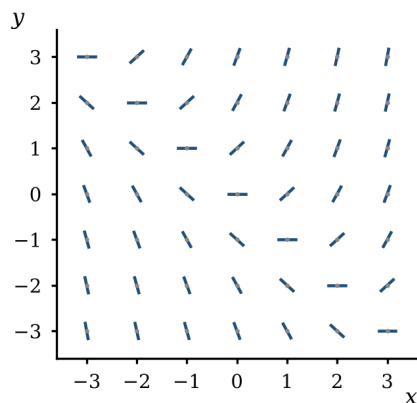


First-Order Differential Equations

- 1 A curve passes through the point (1, 5) and its gradient at (x, y) is given by $\frac{dy}{dx} = 6x^2 - 4x + 3$. Find the equation of the curve. [3]
- 2 Show that $y = x^2 + \frac{3}{x}$ is a solution of the differential equation $x\frac{dy}{dx} + y = 3x^2$. [2]
- 3 Solve the differential equation $\frac{dy}{dx} = 5y$, given that $y = 2$ when $x = 0$. [3]
- 4 Find the general solution of $\frac{dy}{dx} = 3x^2y$, giving y in terms of x . [3]
- 5 Find the general solution of the differential equation $\frac{dy}{dx} = \cos 2x$. [2]
- 6 Given that $y > 0$, solve $\frac{dy}{dx} = \frac{x}{y}$ with $y = 3$ when $x = 0$. [4]
- 7 Solve the differential equation $x\frac{dy}{dx} = y + 1$, for $x > 0$, given that $y = 3$ when $x = 2$. Give y in terms of x . [4]
- 8 Find a differential equation, with no arbitrary constants, satisfied by each of the following families of curves.
 - (a) $y = Ax^3$ [2]
 - (b) $y = Ae^{2x} + 1$ [2]
- 9 Solve $\frac{dy}{dx} = e^{x-y}$, given that $y = \ln 2$ when $x = 0$. Give y in terms of x . [4]
- 10 Solve $\frac{dy}{dx} = 2x(1 + y^2)$, given that $y = 0$ when $x = 0$. Give y in terms of x . [4]
- 11 A curve passes through (1, 2). At each point (x, y) of the curve the gradient is $\frac{3y}{x}$. Find the equation of the curve. [3]
- 12 Given that $\frac{dy}{dx} = 3y(2 - y)$, $0 < y < 2$, and that $y = 1$ when $x = 0$, show that $y = \frac{2}{1 + e^{-6x}}$. [6]
- 13 Solve the differential equation $(1 + x^2)\frac{dy}{dx} = xy$, given that $y = 3$ when $x = 0$. Give y in terms of x . [5]
- 14 Given that $y > 0$, solve $\frac{dy}{dx} = \frac{xe^x}{y}$ with $y = 1$ when $x = 0$. Give y^2 in terms of x . [5]
- 15 The normal at every point P(x, y) of a curve C passes through the origin O.
 - (a) Show that $\frac{dy}{dx} = -\frac{x}{y}$. [2]
 - (b) Given that C passes through (3, 4), find its equation. [3]

- 16** The diagram shows short line segments, drawn at integer points, whose gradients are given by the differential equation $\frac{dy}{dx} = x + y$.



- (a) Write down the gradient of a solution curve at the point (1, 1) and at the point (−2, 1). [2]
 (b) Show that $y = -x - 1$ is a solution of the differential equation. [1]
 (c) Explain why no other solution curve can cross the line $y = -x - 1$. [2]
-
- 17** The area $A \text{ cm}^2$ of a growing fungus at time t days satisfies $\frac{dA}{dt} = 2\sqrt{A}$, where $A > 0$. When $t = 0$, $A = 4$. Find A in terms of t . [4]
-
- 18** Solve $(y^2 + 1)\frac{dy}{dx} = 2x + 3$, given that $y = 1$ when $x = 0$. Give your answer in the form $f(y) = g(x)$. [5]
-
- 19** Solve the differential equation $\cos^2 y \frac{dy}{dx} = \sin^2 x$, given that $y = 0$ when $x = 0$. Give your answer in the form $f(y) = g(x)$. [4]
-
- 20** Consider the differential equation $\frac{dy}{dx} = (2x + y)^2 - 2$, with $y = 1$ when $x = 0$.
 (a) Show that the substitution $u = 2x + y$ transforms the differential equation into $\frac{du}{dx} = u^2$. [2]
 (b) Hence solve the original differential equation, giving y in terms of x . [5]
-
- 21** Consider the differential equation $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$, $x > 0$.
 (a) Use the substitution $y = vx$ to show that $x\frac{dv}{dx} = 1 + v^2$. [3]
 (b) Hence find y in terms of x , given that $y = \sqrt{3}$ when $x = 1$. [4]
-
- 22** A student is asked to solve $\frac{dy}{dx} = 2y + 1$, given that $y = 1$ when $x = 0$, and writes the following working.

$$\int \frac{1}{2y + 1} dy = \int 1 dx$$

$$\frac{1}{2} \ln|2y + 1| = x + c$$

$$2y + 1 = e^{2x} + c$$

$$x = 0, y = 1: 3 = 1 + c, \text{ so } c = 2, \text{ giving } y = \frac{1}{2}(e^{2x} + 1)$$

 (a) Identify the error in the student's working and explain why it is wrong. [2]
 (b) Find the correct solution. [3]

- 23** The differential equation $\frac{dy}{dx} = y^2$ has the solution $y = a$ when $x = 0$, where a is a non-zero constant.
- Find y in terms of a and x . [4]
 - For $a = 2$, state the greatest range of values of x , including $x = 0$, for which the solution is valid. [1]
 - Explain why the method used in part (a) does not produce the solution $y = 0$, even though $y = 0$ satisfies the differential equation. [2]
-
- 24** The family of curves C has equation $y = Ax^2$, where A is a non-zero constant. The family O consists of curves which cross every member of C at right angles.
- Show that every member of C satisfies the differential equation $x\frac{dy}{dx} = 2y$. [2]
 - Write down a differential equation satisfied by the members of O . [2]
 - Hence show that the members of O have equations of the form $x^2 + 2y^2 = k$. [3]
-
- 25** The tangent at a point $P(x, y)$ on a curve C meets the y -axis at the point Q .
- Show that the y -coordinate of Q is $y - x\frac{dy}{dx}$. [3]
 - The y -coordinate of Q is always equal to x^2 , and C passes through $(1, 0)$. Use the substitution $y = vx$ to find the equation of C . [5]
-
- 26** Consider the differential equation $x\frac{dy}{dx} + y = y \ln y$, where $x > 0$ and $y > 0$.
- Use the substitution $z = \ln y$ to show that $x\frac{dz}{dx} = z - 1$. [3]
 - Hence find y in terms of x , given that $y = e^3$ when $x = 1$. [5]
-
- 27** The curve C passes through $(1, 1)$ and satisfies the differential equation $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$, where $x > 0$ and $y > 0$.
- Use the substitution $y = vx$ to show that $2vx\frac{dv}{dx} = -(1 + v^2)$. [4]
 - Hence show that the general solution can be written $x^2 + y^2 = Ax$. [4]
 - Find the equation of C and describe the curve geometrically. [3]
-
- 28** Consider the differential equation $\frac{dy}{dx} = (y - 1)(y - 3)$, with $y = 0$ when $x = 0$.
- Solve the differential equation, giving y in terms of x . [7]
 - Find the exact set of values of x for which this solution is valid and continuous, including $x = 0$. [2]
 - Find the limit of y as $x \rightarrow \infty$ and state what this shows about the line $y = 1$. [2]
-
- 29** Consider the differential equation $\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$, with $y = 2$ when $x = 0$.
- Show that the substitution $u = x + y$ transforms the differential equation into $\frac{du}{dx} = \frac{2u}{u - 1}$. [3]
 - Hence show that $y - x - 2 = \ln(\frac{1}{2}(x + y))$. [7]
-

- 30** Consider the differential equation $\frac{dy}{dx} = \frac{x + 2y - 4}{2x - y - 3}$, for $x > 2$, with $y = 1$ when $x = 3$.
- (a) Find the coordinates of the point where the lines $x + 2y - 4 = 0$ and $2x - y - 3 = 0$ intersect. Hence show that the substitution $X = x - 2$, $Y = y - 1$ transforms the differential equation into $\frac{dY}{dX} = \frac{X + 2Y}{2X - Y}$. [3]
- (b) Use the substitution $Y = VX$ to show that $X \frac{dV}{dX} = \frac{1 + V^2}{2 - V}$. [3]
- (c) Hence show that $4 \tan^{-1}\left(\frac{y-1}{x-2}\right) = \ln((x-2)^2 + (y-1)^2)$. [6]

