

Continuous Random Variables

1 (3 marks)

(a) Time can take any value in an interval

B1

Answer: Continuous

1 mark

(b) Takes only the whole-number values 0, 1, 2, ...

B1

Answer: Discrete

1 mark

(c) Mass can take any value in an interval

B1

Answer: Continuous

1 mark

2 (2 marks)

(a) For a continuous random variable the probability of any single value is zero

B1

Answer: 0

1 mark

(b) $1 - 0.4 = 0.6$

B1

Answer: 0.6

1 mark

3 (2 marks)

Total area must equal 1: $4k = 1$

M1

$$k = \frac{1}{4}$$

A1

Answer: $k = \frac{1}{4}$

2 marks

4 (3 marks)

$f(x) \geq 0$ for all x in the range

B1

$$\int_0^4 \frac{x}{8} dx = \left[\frac{x^2}{16} \right]_0^4$$

M1

$= \frac{16}{16} = 1$, so f is a valid probability density function

A1

Answer: Valid (area = 1)

3 marks

5 (2 marks)

$$P(X < 1) = \int_0^1 \frac{x}{2} dx = \left[\frac{x^2}{4} \right]_0^1$$

M1

$$= \frac{1}{4}$$

A1

Answer: $\frac{1}{4}$

2 marks

6 (2 marks)

Rectangle of width 2 and height 0.25

M1

$$P = 2 \times 0.25 = 0.5$$

A1

Answer: 0.5

2 marks

7 (4 marks)

$$(a) \int_0^3 kx^2 dx = \left[\frac{kx^3}{3} \right]_0^3 = 9k = 1$$

M1

$$k = \frac{1}{9}$$

A1

Answer: $k = \frac{1}{9}$

2 marks

$$(b) P(X > 2) = \int_2^3 \frac{x^2}{9} dx = \left[\frac{x^3}{27} \right]_2^3 = 1 - \frac{8}{27}$$

M1

$$= \frac{19}{27}$$

A1

Answer: $\frac{19}{27}$

2 marks

8 (4 marks)

(a) Area of triangle = $\frac{1}{2} \times 4 \times h = 1$

M1

$$h = \frac{1}{2}$$

A1

Answer: $h = \frac{1}{2}$

2 marks

(b) For $0 \leq x \leq 2$, $f(x) = \frac{x}{4}$, so $f(1) = \frac{1}{4}$

M1

$$P(X < 1) = \frac{1}{2} \times 1 \times \frac{1}{4} = \frac{1}{8}$$

A1

Answer: $\frac{1}{8}$

2 marks

9 (4 marks)

$$P(1 < X < 3) = \int_1^3 \frac{3}{32}(4x - x^2) dx$$

M1

$$= \frac{3}{32} \left[2x^2 - \frac{x^3}{3} \right]_1^3$$

A1

$$= \frac{3}{32} \left((18 - 9) - \left(2 - \frac{1}{3} \right) \right)$$

M1

$$= \frac{3}{32} \times \frac{22}{3} = \frac{11}{16}$$

A1

Answer: $\frac{11}{16}$

4 marks

10 (4 marks)

(a) $f(0) = -\frac{1}{3} < 0$, so f takes negative values

M1

A probability density function can never be negative, so it is not valid

A1

Answer: Not valid (negative values)

2 marks

(b) $\int_0^3 \frac{x^2}{12} dx = \left[\frac{x^3}{36} \right]_0^3 = \frac{27}{36} = \frac{3}{4}$

M1

The total area is not 1, so it is not valid

A1

Answer: Not valid (area is $\frac{3}{4}$)

2 marks

11 (3 marks)

$$P(T > 5) = \int_5^{20} \frac{(20 - t)}{200} dt$$

M1

$$= \left[\frac{t}{10} - \frac{t^2}{400} \right]_5^{20} = (2 - 1) - \left(\frac{1}{2} - \frac{1}{16} \right)$$

