dt^2} + 25x = 10 \cos 4t$, with $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

- (a) Show that $x = \frac{10}{9}(\cos 4t - \cos 5t)$. [4]
 (b) Using $\cos P - \cos Q = 2 \sin \frac{P+Q}{2} \sin \frac{Q-P}{2}$, show that $x = \frac{20}{9} \sin(\frac{1}{2}t) \sin(\frac{9}{2}t)$. [2]
 (c) Describe the motion, giving the maximum possible amplitude and the time between successive times when the amplitude is zero (the “beat” envelope). [3]