First-Order Linear Differential Equations

- 1** Find the integrating factor for the differential equation $\frac{dy}{dx} + 4y = e^x$. [2]
- 2** For $x > 0$, write $x \frac{dy}{dx} + 3y = 5x$ in the form $\frac{dy}{dx} + P(x)y = Q(x)$, and find an integrating factor. [3]
- 3** Find an integrating factor for $\frac{dy}{dx} - y \tan x = \cos x$, where $0 < x < \frac{\pi}{2}$. [2]
- 4** Show that e^{x^2} is an integrating factor for $\frac{dy}{dx} + 2xy = x$. [2]
- 5** After multiplying by an integrating factor, a differential equation becomes $\frac{d}{dx}(e^{3x}y) = 6e^{3x}$. Find the general solution for y . [3]
- 6** Find the general solution of $\frac{dy}{dx} + 3y = 12$. [5]
- 7** Find the general solution of $\frac{dy}{dx} + \frac{y}{x} = 3x$, where $x > 0$. [5]
- 8** Find the general solution of $\frac{dy}{dx} + 2y = e^{3x}$. [5]
- 9** Find the general solution of $\frac{dy}{dx} + y \tan x = \sec x$, for $0 < x < \frac{\pi}{2}$. [5]
- 10** Find the general solution of $x \frac{dy}{dx} - 3y = x^4$, where $x > 0$. [5]
- 11** Find the general solution of $\frac{dy}{dx} + y \cot x = 3 \sin^2 x \cos x$, for $0 < x < \pi$. [5]
- 12** Solve $\frac{dy}{dx} - y = e^{2x}$, given that $y = 3$ when $x = 0$. [7]
- 13** Find the general solution of $\frac{dy}{dx} + \frac{3y}{x} = \ln x$, where $x > 0$. [6]
- 14** Given that $\frac{dy}{dx} + y \tan x = 2\cos^2 x$, $0 < x < \frac{\pi}{2}$, and $y = 3$ when $x = 0$, show that $y = \sin 2x + 3 \cos x$. [7]
- 15** Given that $\frac{dy}{dx} = \frac{1}{x+y}$, find the general solution, giving x in terms of y . (Hint: consider $\frac{dx}{dy}$.) [7]

- 16** Consider the differential equation $\frac{dy}{dx} + 2y = 6x$.
- (a) Find the general solution. [6]
- (b) Write down the equation of the oblique asymptote to the solution curves as $x \rightarrow \infty$. [1]
-
- 17** Consider the differential equation $(x^2 + 1)\frac{dy}{dx} + 2xy = 3x^2 + 1$.
- (a) Show that the left-hand side is equal to $\frac{d}{dx}[(x^2 + 1)y]$. [1]
- (b) Hence find y in terms of x , given that $y = 2$ when $x = 0$. [4]
-
- 18** Find the general solution of the differential equation $\frac{dy}{dx} + y \coth x = 2 \cosh x$, where $x > 0$. [5]
-
- 19** Find y in terms of x , given that $\frac{dy}{dx} + (1 + \frac{1}{x})y = e^{-x}$, where $x > 0$, and that $y = \frac{3}{2e}$ when $x = 1$. [6]
-
- 20** Consider the differential equation $\frac{dy}{dx} + \frac{y}{x} = xy^2$, where $x > 0$ and $y > 0$.
- (a) Show that the substitution $z = \frac{1}{y}$ transforms the equation into $\frac{dz}{dx} - \frac{z}{x} = -x$. [2]
- (b) Solve this equation for z in terms of x . [5]
- (c) Given that $y = \frac{1}{2}$ when $x = 1$, find y in terms of x . [2]
-
- 21** Consider the differential equation $y\frac{dy}{dx} + xy^2 = \frac{x}{2}$, with $y > 0$.
- (a) Show that the substitution $z = y^2$ transforms the equation into $\frac{dz}{dx} + 2xz = x$. [2]
- (b) Find z in terms of x . [5]
- (c) Given that $y = 1$ when $x = 0$, find y in terms of x . [2]
-
- 22** A student is asked to find the general solution of $x\frac{dy}{dx} + 3y = 6x^2$, for $x > 0$, and writes:
- $$I = e^{\int 3 dx} = e^{3x}$$
- $$\frac{d}{dx}(e^{3x}y) = 6x^2e^{3x}$$
- (a) Explain the error in the student's first line. [2]
- (b) Find the correct general solution. [5]
-
- 23** Consider the differential equation $\frac{dy}{dx} + ky = e^{-kx}$, where k is a constant with $0 < k < 1$.
- (a) Find the general solution. [5]
- (b) Given that $y = 1$ when $x = 0$, find the coordinates of the stationary point of the solution curve. [4]
-
- 24** Consider the differential equation $\frac{dy}{dx} + 2y = 5 \sin x$.
- (a) Find the general solution. [6]
- (b) For large positive x , every solution is approximately $R \sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Find the exact value of R and the value of α to 3 decimal places. [2]
-
- 25** Given that $\frac{dy}{dx} + 2y \cot 2x = 4 \cos 2x$, $0 < x < \frac{\pi}{2}$, and $y = 1$ when $x = \frac{\pi}{4}$, show that $y = \sin 2x$. [7]
-

- 26** The curve C passes through the point (2, 2). At every point P(x, y) of C, where $x > 0$, the gradient of C is x^2 more than the gradient of the line OP, where O is the origin.
- (a) Show that $\frac{dy}{dx} - \frac{y}{x} = x^2$, and hence find the equation of C. [6]
- (b) Find the coordinates of the stationary point of C for $x > 0$, and determine its nature. [4]
-
- 27** The functions $z(x)$ and $y(x)$ satisfy $\frac{dz}{dx} + 3z = 0$ and $\frac{dy}{dx} + 3y = z$, with $z = 4$ and $y = -1$ when $x = 0$.
- (a) Find z in terms of x . [2]
- (b) Hence find y in terms of x . [7]
- (c) Find the exact coordinates of the maximum point of the graph of y for $x \geq 0$. [3]
-
- 28** Consider the differential equation $\frac{dy}{dx} + \frac{y}{x \ln x} = \frac{6}{x}$, for $x > 1$, with $y = 5$ when $x = e$.
- (a) Find y in terms of x . [7]
- (b) Find the exact minimum value of y , and justify that it is a minimum. [5]
-
- 29** Let $y_1(x)$ and $y_2(x)$ be two different solutions of $\frac{dy}{dx} + 2y = f(x)$, where f is a continuous function.
- (a) Show that $u = y_1 - y_2$ satisfies $\frac{du}{dx} + 2u = 0$. [2]
- (b) Hence show that $u = Ae^{-2x}$, and explain what this shows about the solution curves for large x . [3]
- (c) For $f(x) = 4x$, the solutions satisfying $y_1(0) = 0$ and $y_2(0) = 1$ are y_1 and y_2 . Find $y_1 - y_2$ and the exact value of $y_1 - y_2$ when $x = \ln 2$. [4]
-
- 30** Consider the differential equation $\frac{dy}{dx} + y = g(x)$, where $y = 0$ when $x = 0$.
- (a) Show that $y = e^{-x} \int_0^x e^t g(t) dt$. [4]
- (b) Hence find y in terms of x when $g(x) = \cos x$. [4]

Second-Order Differential Equations

- 1** Show that $y = e^{3x}$ is a solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$. [2]
-
- 2** Write down the auxiliary equation of $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ and solve it. [2]
-
- 3** Show that $y = 3 \cos 2x + \sin 2x$ is a solution of $\frac{d^2y}{dx^2} + 4y = 0$. [3]
-
- 4** Find the general solution of $\frac{d^2y}{dx^2} = 6x$. [2]
-
- 5** Find the general solution of $\frac{d^2y}{dx^2} - 9y = 0$. [2]
-
- 6** State the order of each differential equation, and say whether it is linear or non-linear.
- (a) $x^2 \frac{d^2y}{dx^2} + y = 0$ [1]
- (b) $\left(\frac{dy}{dx}\right)^2 + y = x$ [1]
- (c) $\frac{d^3y}{dx^3} + 2\frac{dy}{dx} = \sin x$ [1]