M1

$$= 1 - \frac{7}{16} = \frac{9}{16}$$

A1

Answer: $\frac{9}{16}$

3 marks

12 (3 marks)

$$\int_0^a \frac{3x^2}{8} dx = \frac{a^3}{8}$$

M1

$$\frac{a^3}{8} = \frac{1}{4}, \text{ so } a^3 = 2$$

M1

$$a = \sqrt[3]{2} \approx 1.26$$

A1

Answer: $a = \sqrt[3]{2} \approx 1.26$

3 marks

13 (4 marks)

(a) $\int_0^\infty k e^{-2x} dx = \left[-\frac{k}{2} e^{-2x} \right]_0^\infty = \frac{k}{2}$

M1

$$\frac{k}{2} = 1, \text{ so } k = 2$$

A1

Answer: $k = 2$

2 marks

$$(b) P(X > 1) = [-e^{-2x}]_1^{\infty}$$

M1

$$= e^{-2}$$

A1

Answer: e^{-2}

2 marks

14 (6 marks)

$$(a) f(x) \geq 0 \text{ on } [-1, 1] \text{ since } x^2 \leq 1$$

B1

$$\int_{-1}^1 \frac{3}{4}(1 - x^2) dx = \frac{3}{4}[x - \frac{x^3}{3}]_{-1}^1$$

M1

$$= \frac{3}{4} \times \frac{4}{3} = 1$$

A1

Answer: Valid

3 marks

$$(b) \int_{1/2}^1 \frac{3}{4}(1 - x^2) dx = \frac{3}{4}[x - \frac{x^3}{3}]_{1/2}^1$$

M1

$$= \frac{3}{4}(\frac{2}{3} - (\frac{1}{2} - \frac{1}{24}))$$

M1

$$= \frac{3}{4} \times \frac{5}{24} = \frac{5}{32}$$

A1

Answer: $\frac{5}{32}$

3 marks

15 (5 marks)

$$\int_1^2 kx^2 dx = k[\frac{x^3}{3}]_1^2 = \frac{7k}{3}$$

M1

$$\frac{7k}{3} = 1, \text{ so } k = \frac{3}{7}$$

A1

$$P(X < 1.5) = \int_1^{1.5} \frac{3x^2}{7} dx$$

M1

$$= [\frac{x^3}{7}]_1^{1.5} = \frac{1}{7}(\frac{27}{8} - 1)$$

M1

$$= \frac{19}{56}$$

A1

Answer: $k = \frac{3}{7}$, $P = \frac{19}{56}$

5 marks

16 (5 marks)

$$\int_0^a \frac{x}{18} dx = [\frac{x^2}{36}]_0^a = \frac{a^2}{36}$$

M1

$$\frac{a^2}{36} = 1, \text{ so } a = 6 \text{ (} a > 0 \text{)}$$

A1

$$P(X > 4) = \int_4^6 \frac{x}{18} dx$$

M1

$$= [\frac{x^2}{36}]_4^6 = 1 - \frac{16}{36}$$

M1

$$= \frac{5}{9}$$

A1

Answer: $a = 6$, $P(X > 4) = \frac{5}{9}$

5 marks

17 (4 marks)

$$P(X > c) = \int_c^2 \frac{3x^2}{8} dx = [\frac{x^3}{8}]_c^2 = 1 - \frac{c^3}{8}$$

M1

$$P(X > 1.5) = 1 - \frac{27}{64} = \frac{37}{64}, P(X > 1) = 1 - \frac{1}{8} = \frac{7}{8}$$

A1

$$P(X > 1.5 | X > 1) = \frac{P(X > 1.5)}{P(X > 1)} = \frac{37}{64} \div \frac{7}{8}$$