- 7 Find the general solution of $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$. [2]
- 8 Find the general solution of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$. [2]
- 9 Find the general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$. [2]
- 10 The general solution of a differential equation is $y = Ae^{2x} + Be^{-x}$. By eliminating A and B, show that $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$. [4]
- 11 Given that $\frac{d^2y}{dx^2} = 12x^2 - 2$, with $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$, find y in terms of x . [4]
- 12 Given that $y = e^{mx}$ is a solution of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$, find the possible values of m . [3]
- 13 Find the general solution of $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 4x$, where $x > 0$. [4]
- 14 $y_1(x)$ and $y_2(x)$ are both solutions of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$. Show that $y = \alpha y_1 + \beta y_2$, where α and β are constants, is also a solution. [4]
- 15 The general solution of a differential equation is $y = e^x(A \cos 2x + B \sin 2x)$. Find a differential equation of the form $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ satisfied by this family of curves. [6]
- 16 Given that $\frac{d^2y}{dx^2} = e^{2x} + \sin x$, and that $y = 0$ and $\frac{dy}{dx} = 3$ at $x = 0$, find y in terms of x . [5]
- 17 The displacement x metres of a particle at time t seconds satisfies $\frac{d^2x}{dt^2} = -16x$. When $t = 0$, $x = 0.5$ and $\frac{dx}{dt} = 0$.
(a) Find x in terms of t . [4]
(b) Find the period of the motion. [1]
- 18 Find the general solution of $x\frac{d^2y}{dx^2} = 2\frac{dy}{dx}$, for $x > 0$, by first letting $p = \frac{dy}{dx}$. [4]
- 19 Consider the differential equation $\frac{d^2y}{dx^2} + k\frac{dy}{dx} + 4y = 0$, where k is a positive constant. State, with justification, the range of values of k for which the general solution
(a) contains two distinct real exponentials, [2]
(b) has the form $(A + Bx)e^{mx}$, [1]
(c) involves oscillating terms, and give the angular frequency in this case. [3]
- 20 Using $x = e^t$, it can be shown that $x\frac{dy}{dx} = \frac{dy}{dt}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.
Use these results to solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$, for $x > 0$, giving y in terms of x . [6]

- 21** Consider the equation $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$, where $y > 0$.
- (a) Show that the left-hand side equals $\frac{d}{dx}\left(y \frac{dy}{dx}\right)$. [2]
- (b) Hence find y in terms of x , given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$. [4]
-
- 22** A student is asked to solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ and writes: " $m^2 + 4m + 4 = 0$, so $m = -2$ twice, and the general solution is $y = Ae^{-2x} + Be^{-2x}$."
- (a) Explain why the student's answer cannot be the general solution. [2]
- (b) Show that $y = xe^{-2x}$ is a solution of the differential equation. [3]
- (c) Write down the correct general solution. [1]
-
- 23** Show that the substitution $y = ue^{2x}$, where u is a function of x , transforms $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ into $\frac{d^2u}{dx^2} = 0$. Hence find the general solution for y . [7]
-
- 24** A particle moves on a straight line with displacement x metres from O and velocity v m s⁻¹ at time t seconds, where $\frac{d^2x}{dt^2} = -4x$.
- (a) Show that $\frac{d^2x}{dt^2} = v \frac{dv}{dx}$. [2]
- (b) Given that $v = 0$ when $x = 3$, show that $v^2 = 4(9 - x^2)$. [4]
- (c) Find the maximum speed of the particle. [1]
-
- 25** Consider the differential equation $\frac{d^2y}{dx^2} = 2y^3$, with $y = 1$ and $\frac{dy}{dx} = -1$ when $x = 0$.
- (a) By writing $p = \frac{dy}{dx}$ and showing that $\frac{d^2y}{dx^2} = p \frac{dp}{dy}$, show that $p^2 = y^4$. [4]
- (b) Hence find y in terms of x . [4]
-
- 26** Consider the differential equation $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \ln x$, for $x > 0$.
- (a) Use the substitution $x = e^t$, with $x \frac{dy}{dx} = \frac{dy}{dt}$ and $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$, to show that $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = t$. [3]
- (b) Hence find the general solution for y in terms of x . [6]
-
- 27** Consider the differential equation $(x^2 + 1) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$, for $x > 0$.
- (a) Show that $y = x$ is a solution. [1]
- (b) Use the substitution $y = xv$ to show that $x(x^2 + 1) \frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0$. [3]
- (c) Hence find the general solution for y . [6]
-
- 28** Consider the differential equation $\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + (k^2 + 9)y = 0$, where k is a constant.
- (a) Show that $y = e^{kx} \sin 3x$ is a solution. [5]
- (b) Find the general solution. [2]
- (c) The solution $y = e^{kx} \sin 3x$ has its first stationary point for $x > 0$ at $x = \frac{\pi}{12}$. Find the value of k . [2]

- 29** Consider the differential equation $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$, with $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$.
- (a) By letting $p = \frac{dy}{dx}$, find y in terms of x . [6]
- (b) State the largest set of values of x for which this solution is valid, and describe the behaviour of y at the end of this interval. [2]
-
- 30** Consider the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4e^{2x}y = 0$, and let $z = e^x$.
- (a) Show that $\frac{dy}{dx} = z \frac{dy}{dz}$ and $\frac{d^2y}{dx^2} = z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz}$. [4]
- (b) Hence show that the differential equation becomes $\frac{d^2y}{dz^2} - 4y = 0$, and find the general solution for y in terms of x . [4]
- (c) Given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$, find y in terms of x . [3]
-

Homogeneous Differential Equations

- 1** The auxiliary equation of a second-order differential equation is $m^2 - 2m - 15 = 0$. Write down the general solution of the differential equation. [2]
-
- 2** Write down the general solution of $\frac{d^2y}{dx^2} + 25y = 0$. [2]
-
- 3** A second-order homogeneous differential equation has auxiliary equation with roots $m = 2$ and $m = -5$. Find the differential equation, in the form $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$. [2]
-
- 4** Find the general solution of $\frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 8y = 0$. [2]
-
- 5** Find the general solution of $4 \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + y = 0$. [2]
-
- 6** Without solving the equations, state which of the following differential equations have solutions that oscillate. Give a reason in each case.
- (a) $\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 3y = 0$ [1]
- (b) $\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 8y = 0$ [1]
- (c) $\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 4y = 0$ [1]
-
- 7** Find the general solution of $3 \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} - 2y = 0$. [2]
-
- 8** Find the general solution of $\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 10y = 0$. [2]
-
- 9** A second-order homogeneous differential equation has auxiliary equation roots $m = -1 + 2i$ and $m = -1 - 2i$. Find the differential equation, and write down its general solution. [3]
-
- 10** Find the general solution of $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} = 0$. [2]
-

- 11** A particle moves so that its displacement x metres at time t seconds satisfies $\frac{d^2x}{dt^2} + 9x = 0$.
When $t = 0$, $x = 2$ and $\frac{dx}{dt} = 0$. Find x in terms of t . [4]
- 12** Use the substitution $y = vx$ to solve $\frac{dy}{dx} = \frac{x+y}{x}$, for $x > 0$, given that $y = 2$ when $x = 1$. [4]
- 13** Show that the general solution $y = Ae^x + Be^{-x}$ of $\frac{d^2y}{dx^2} = y$ can also be written as $y = C \cosh x + D \sinh x$, where C and D are constants to be found in terms of A and B . [3]
- 14** Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$, given that $y = 1$ and $\frac{dy}{dx} = 8$ when $x = 0$. [6]
- 15** Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$, given that $y = 2$ and $\frac{dy}{dx} = -1$ when $x = 0$. [6]
- 16** Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$, given that $y = 1$ and $\frac{dy}{dx} = 3$ when $x = 0$. [6]
- 17** Use the substitution $y = vx$ to show that the general solution of $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$, for $x > 0$ and $y > 0$, is $x^2 + y^2 = Ax^3$. Hence find the solution which passes through $(1, 1)$. [6]
- 18** The displacement x metres of a particle at time t seconds satisfies $\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + 9x = 0$, where λ is a positive constant. In each case, find the general solution and state whether the motion is overdamped, critically damped or underdamped.
(a) $\lambda = 5$ [2]
(b) $\lambda = 3$ [2]
(c) $\lambda = 1$ [2]
- 19** The auxiliary equation of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + (a+3)y = 0$ has equal roots, where a is a positive constant. Find a and solve the differential equation. [5]
- 20** The curve C with equation $y = e^{-2x} \sin x$, for $x \geq 0$, is a solution of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$.
Find the exact coordinates of the first maximum point of C . [5]
- 21** The displacement x metres of a particle at time t seconds satisfies $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 29x = 0$. When $t = 0$, $x = 1$ and $\frac{dx}{dt} = 0$.
(a) Find x in terms of t . [6]
(b) Describe the long-term behaviour of the particle. [1]
- 22** Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$, for $x > 0$, given that $y = \frac{\pi}{6}$ when $x = 1$. Give y in terms of x . [7]
- 23** Find the general solution of $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$, given that $m = -1$ is a root of the auxiliary equation. [6]

- 24** The general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$ is $y = e^{-x}(A \cos 2x + B \sin 2x)$.
- (a) Show that this can be written in the form $y = Re^{-x} \cos(2x - \varphi)$, and find R and φ in terms of A and B . [3]
- (b) Hence state the time between successive maximum values of $\cos(2x - \varphi)$, and the factor by which the amplitude Re^{-x} is multiplied over this interval. [2]
-
- 25** Find the values of m for which $y = x^m$ is a solution of $x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 4y = 0$, for $x > 0$.
- Given that the general solution is $y = \frac{A + B \ln x}{x^2}$, show that $y = x^{-2}$ is the only solution of the form x^m , and explain why. [5]
-
- 26** Consider the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + ky = 0$, where k is a real constant.
- (a) Find the roots of the auxiliary equation in terms of k . [2]
- (b) Find the range of values of k for which the solutions do not oscillate and every solution tends to 0 as $x \rightarrow \infty$. [3]
- (c) Describe the behaviour of the solutions when $k > 1$. [2]
-
- 27** Consider the differential equation $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + (a^2 + \omega^2)y = 0$, where a and ω are positive constants.
- (a) Show that $y = e^{-ax} \sin \omega x$ is a solution. [5]
- (b) Show that successive stationary points of y are a distance $\frac{\pi}{\omega}$ apart in x , and find the exact value of y at the first stationary point with $x > 0$ in terms of a and ω . [4]
-
- 28** Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$, given that $y = 3$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = -4$ when $x = 0$. [9]
-
- 29** Consider the differential equation $x \frac{dy}{dx} = y + \sqrt{x^2 + y^2}$, for $x > 0$, with $y = 0$ when $x = 1$.
- (a) Use the substitution $y = vx$ to show that $x \frac{dv}{dx} = \sqrt{1 + v^2}$. [3]
- (b) Hence show that $y = \frac{1}{2}(x^2 - 1)$. [6]
-
- 30** A differential equation $\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = 0$ has real constants $\alpha \neq \beta$.
- (a) Find the solution y_1 with $y_1 = 0$ and $\frac{dy_1}{dx} = 1$ when $x = 0$. [4]
- (b) Show that, as $\beta \rightarrow \alpha$, $y_1 \rightarrow xe^{\alpha x}$, and explain how this relates to the case of a repeated root. [4]

Non-Homogeneous Differential Equations

- 1** The complementary function of $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12$ is $y = Ae^{-2x} + Be^{-3x}$. Show that $y = 2$ is a particular integral, and write down the general solution. [2]
-
- 2** Show that $y = x$ is a particular integral of $\frac{d^2y}{dx^2} + y = x$. [2]

- 3** Find the value of the constant p for which $y = pe^x$ is a particular integral of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 8e^x$. [2]
- 4** Explain why the general solution of a second-order non-homogeneous linear differential equation is the sum of the complementary function and a particular integral, and why it contains two arbitrary constants. [2]
- 5** The complementary function of a differential equation is $y = e^{-x}(A \cos 2x + B \sin 2x)$ and a particular integral is $y = 3x - 1$. Write down the general solution. [2]
- 6** Find the general solution of $\frac{d^2y}{dx^2} + 9y = 18$. [3]
- 7** Find the general solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4e^x$. [5]
- 8** Find the general solution of $\frac{d^2y}{dx^2} + 4y = 8x$. [5]
- 9** Find the general solution of $\frac{d^2y}{dx^2} - y = 2 \sin x$. [5]
- 10** Find the general solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 6$. [5]
- 11** Find the general solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$. [5]
- 12** Find the general solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 6x^2 - 4x$. [5]
- 13** Find the general solution of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3x + 4$. [6]
- 14** Find the general solution of $\frac{d^2y}{dx^2} + 9y = 5 \cos 2x$. [6]
- 15** Find the general solution of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8e^{2x}$. [6]
- 16** Find the general solution of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 6x + 2e^{-x}$. [7]
- 17** Find constants a and b such that $y = ax + b$ is a particular integral of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x - 1$. Hence find the general solution. [5]
- 18** Consider the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$.
 (a) Show that a trial particular integral of the form $y = pe^{2x}$ cannot work. [2]
 (b) Find the general solution. [4]
- 19** Find the general solution of $\frac{d^2y}{dx^2} + 4y = 8 \sin 2x$. [6]

- 20** Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 6e^{-x}$, given that $y = 2$ and $\frac{dy}{dx} = 1$ when $x = 0$. [9]
-
- 21** Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 26$, given that $y = 3$ and $\frac{dy}{dx} = 1$ when $x = 0$. [9]
-
- 22** Given that $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 12e^{-3x}$, with $y = 1$ and $\frac{dy}{dx} = -2$ when $x = 0$, show that $y = (6x^2 + x + 1)e^{-3x}$. [9]
-
- 23** Consider the differential equation $\frac{d^2y}{dx^2} + 9y = \sin kx$, where k is a positive constant.
- (a) For $k \neq 3$, find a particular integral. [3]
- (b) For $k = 3$, find a particular integral, and explain why a different form is needed. [4]
- (c) Describe how the size of the oscillations in the solution to part (b) changes as x increases. [1]
-
- 24** Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 5 \sin x$, given that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$. [9]
-
- 25** Find the general solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2xe^{3x}$. [6]
-
- 26** The curve C satisfies the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5x^2 + 18x + 10$. The curve passes through the origin and has a stationary point there.
- (a) Show that the origin is a local minimum of C . [2]
- (b) Find the equation of C . [8]
-
- 27** Consider the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 4e^{-x} \cos 2x$.
- (a) Explain why the particular integral must have the form $y = xe^{-x}(p \cos 2x + q \sin 2x)$, and find p and q . [5]
- (b) Find the solution for which $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$. [5]
- (c) Explain why this solution is oscillatory but tends to 0 as $x \rightarrow \infty$ even though the amplitude factor xe^{-x} starts by increasing. [2]
-
- 28** Show that the solution of $\frac{d^2y}{dx^2} - 4y = 8 \cosh 2x$, with $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$, is $y = 2x \sinh 2x$. [10]
-
- 29** Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 6xe^{-x}$, given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$. [10]
-
- 30** The functions $y = x$, $y = x + e^x$ and $y = x + e^{2x}$ are all solutions of the differential equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$, where a and b are constants and f is a function of x .
- (a) Explain why e^x and e^{2x} are solutions of the corresponding homogeneous equation. [2]
- (b) Find a and b . [2]
- (c) Find $f(x)$, and write down the general solution of the differential equation. [3]
- (d) Find the solution for which $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$. [3]
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Complementary Functions

- 1 Find the complementary function of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 10 \sin x$. [2]
- 2 Write down the complementary function of $\frac{d^2y}{dx^2} + 49y = e^x$. [2]
- 3 Find the complementary function of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 3x$. [2]
- 4 Find the complementary function of the first-order equation $\frac{dy}{dx} + 4y = x^2$. [2]
- 5 A second-order homogeneous differential equation has complementary function $y = (A + Bx)e^{2x}$. Find the differential equation, in the form $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$. [2]
- 6 The homogeneous equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ is given. State whether each of the following functions is a solution.
 - (a) $y = e^{3x}$ [1]
 - (b) $y = e^{-2x}$ [1]
 - (c) $y = e^{2x}$ [1]
- 7 Find the complementary function of $3\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 3y = 6x$. [2]
- 8 Find the complementary function of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 2e^x$. [2]
- 9 Find the complementary function of $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 5$. [2]
- 10 Find the complementary function of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 4$. [2]
- 11 Find the complementary function of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + ky = 0$ in each of the cases
 - (a) $k = 5$, [2]
 - (b) $k = 9$. [2]
- 12 The complementary function of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 5$ is $y = Ae^{-x} + Be^{3x}$. Find the values of a and b . [3]
- 13 Find the complementary function of $\frac{dy}{dx} - \frac{2y}{x} = x^2$, where $x > 0$. [3]
- 14 The displacement x of a forced oscillator satisfies $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 50 \cos 3t$.
 - (a) Find the complementary function. [3]
 - (b) Explain why the complementary function is called the transient part of the solution. [1]