M1

$$= \frac{37}{56}$$

A1

Answer: $\frac{37}{56}$

4 marks

18 (6 marks)

$$(a) \text{ Let } u = 1 - 9l: \int_0^2 ku(2 - u) du = k[u^2 - \frac{u^3}{3}]_0^2 \quad \text{M1}$$

$$= k(4 - \frac{8}{3}) = \frac{4k}{3} \quad \text{A1}$$

$$\frac{4k}{3} = 1, \text{ so } k = \frac{3}{4} \quad \text{A1}$$

$$\text{Answer: } k = \frac{3}{4} \quad \text{3 marks}$$

$$(b) P(9.5 < L < 10.5) = \int_{9.5}^{10.5} \frac{3}{4}(1 - 9l)(11 - l) dl, \text{ and } (1 - 9l)(11 - l) = 1 - (l - 10)^2 \quad \text{M1}$$

$$\text{Let } w = l - 10: \frac{3}{4}[w - \frac{w^3}{3}]_{-0.5}^{0.5} \quad \text{M1}$$

$$= \frac{3}{4}(1 - \frac{1}{12}) = \frac{11}{16} \quad \text{A1}$$

$$\text{Answer: } \frac{11}{16} \quad \text{3 marks}$$

19 (5 marks)

The student has set the largest value of f equal to 1; it is the total area under f that must equal 1, not a value of f E1

$$\int_0^3 kx dx = [\frac{kx^2}{2}]_0^3 = \frac{9k}{2} = 1 \quad \text{M1}$$

$$k = \frac{2}{9} \quad \text{A1}$$

$$P(X \geq 2) = \int_2^3 \frac{2x}{9} dx = [\frac{x^2}{9}]_2^3 \quad \text{M1}$$

$$= 1 - \frac{4}{9} = \frac{5}{9} \quad \text{A1}$$

$$\text{Answer: } k = \frac{2}{9}, P(X \geq 2) = \frac{5}{9} \quad \text{5 marks}$$

20 (5 marks)

$$(a) \int_0^{1/2} kx dx = [\frac{kx^2}{2}]_0^{1/2} = \frac{k}{8} \quad \text{M1}$$

$$k = 8 \quad \text{A1}$$

$$\text{Answer: } k = 8 \quad \text{2 marks}$$

$$(b) f(\frac{1}{2}) = 8 \times \frac{1}{2} = 4 \quad \text{B1}$$

$f(x)$ is not a probability; it is a density, and probabilities are areas under the graph E1

Here the base of the region has width at most $\frac{1}{2}$, so every probability is at most 1 (total area = 1) E1

$$\text{Answer: } f(\frac{1}{2}) = 4; \text{ a density value can exceed 1} \quad \text{3 marks}$$

21 (6 marks)

$$(a) \int_0^5 k(5x - x^2) dx = k[\frac{5x^2}{2} - \frac{x^3}{3}]_0^5 \quad \text{M1}$$

$$= k(\frac{125}{2} - \frac{125}{3}) = \frac{125k}{6} \quad \text{A1}$$

$$\frac{125k}{6} = 1, \text{ so } k = \frac{6}{125} \quad \text{A1}$$

$$\text{Answer: } k = \frac{6}{125} \quad \text{3 marks}$$

(b) The graph of f is a symmetrical quadratic with maximum at $x = 2.5$, so the mode is 2.5 B1

$$\text{Answer: } 2.5 \quad \text{1 mark}$$

$$(c) \frac{6}{125}[\frac{5x^2}{2} - \frac{x^3}{3}]_1^2 = \frac{6}{125}((10 - \frac{8}{3}) - (\frac{5}{2} - \frac{1}{3})) \quad \text{M1}$$

$$= \frac{6}{125} \times \frac{31}{6} = \frac{31}{125} \quad \text{A1}$$

Answer: $\frac{31}{125}$

2 marks

22 (6 marks)