- 15** The function $y_c(x)$ satisfies $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$, and the function $y_p(x)$ satisfies $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x)$, where p and q are constants. Show that $y = y_c + y_p$ satisfies $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x)$. [4]
- 16** Consider the differential equation $\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + (k + 6)y = 4x$, where k is a constant.
- (a) Show that the complementary function contains oscillating terms when $k^2 - k - 6 < 0$. [2]
 (b) Find the range of values of k for which this happens. [2]
 (c) Find the complementary function when $k = 1$. [2]
- 17** Find the complementary function of $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = e^x$. [4]
- 18** The homogeneous equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$ has a solution curve with $y = 3$ and $\frac{dy}{dx} = 0$ when $x = 0$.
- (a) Find the equation of this solution curve. [4]
 (b) Write $3 \cos 3x + 2 \sin 3x$ in the form $R \cos(3x - \phi)$, giving R exactly and ϕ to 3 decimal places. [3]
- 19** The temperature excess θ °C of a cooling object satisfies $\frac{d\theta}{dt} + 0.2\theta = 5 + 3 \sin t$, where t is time in minutes. The complementary function describes the transient part of the solution.
- (a) Find the complementary function. [2]
 (b) Find the time, to 3 significant figures, for the complementary function to halve in size. [2]
- 20** A student claims that the complementary function of $\frac{d^2y}{dx^2} + 9y = 5e^{3x}$ is $y = Ae^{3x} + Be^{-3x}$.
- (a) Explain the error in the student's claim. [2]
 (b) Write down the correct complementary function. [1]
 (c) Find a particular integral of the differential equation. [2]
- 21** The general solution of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$ is $y = e^{-2x}(A \cos x + B \sin x) + 2x - 1$, where a and b are constants. Find a , b and $f(x)$. [6]
- 22** Find the general solution of $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$. [6]
- 23** Find the values of m for which $y = x^m$ is a solution of $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = 0$, for $x > 0$, and hence state the general solution. [5]
- 24** The functions $y_1 = e^{2x}$ and $y_2 = e^{-x}$ are solutions of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$. Show that they are linearly independent, by proving that if $Ay_1 + By_2 = 0$ for all x then $A = B = 0$. [5]
- 25** The displacement x of a forced oscillator satisfies $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10 \sin t$.
- (a) Find the complementary function. [2]
 (b) Find a particular integral. [3]
 (c) For large t the motion is approximately the particular integral. State its amplitude. [1]
 (d) The complementary function has initial amplitude 1 and decays like e^{-t} . Find the time after which its amplitude is below 0.01. [1]

- 26** Show that the general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2y = 0$ can be written as $y = e^{-x}(C \cosh \sqrt{3}x + D \sinh \sqrt{3}x)$. [5]
-
- 27** Consider the fourth-order differential equation $\frac{d^4y}{dx^4} - 16y = 0$.
- (a) Show that the auxiliary equation can be written $(m - 2)(m + 2)(m^2 + 4) = 0$, and find the general solution. [4]
- (b) Find the solution which is bounded for all real x and satisfies $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$. [4]
-
- 28** Let a , b and c be real constants, and suppose that a complex-valued function $y = u + iv$, where u and v are real functions of x , satisfies $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.
- (a) Show that u and v are also solutions of the same differential equation. [3]
- (b) Verify that $y = e^{(-1 + 2i)x}$ is a solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$. [2]
- (c) Use $e^{i\theta} = \cos \theta + i \sin \theta$ to write down two real solutions of the equation in part (b). [3]
-
- 29** Consider the differential equation $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$, for $x > 0$.
- (a) Show that $y = x^2$ is a solution. [2]
- (b) Use the substitution $y = x^2v$, where v is a function of x , to show that $x\frac{d^2v}{dx^2} + \frac{dv}{dx} = 0$. [4]
- (c) Hence find the general solution of the differential equation. [3]
-
- 30** The functions y_1 and y_2 both satisfy $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$, where p and q are constants. Their Wronskian is defined as $W = y_1y_2' - y_1'y_2$.
- (a) Show that $\frac{dW}{dx} = -pW$. [4]
- (b) Deduce that $W = Ce^{-px}$, and explain why W is either zero for every x or non-zero for every x . [3]
- (c) For $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$, with $y_1 = e^x$ and $y_2 = e^{3x}$, find W and check that it has the form in part (b). [2]

Particular Integrals

- 1** Find a particular integral of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 10$. [2]
-
- 2** The differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = f(x)$ is to be solved. State the form of particular integral to try, without finding the constants, when $f(x)$ is
- (a) $3x + 1$, [1]
- (b) $5e^{2x}$, [1]
- (c) $2 \sin 3x$. [1]
-
- 3** Find a particular integral of $\frac{d^2y}{dx^2} + y = 5e^{2x}$. [2]

- 4** Find a particular integral of $\frac{d^2y}{dx^2} + 4y = 8x$. [3]

- 5** Find a particular integral of $\frac{d^2y}{dx^2} - 4y = 6 \cos x$. [3]

- 6** Find a particular integral of $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 6e^{-2x}$. [2]

- 7** Find a particular integral of $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 6x^2 + 10x + 2$. [4]

- 8** Find a particular integral of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 10 \sin x$. [3]

- 9** Find a particular integral of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 6e^x$. [3]

- 10** Find a particular integral of $\frac{d^2y}{dx^2} + 9y = 6 \sin 3x$. [3]

- 11** Find a particular integral of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 6x$. [3]

- 12** Find a particular integral of $\frac{d^2y}{dx^2} - y = 2e^{2x} + 3$. [3]

- 13** Find the general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 17 \cos 2x$, including the particular integral. [6]

- 14** Explain why neither $y = pe^{-2x}$ nor $y = pxe^{-2x}$ can be a particular integral of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 6e^{-2x}$, and find a particular integral. [5]

- 15** Find a particular integral of $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 12x + 2 + 4e^x$. [4]

- 16** Find the general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 8e^x \cos x$. [6]

- 17** Find the general solution of $\frac{d^2y}{dx^2} + 4y = (5x + 7)e^x$. [6]

- 18** Show that $y = \frac{1}{2}x^2e^{3x}$ is a particular integral of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$. [4]

- 19** The function $y = x^2$ is a particular integral of the differential equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 6x^2 + 8x + 2$, where a and b are constants. Find the values of a and b , and hence find the general solution. [6]

- 20** State the form of particular integral that you would try, without finding the constants, and give a reason, for each of the following differential equations.
- (a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 5e^x$ [2]
- (b) $\frac{d^2y}{dx^2} + 4y = 2 \sin 2x$ [2]
- (c) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3e^{2x}$ [2]
-
- 21** Find the general solution of $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6xe^x$. [6]
-
- 22** Find a particular integral of $\frac{d^2y}{dx^2} + 4y = \sin^2 x$. [7]
-
- 23** Consider the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{ax}$, where a is a constant.
- (a) For $a \neq 1$, show that a particular integral is $y = \frac{e^{ax}}{(a-1)^2}$. [3]
- (b) For $a = 1$, find a particular integral. [3]
-
- 24** Find the general solution of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 8e^{-3x} \sin 2x$. [6]
-
- 25** Consider the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10e^{ix}$, where $i^2 = -1$.
- (a) Find a particular integral of the form $y = pe^{ix}$, where p is a complex constant. [3]
- (b) Hence find a particular integral of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10 \cos x$. [2]
- (c) Hence write down a particular integral of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10 \sin x$. [2]
-
- 26** Consider the first-order equation $\frac{dy}{dx} + 3y = 4xe^{-3x}$.
- (a) Find a particular integral. [5]
- (b) Hence state the general solution. [1]
-
- 27** Consider the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 12xe^x$.
- (a) Show that the substitution $y = ue^x$, where u is a function of x , transforms the equation into $\frac{d^2u}{dx^2} = 12x$. [4]
- (b) Hence find the general solution for y . [3]
- (c) State a particular integral, and explain why the trial form pe^x , $(px + q)e^x$ or $x(px + q)e^x$ would all have failed. [2]
-
- 28** Solve $\frac{d^2y}{dx^2} - y = x^2e^x$, given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$. [11]
-

29 Consider the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{ax}$, where a is a constant different from 1 and 2.

(a) Show that $y_a = \frac{e^{ax} - e^{2x}}{(a-1)(a-2)}$ is a particular integral. [3]

(b) By considering the limit of y_a as $a \rightarrow 2$, show that a particular integral when $a = 2$ is $y = xe^{2x}$. [4]

(c) Verify directly that $y = xe^{2x}$ is a particular integral when $a = 2$. [2]

30 A system is modelled by $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = \cos \omega x$, where $\omega > 0$ is a constant. For large x the solution is approximately the particular integral.