(a) $\int_0^2 kx \, dx = 2k$

B1

$\int_2^4 2k \, dx = 4k$, so $2k + 4k = 1$

M1

$k = \frac{1}{6}$

A1

Answer: $k = \frac{1}{6}$

3 marks

(b) $\int_1^2 \frac{x}{6} \, dx = \left[\frac{x^2}{12} \right]_1^2 = \frac{1}{4}$

M1

$\int_2^3 \frac{1}{3} \, dx = \frac{1}{3}$

M1

$P = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$

A1

Answer: $\frac{7}{12}$

3 marks

23 (7 marks)

(a) $\int_1^\infty \frac{k}{x^2} \, dx = \left[-\frac{k}{x} \right]_1^\infty = k$

M1

$k = 1$

A1

Answer: $k = 1$

2 marks

(b) $\left[-\frac{1}{x} \right]_2^5 = \frac{1}{2} - \frac{1}{5}$

M1

$= \frac{3}{10}$

A1

Answer: $\frac{3}{10}$

2 marks

(c) $P(X > m) = \left[-\frac{1}{x} \right]_m^\infty = \frac{1}{m}$

M1

$\frac{1}{m} = 0.05$

M1

$m = 20$

A1

Answer: $m = 20$

3 marks

24 (8 marks)

(a) Continuity at $x = 2$: $2a = b$

B1

$\int_0^2 ax \, dx = 2a$

M1

$\int_2^4 b \, dx = 2b$

M1

$2a + 2b = 1$; substituting $b = 2a$ gives $6a = 1$

M1

$a = \frac{1}{6}$, $b = \frac{1}{3}$

A1

Answer: $a = \frac{1}{6}$, $b = \frac{1}{3}$

5 marks

(b) $\int_{1.5}^2 \frac{x}{6} \, dx = \left[\frac{x^2}{12} \right]_{1.5}^2 = \frac{1}{3} - \frac{3}{16} = \frac{7}{48}$

M1

$\int_2^4 \frac{1}{3} \, dx = \frac{2}{3}$

M1

$P = \frac{7}{48} + \frac{2}{3} = \frac{13}{16}$

A1

Answer: $\frac{13}{16}$

3 marks

25 (7 marks)

$$(a) P(X > 1) = \int_1^2 \frac{3x^2}{8} dx = \left[\frac{x^3}{8} \right]_1^2 = \frac{7}{8} \quad \text{M1 A1}$$

$$P(\text{exactly one}) = 2 \times \frac{7}{8} \times \frac{1}{8} \quad \text{M1}$$

$$= \frac{7}{32} \quad \text{A1}$$

$$\text{Answer: } \frac{7}{32} \quad \text{4 marks}$$

$$(b) Y \sim B(10, \frac{7}{8}): P(Y \geq 9) = P(Y = 9) + P(Y = 10) \quad \text{M1}$$

$$= 10 \times \left(\frac{7}{8}\right)^9 \times \frac{1}{8} + \left(\frac{7}{8}\right)^{10} \quad \text{M1}$$

$$= 0.639 \quad \text{A1}$$

$$\text{Answer: } 0.639 \quad \text{3 marks}$$

26 (6 marks)

$$(a) \int_0^\pi k \sin x \, dx = [-k \cos x]_0^\pi \quad \text{M1}$$

$$= k(1 + 1) = 2k \quad \text{A1}$$

$$2k = 1, \text{ so } k = \frac{1}{2} \quad \text{A1}$$

$$\text{Answer: } k = \frac{1}{2} \quad \text{3 marks}$$

$$(b) P = \int_{\pi/3}^\pi \frac{1}{2} \sin x \, dx = \left[-\frac{1}{2} \cos x \right]_{\pi/3}^\pi \quad \text{M1}$$

$$= \frac{1}{2} + \frac{1}{2} \cos \frac{\pi}{3} = \frac{1}{2} + \frac{1}{4} \quad \text{M1}$$

$$= \frac{3}{4} \quad \text{A1}$$

$$\text{Answer: } \frac{3}{4} \quad \text{3 marks}$$

27 (9 marks)

$$(a) \int_0^a kx^2 \, dx = \frac{ka^3}{3} = 1 \quad \text{M1 A1}$$

$$P(X < 1) = \int_0^1 kx^2 \, dx = \frac{k}{3} = \frac{1}{8} \quad \text{M1}$$

$$k = \frac{3}{8} \quad \text{A1}$$

$$\frac{3}{8} \times \frac{a^3}{3} = 1, \text{ so } a^3 = 8 \quad \text{M1}$$

$$a = 2 \quad \text{A1}$$

$$\text{Answer: } k = \frac{3}{8}, a = 2 \quad \text{6 marks}$$

$$(b) P(X > 1.5) = \left[\frac{x^3}{8} \right]_{1.5}^2 \quad \text{M1}$$

$$= 1 - \frac{27}{64} \quad \text{M1}$$

$$= \frac{37}{64} \quad \text{A1}$$

$$\text{Answer: } \frac{37}{64} \quad \text{3 marks}$$

28 (10 marks)

(a) Graph: triangle with peak at (0, 2k), zero at $x = \pm 2$, symmetric about the y-axis

B1

$$\text{Area} = \frac{1}{2} \times 4 \times 2k = 4k$$

M1 A1

$$4k = 1, \text{ so } k = \frac{1}{4}$$

A1

Answer: Triangle, peak $2k = \frac{1}{2}$ at $x = 0$; $k = \frac{1}{4}$

4 marks

(b) For $x > 1$, $f(x) = \frac{(2-x)}{4}$: area of triangle between $x = 1$ and 2 is $\frac{1}{2} \times 1 \times \frac{1}{4} = \frac{1}{8}$

M1

$$\text{By symmetry } P(|X| > 1) = 2 \times \frac{1}{8} = \frac{1}{4}$$

A1

Answer: $\frac{1}{4}$

2 marks

$$(c) P(X < 1) = 1 - \frac{1}{8} = \frac{7}{8}$$

M1 A1

$$P(-1 < X < 1) = 1 - \frac{1}{4} = \frac{3}{4}$$

M1

$$P = \frac{3}{4} \div \frac{7}{8} = \frac{6}{7}$$

A1

Answer: $\frac{6}{7}$

4 marks

29 (9 marks)

$$(a) \int_0^{\infty} kxe^{-x^2} dx = \left[-\frac{k}{2}e^{-x^2}\right]_0^{\infty}$$

M1 A1

$$= \frac{k}{2} = 1, \text{ so } k = 2$$

A1

Answer: $k = 2$

3 marks

$$(b) P(X > 1) = [-e^{-x^2}]_1^{\infty}$$

M1

$$= e^{-1}$$

A1

Answer: e^{-1}

2 marks

$$(c) P(X < a) = [-e^{-x^2}]_0^a = 1 - e^{-a^2}$$

M1 A1

$$1 - e^{-a^2} = \frac{1}{2}, \text{ so } e^{-a^2} = \frac{1}{2}$$

M1

$$a^2 = \ln 2, a = \sqrt{\ln 2}$$

A1

Answer: $a = \sqrt{\ln 2}$

4 marks

30 (9 marks)

$$(a) \int_0^2 (ax + b) dx = 2a + 2b = 1$$

M1 A1

$$\int_0^1 (ax + b) dx = \frac{a}{2} + b = 0.3$$

M1 A1

From the second equation $b = 0.3 - \frac{a}{2}$; substituting into the first, $2a + 0.6 - a = 1$, so $a = 0.4$ and

A1

$$b = 0.1$$

Answer: $a = 0.4$, $b = 0.1$

5 marks

(b) f is a straight line with positive gradient and $f(0) = 0.1 > 0$, so $f(x) > 0$ on $[0, 2]$