(a) Show that a particular integral is $y = p \cos \omega x + q \sin \omega x$, where $p = \frac{5 - \omega^2}{D}$, $q = \frac{2\omega}{D}$ and $D = (5 - \omega^2)^2 + 4\omega^2$. [4]

(b) Show that the amplitude of this particular integral is $\frac{1}{\sqrt{D}}$. [2]

(c) Find the value of ω for which the amplitude is greatest, and the greatest amplitude. [4]

Initial and Boundary Conditions

1 The general solution of a differential equation is $y = Ae^{2x}$. Find the particular solution for which $y = 5$ when $x = 0$. [2]

2 The general solution of $\frac{d^2y}{dx^2} + 4y = 0$ is $y = A \cos 2x + B \sin 2x$. Find A and B given that $y = 3$ and $\frac{dy}{dx} = 4$ when $x = 0$. [3]

3 The general solution of $\frac{d^2y}{dx^2} = y$ is $y = Ae^x + Be^{-x}$. Find A and B given that $y = 3$ and $\frac{dy}{dx} = 1$ when $x = 0$. [3]

4 State whether each of the following sets of conditions on a second-order differential equation are initial conditions or boundary conditions, giving a reason.

(a) $y(0) = 1$ and $y'(0) = 2$ [1]

(b) $y(0) = 1$ and $y(2) = 3$ [1]

(c) $y(1) = 0$ and $y'(1) = 5$ [1]

5 Explain why a particular solution of a second-order linear differential equation normally needs two conditions, but a first-order equation needs only one. [2]

6 Find the equation of the curve which passes through $(0, 3)$ and has gradient $\frac{dy}{dx} = 2x$. [2]

7 The general solution of $\frac{dy}{dx} + 2y = 6$ is $y = 3 + Ce^{-2x}$. Find the solution which passes through $(0, 1)$, and write down the value that y approaches as $x \rightarrow \infty$. [3]

8 The general solution of $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ is $y = Ae^{2x} + Be^{3x}$. Find the solution with $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. [4]

- 9** The general solution of $\frac{d^2y}{dx^2} + y = 0$ is $y = A \cos x + B \sin x$. Find the solution satisfying $y(0) = 1$ and $y(\frac{\pi}{2}) = 3$. [4]
-
- 10** The general solution of $\frac{d^2y}{dx^2} = 4y$ is $y = Ae^{2x} + Be^{-2x}$. Find the solution with $y(0) = 0$ and $y(1) = e^2 - e^{-2}$. [4]
-
- 11** A curve satisfies $\frac{d^2y}{dx^2} = 6x$ and has gradient 2 at the point (1, 5). Find the equation of the curve. [4]
-
- 12** The general solution of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$ is $y = Ae^{-x} + Be^{2x}$. Find the solution which is bounded as $x \rightarrow \infty$ and satisfies $y = 3$ when $x = 0$. [3]
-
- 13** The curve $y = Ae^{2x} + Be^{-x}$ has a stationary point at (0, 3). Find the values of A and B. [4]
-
- 14** Solve $\frac{d^2y}{dx^2} + 4y = 0$, given that $y = 1$ when $x = 0$ and $y = 3$ when $x = \frac{\pi}{4}$. [6]
-
- 15** Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$, given that $y = 1$ when $x = 0$ and $y = 3e$ when $x = 1$. [6]
-
- 16** Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$, given that $y = 0$ when $x = 0$ and $y = e^{-\pi/4}$ when $x = \frac{\pi}{4}$. [6]
-
- 17** Find the solution of $x\frac{dy}{dx} + y = 2x$, for $x > 0$, which passes through the point (1, 3). [7]
-
- 18** Consider the differential equation $\frac{d^2y}{dx^2} - 9y = 0$.
 (a) Find the solution which is bounded as $x \rightarrow \infty$ and passes through (0, 4). [3]
 (b) Write down the gradient of this solution at $x = 0$. [1]
-
- 19** Consider the differential equation $\frac{d^2y}{dx^2} + y = 0$ with boundary conditions $y(0) = 0$.
 (a) Show that there is no solution which also satisfies $y(\pi) = 1$. [3]
 (b) Describe all solutions which satisfy $y(0) = 0$ and $y(\pi) = 0$. [2]
-
- 20** A solution curve of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ passes through (0, 2) and has a stationary point at $x = 1$. Find its equation and the value of y at the stationary point. [5]
-
- 21** Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 18$, given that $y = 3$ and $\frac{dy}{dx} = 1$ when $x = 0$. [9]
-
- 22** Find the values of the positive constant k for which the equation $\frac{d^2y}{dx^2} + ky = 0$, with $y(0) = 0$ and $y'(1) = 0$, has a solution other than $y = 0$. [6]
-
- 23** Solve $\frac{d^2y}{dx^2} + y = 2$, given that $y = 0$ when $x = 0$ and $y = 3$ when $x = \frac{\pi}{2}$. [9]
-

- 24** The function $y(x)$ satisfies $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10$, with $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$.
- By differentiating the equation, find expressions for $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$ in terms of lower derivatives. [2]
 - Hence find the values of the first four derivatives of y at $x = 0$. [2]
 - Write down the Maclaurin series for y up to and including the term in x^4 , and use it to estimate $y(0.1)$. [3]
-
- 25** Consider the differential equation $\frac{dy}{dx} = 4 - 2y$, with $y = a$ when $x = 0$, where a is a constant.
- Show that $y = 2 + (a - 2)e^{-2x}$. [4]
 - Find the range of values of a for which y is increasing for all x , and describe the solution when $a = 2$. [3]
-
- 26** The curve C is a solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ and passes through the origin. Its first maximum point has y -coordinate 1. Find the equation of C . [6]
-
- 27** Consider the boundary value problem $\frac{d^2y}{dx^2} + \lambda y = 0$, with $y(0) = 0$ and $y(\pi) = 0$, where λ is a constant.
- Show that if $\lambda \leq 0$ then the only solution is $y = 0$. [4]
 - For $\lambda > 0$ find the values of λ for which there is a non-zero solution, and write down the corresponding solutions. [5]
-
- 28** Consider the differential equation $\frac{d^2y}{dx^2} + y = \sin x$.
- Find the general solution. [5]
 - Show that no solution satisfies $y(0) = 0$ and $y(\pi) = 0$. [3]
 - Show that $y(0) = 0$ and $y(\pi) = \frac{\pi}{2}$ are satisfied by infinitely many solutions. [2]
-
- 29** Consider the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.
- Show that the only solution with $y(0) = 0$ and $y'(0) = 0$ is $y = 0$. [3]
 - Two functions y_1 and y_2 both satisfy $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$, with $y_1(0) = y_2(0)$ and $y_1'(0) = y_2'(0)$. Prove that $y_1 = y_2$ for all x . [4]
-
- 30** The function y satisfies $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, with $y = 3$ and $\frac{dy}{dx} = k$ when $x = 0$.
- Find y in terms of k and x . [4]
 - Find the value of k for which y remains bounded as $x \rightarrow \infty$, and give the solution in this case. [2]
 - Find the value of k for which y has a stationary point at $x = \ln 2$. [4]
-

Coupled Differential Equations

- 1** Solve the uncoupled equations $\frac{dx}{dt} = 3x$ and $\frac{dy}{dt} = -2y$, given that $x = 4$ and $y = 5$ when $t = 0$. [3]
-
- 2** Show that $x = 2e^t$, $y = e^t$ satisfy the coupled equations $\frac{dx}{dt} = 3x - 4y$ and $\frac{dy}{dt} = x - y$. [2]
-

- 3** Given $\frac{dx}{dt} = 2x + y$, write y in terms of x and $\frac{dx}{dt}$. [1]
-
- 4** Write the coupled equations $\frac{dx}{dt} = 2x - y$ and $\frac{dy}{dt} = 3x + 4y$ in the matrix form $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}$,
stating the matrix M . [2]
-
- 5** Given that $\frac{dx}{dt} = 5x + 2y$ and $\frac{dy}{dt} = x + 4y$, show that $\frac{d^2x}{dt^2} = 5\frac{dx}{dt} + 2(x + 4y)$. [2]
-
- 6** Solve $\frac{dx}{dt} = x + 4$, $\frac{dy}{dt} = 2y$, given that $x = 1$ and $y = 3$ when $t = 0$. [3]
-
- 7** $\frac{dx}{dt} = 2x + y$, $\frac{dy}{dt} = 2y + x$
Find the general solution for x and y in terms of t . [5]
-
- 8** $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = y + 4x$
Find the general solution for x and y in terms of t . [5]
-
- 9** $\frac{dx}{dt} = 4x + y$, $\frac{dy}{dt} = 2y + 3x$
Find the general solution for x and y in terms of t . [5]
-
- 10** $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -4x$
Find the general solution for x and y in terms of t . [5]
-
- 11** $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} = -x$
Find the general solution for x and y in terms of t . [5]
-
- 12** $\frac{dx}{dt} = x - 2y$, $\frac{dy}{dt} = -y + 5x$
Find the general solution for x and y in terms of t . [5]
-
- 13** $\frac{dx}{dt} = 5x + 3y$, $\frac{dy}{dt} = 4y + 2x$
Find the general solution for x and y in terms of t . [5]
-
- 14** $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = -y + x$
Given that $x = 2$ and $y = 1$ when $t = 0$, find x and y in terms of t . [7]
-
- 15** $\frac{dx}{dt} = x + 2y$, $\frac{dy}{dt} = y + 2x + 4e^t$
Given that $x = 1$ and $y = 0$ when $t = 0$, find x and y in terms of t . [7]
-
- 16** $\frac{dx}{dt} = 2x + y + 6$, $\frac{dy}{dt} = 2y + x$
Given that $x = 0$ and $y = 0$ when $t = 0$, find x and y in terms of t . [7]
-
- 17** $\frac{dx}{dt} = 4x - 2y$, $\frac{dy}{dt} = y + x$
Find the general solution for x and y in terms of t . [5]
-

- 18** A student solves $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = x + 3y$ and obtains $x = Ae^{4t} + Be^{2t}$, then writes $y = Ae^{4t} + Be^{2t}$ "as the equations are symmetric".
- (a) Show that the student's y does not satisfy the first equation. [2]
- (b) Find the correct y in terms of A and B . [2]
-
- 19** The variables x and y satisfy $\frac{dx}{dt} = -2x + 3y$ and $\frac{dy}{dt} = 4x - 3y$.
- (a) Show that $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 6x = 0$. [4]
- (b) Hence find x and y , given that $x = 3$ and $y = 1$ when $t = 0$. [5]
-
- 20** Two tanks A and B each hold 100 litres of liquid. Liquid flows from A to B at 5 litres per minute and from B to A at 5 litres per minute, so the volumes stay constant, and the contents of each tank are kept well mixed. At time t minutes the masses of salt in A and B are x kg and y kg respectively.
- (a) Show that $\frac{dx}{dt} = -0.05x + 0.05y$ and $\frac{dy}{dt} = 0.05x - 0.05y$. [2]
- (b) Find x and y given that $x = 20$ and $y = 0$ when $t = 0$. [5]
- (c) State the limiting amounts of salt in each tank, and the time for the difference $x - y$ to fall to 5 kg. [2]
-
- 21** Consider the coupled equations $\frac{dx}{dt} = 4x + 2y + 10$ and $\frac{dy}{dt} = x + 3y$.
- (a) Find the constant solution $x = p$, $y = q$ (the equilibrium point). [2]
- (b) By letting $x = -3 + u$ and $y = 1 + v$, or otherwise, find the general solution. [6]
-
- 22** $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = -2x$
Given that $x = 1$ and $y = 0$ when $t = 0$, find x and y in terms of t . [9]
-
- 23** $\frac{dx}{dt} = x + 4y$, $\frac{dy}{dt} = y - x$
Given that $x = 2$ and $y = 1$ when $t = 0$, find x and y in terms of t . [9]
-
- 24** $\frac{dx}{dt} = x + 2y + 3e^{2t}$, $\frac{dy}{dt} = y + 2x$
Given that $x = 0$ and $y = 1$ when $t = 0$, find x and y in terms of t . [9]
-
- 25** A drug is injected into the bloodstream. The mass x mg of the drug in the blood and the mass y mg of the drug in the body tissue at time t hours satisfy $\frac{dx}{dt} = -0.5x$ and $\frac{dy}{dt} = 0.5x - 0.2y$. Initially $x = 40$ and $y = 0$.
- (a) Find x in terms of t . [2]
- (b) Show that $\frac{dy}{dt} + 0.2y = 20e^{-0.5t}$, and hence find y in terms of t . [6]
- (c) Find the time at which the mass of the drug in the body tissue is greatest. [3]
-
- 26** The variables x and y satisfy $\frac{dx}{dt} = y + 2$ and $\frac{dy}{dt} = 2t - x$, with $x = 1$ and $y = 3$ when $t = 0$.
- (a) Show that $\frac{d^2x}{dt^2} + x = 2t$. [2]
- (b) Hence find x and y in terms of t . [5]
- (c) Show that $(x - 2t)^2 + y^2$ is constant, and state its value. [2]

- 27 Consider the system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -x - 2y$.
- (a) Show that the system leads to $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$, and find the general solution for x and y . [6]
- (b) Given that $x = 1$ and $y = 1$ when $t = 0$, find x and y , and show that x has a maximum at $t = \frac{1}{2}$. [5]
-
- 28 $\frac{dx}{dt} = 2x - y$, $\frac{dy}{dt} = -2y + 5x$
Given that $x = 1$ and $y = 2$ when $t = 0$, find x and y in terms of t . [10]
-
- 29 Small deviations x and y from an equilibrium in a predator-prey model satisfy $\frac{dx}{dt} = -y$ and $\frac{dy}{dt} = 4x$, with $x = 1$ and $y = 0$ when $t = 0$.
- (a) Show that $x = \cos 2t$ and find y . [6]
- (b) Show that $4x^2 + y^2$ is constant, and describe the path in the (x, y) plane. [4]
-
- 30 The variables x and y satisfy $\frac{dx}{dt} = ax + y$ and $\frac{dy}{dt} = -x + ay$, where a is a constant, with $x = 1$ and $y = 0$ when $t = 0$.
- (a) Show that $\frac{d^2x}{dt^2} - 2a\frac{dx}{dt} + (a^2 + 1)x = 0$. [4]
- (b) Find x and y in terms of a and t . [5]
- (c) Find the values of a for which x and y both tend to 0 as $t \rightarrow \infty$, and describe the motion in the (x, y) plane when $a = 0$. [3]

Differential Equation Modelling

- 1 A population P of bacteria at time t hours satisfies $\frac{dP}{dt} = 0.03P$, and $P = 500$ when $t = 0$. Find P in terms of t , and the population after 10 hours, to the nearest whole number. [3]
-
- 2 The temperature θ °C of a cooling object in a room at 20 °C satisfies $\frac{d\theta}{dt} = -k(\theta - 20)$, where k is a positive constant. Show that $\theta = 20 + Ae^{-kt}$, where A is a constant. [2]
-
- 3 A radioactive substance has mass N grams at time t years, where $\frac{dN}{dt} = -\lambda N$. Given that its half-life is 5 years, find the exact value of λ . [2]
-
- 4 A stone is dropped from rest and its velocity v m s⁻¹ downwards satisfies $\frac{dv}{dt} = 9.8$. Find v after 3 seconds. [2]
-
- 5 The displacement x metres of a particle satisfies $\frac{d^2x}{dt^2} = -16x$. State the angular frequency and the period of the motion. [2]
-
- 6 A cup of tea at 90 °C is left in a room at 20 °C. Its temperature θ °C after t minutes satisfies $\frac{d\theta}{dt} = -0.1(\theta - 20)$. Find the time, to 3 significant figures, for the tea to cool to 50 °C. [4]

- 7** A drug is infused into a patient so that its concentration C mg per litre in the blood satisfies $\frac{dC}{dt} = 5 - 0.2C$, with $C = 0$ when $t = 0$, t in hours. Find C in terms of t , and the value approached as $t \rightarrow \infty$. [4]
- 8** A skydiver falls with velocity v m s⁻¹, with $\frac{dv}{dt} = 9.8 - 0.5v$, and $v = 0$ when $t = 0$. Take $g = 9.8$ m s⁻². Find v in terms of t , and the terminal velocity. [4]
- 9** A mass on a spring has displacement x metres satisfying $\frac{d^2x}{dt^2} + 25x = 0$, with $x = 0.2$ and $\frac{dx}{dt} = 0$ when $t = 0$. Find x in terms of t , the maximum speed of the mass, and the period. [4]
- 10** A fish population P (in thousands) is harvested so that $\frac{dP}{dt} = 0.1P - 50$, measured in thousands per year, with $P = 400$ initially. Find P in terms of t , and how long it takes for the fish to die out, to 3 significant figures. [4]
- 11** A boat of mass 200 kg is moving at 8 m s⁻¹ through still water when its engine is switched off. The water exerts a resistive force of $40v$ N, where v m s⁻¹ is the speed of the boat t seconds later. Find v in terms of t , and find the total distance the boat travels before it comes to rest. [5]
- 12** A tank contains 200 litres of pure water. Brine containing 0.5 kg of salt per litre flows in at 4 litres per minute, and the well-mixed solution flows out at 4 litres per minute. Let x kg be the mass of salt after t minutes.
 (a) Show that $\frac{dx}{dt} = 2 - \frac{x}{50}$. [2]
 (b) Find x in terms of t , and the time to reach 50 kg of salt. [4]
- 13** A population P satisfies the logistic equation $\frac{dP}{dt} = 0.2P(1 - \frac{P}{1000})$, with $P = 100$ when $t = 0$.
 (a) Show that $P = \frac{1000}{1 + 9e^{-0.2t}}$. [4]
 (b) Find the time, to 3 significant figures, at which the population reaches 500. [2]
- 14** A constant force acts on a damped oscillator so that its displacement x metres satisfies $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 16$, with $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$. Find x in terms of t , and state its long-term value. [5]
- 15** A room is heated so that its temperature rises steadily as $15 + t$ °C, where t is time in hours. An object in the room has temperature θ °C satisfying $\frac{d\theta}{dt} = 0.5(15 + t - \theta)$, with $\theta = 5$ when $t = 0$. Find θ in terms of t . [5]
- 16** A tank initially holds 100 litres of pure water. Brine containing 0.5 kg of salt per litre enters at 6 litres per minute and the well-mixed liquid leaves at 4 litres per minute. Let x kg be the mass of salt after t minutes.
 (a) Show that $\frac{dx}{dt} + \frac{2x}{50 + t} = 3$. [2]
 (b) Solve the differential equation, and find the mass of salt after 10 minutes to 3 significant figures. [5]
- 17** A metal bar at 80 °C is placed in a room at 20 °C. After 5 minutes its temperature is 50 °C. Assuming Newton's law of cooling, $\frac{d\theta}{dt} = -k(\theta - 20)$, find the exact value of k and the temperature after 10 minutes. [5]

- 18** A loan of £20 000 accrues interest continuously at 4% per year and is repaid continuously at £1200 per year, so the amount owed B satisfies $\frac{dB}{dt} = 0.04B - 1200$. Find B in terms of t , and how long, to the nearest 0.1 year, it takes to repay the loan. [5]
-
- 19** A particle of mass 2 kg moves along the positive x -axis, away from the origin O , under the action of a single force of magnitude $\frac{40}{(x+1)^2}$ N directed away from O , where x metres is its distance from O . When $x = 1$ the speed of the particle is 1 m s^{-1} . Let $v \text{ m s}^{-1}$ be its speed at distance x .
- (a) Show that $v^2 = 21 - \frac{40}{x+1}$. [4]
 (b) Find the speed of the particle when $x = 7$. [2]
 (c) Find the limit of the speed as $x \rightarrow \infty$. [1]
-
- 20** A parachutist falls with velocity $v \text{ m s}^{-1}$ satisfying $\frac{dv}{dt} = 9.8 - 0.1v^2$, with $v = 0$ when $t = 0$. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Find the terminal velocity, in exact form. [2]
 (b) Show that $v = 7\sqrt{2} \tanh(0.7\sqrt{2}t)$. [5]
 (c) Find the time for the velocity to reach 90% of the terminal velocity, to 3 significant figures. [2]
-
- 21** A damped system is driven so that its displacement satisfies $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10 \cos 2t$, where $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.
- (a) Find the steady-state solution (the particular integral) $x = p \cos 2t + q \sin 2t$. [4]
 (b) Show that the transient part of the solution is $e^{-t}(-\frac{10}{17} \cos 2t - \frac{45}{17} \sin 2t)$. [4]
 (c) State the amplitude of the steady-state oscillation to 3 significant figures. [1]
-
- 22** In a series circuit the charge q coulombs on a capacitor at time t seconds satisfies $\frac{d^2q}{dt^2} + 6\frac{dq}{dt} + 8q = 12$, with $q = 0$ and the current $i = \frac{dq}{dt} = 0$ when $t = 0$.
- (a) Find q in terms of t . [4]
 (b) Show that the current is greatest when $t = \frac{1}{2} \ln 2$, and find the greatest current. [4]
-
- 23** A mass of 0.5 kg hangs on a spring of stiffness 16 N m^{-1} , in a medium which exerts a resistive force of $4v$ newtons, where $v \text{ m s}^{-1}$ is the velocity. The mass is released from rest at a displacement of 0.1 m from equilibrium. Let x metres be the displacement at time t seconds.
- (a) Show that $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 32x = 0$. [3]
 (b) Find x in terms of t . [4]
 (c) Find the ratio of the sizes of successive maximum displacements on the same side of equilibrium, to 3 significant figures. [2]
-
- 24** Water drains from a tank so that the depth h metres at time t minutes satisfies $\frac{dh}{dt} = -k\sqrt{h}$, where k is a positive constant. Initially $h = 4$, and the tank is empty after 10 minutes.
- (a) Find k and show that $h = (2 - \frac{t}{5})^2$. [5]
 (b) Find the time, to 3 significant figures, for the depth to halve. [2]
-

- 25** A rumour spreads through a population of 1000 people. The number N who have heard it at time t days satisfies $\frac{dN}{dt} = kN(1000 - N)$, where k is a constant. Initially $N = 100$, and after 2 days $N = 400$.
- (a) Show that $N = \frac{1000}{1 + 9e^{-1000kt}}$. [4]
- (b) Find the exact value of $1000k$, and the number who have heard the rumour after 4 days. [3]
-
- 26** A hot drink cools according to Newton's law of cooling $\frac{d\theta}{dt} = -k(\theta - \theta_0)$, where θ °C is its temperature after t minutes, θ_0 °C is the constant temperature of the room and k is a positive constant. The temperature of the drink is 90 °C when $t = 0$, 60 °C when $t = 5$ and 45 °C when $t = 10$.
- (a) Show that $\theta = \theta_0 + (90 - \theta_0)e^{-kt}$. [3]
- (b) Find the value of θ_0 . [4]
- (c) Find the exact value of k . [2]
-
- 27** A frictionless oscillator is driven at its natural frequency so that $\frac{d^2x}{dt^2} + 4x = 8 \cos 2t$, with $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.
- (a) Find x in terms of t . [6]
- (b) Show that the stationary points satisfy $\tan 2t = -2t$, and find the greatest value of x for $0 \leq t \leq \pi$, to 3 significant figures. [4]
- (c) Explain what happens to the size of the oscillations for large t . [2]
-
- 28** In a chemical reaction the mass x grams of a product at time t minutes satisfies $\frac{dx}{dt} = 0.1(4 - x)(2 - x)$, with $x = 0$ when $t = 0$.
- (a) Show that $x = \frac{4(e^{0.2t} - 1)}{2e^{0.2t} - 1}$. [6]
- (b) Find the limit of x as $t \rightarrow \infty$, and the time to reach 1.5 grams, to 3 significant figures. [3]
-
- 29** The temperature θ °C of a greenhouse satisfies $\frac{d\theta}{dt} + 0.5\theta = 5 + 2 \sin t$, where t is time in hours, and $\theta = 8$ when $t = 0$.
- (a) Find θ in terms of t . [7]
- (b) For large t the temperature oscillates about a mean value. State this mean value, and find the greatest value reached, to 3 significant figures. [3]
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- 30** A particle of unit mass is projected vertically upwards with speed 20 m s^{-1} in a medium which exerts a resistance of $0.1v^2$ newtons. Take $g = 9.8 \text{ m s}^{-2}$. Let $v \text{ m s}^{-1}$ be the upward speed when the height is x metres.
- (a) Show that $v \frac{dv}{dx} = -(9.8 + 0.1v^2)$. [3]
- (b) Find the greatest height reached, to 3 significant figures. [4]
- (c) Without calculation, state whether the greatest height would be greater or less if there were no resistance, giving a reason, and calculate the height in that case. [3]