B1

$$P(X > 1.5) = \int_{1.5}^2 (0.4x + 0.1) dx = [0.2x^2 + 0.1x]_{1.5}^2$$

M1

$$= (0.8 + 0.2) - (0.45 + 0.15)$$

M1

$$= 0.4$$

A1

Answer: 0.4

4 marks

Probability Density Functions

1 (2 marks)

$$P = \int_1^3 \frac{x}{8} dx = \left[\frac{x^2}{16} \right]_1^3 \quad \text{M1}$$

$$= \frac{9}{16} - \frac{1}{16} = \frac{1}{2} \quad \text{A1}$$

Answer: $\frac{1}{2}$ 2 marks

2 (2 marks)

$$\int_0^3 k(3 - x) dx = k \left[3x - \frac{x^2}{2} \right]_0^3 = \frac{9k}{2} \quad \text{M1}$$

$$\frac{9k}{2} = 1, \text{ so } k = \frac{2}{9} \quad \text{A1}$$

Answer: $k = \frac{2}{9}$ 2 marks

3 (2 marks)

The mode is where $f(x)$ is a maximum; $f'(x) = \frac{3}{32}(4 - 2x) = 0$ M1

$$x = 2 \quad \text{A1}$$

Answer: 2 2 marks

4 (2 marks)

Straight line through the origin from (0, 0) to (2, 1) B1

$f(x) = 0$ elsewhere, so the graph runs along the x-axis outside $0 \leq x \leq 2$ (jump down to 0 at $x = 2$) B1

Answer: Line from (0, 0) to (2, 1); zero elsewhere 2 marks

5 (2 marks)

$$P(X > 2) = [-e^{-x/2}]_2^\infty \quad \text{M1}$$

$$= e^{-1} \approx 0.368 \quad \text{A1}$$

Answer: $e^{-1} \approx 0.368$ 2 marks

6 (2 marks)

Area of rectangle from 0.5 to 1: $0.5 \times 0.4 = 0.2$; area from 1 to 2: $1 \times 0.6 = 0.6$ M1

$$P = 0.2 + 0.6 = 0.8 \quad \text{A1}$$

Answer: 0.8 2 marks

7 (5 marks)

$$\text{(a) } \int_0^2 k(4 - x^2) dx = k \left[4x - \frac{x^3}{3} \right]_0^2 \quad \text{M1}$$

$$= k \left(8 - \frac{8}{3} \right) = \frac{16k}{3} \quad \text{A1}$$

$$k = \frac{3}{16} \quad \text{A1}$$

Answer: $k = \frac{3}{16}$ 3 marks

$$\text{(b) } \frac{3}{16} \left[4x - \frac{x^3}{3} \right]_0^1 = \frac{3}{16} \times \frac{11}{3} \quad \text{M1}$$

$$= \frac{11}{16} \quad \text{A1}$$

Answer: $\frac{11}{16}$ 2 marks

8 (4 marks)

$$\int_0^2 kx^3 \, dx = \left[\frac{kx^4}{4} \right]_0^2 = 4k = 1$$

M1

$$k = \frac{1}{4}$$

A1

$$P(X > 1) = \left[\frac{x^4}{16} \right]_1^2$$

M1

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

A1

$$\text{Answer: } k = \frac{1}{4}, P(X > 1) = \frac{15}{16}$$

4 marks

9 (3 marks)

$$\text{Let the median be } m: \int_0^m \frac{x}{8} \, dx = \frac{m^2}{16}$$

M1

$$\frac{m^2}{16} = \frac{1}{2}, \text{ so } m^2 = 8$$

M1

$$m = 2\sqrt{2} \approx 2.83$$

A1

$$\text{Answer: } 2\sqrt{2} \approx 2.83$$

3 marks

10 (3 marks)

$$P(X < q) = [-e^{-x/2}]_0^q = 1 - e^{-q/2}$$

M1

$$1 - e^{-q/2} = \frac{1}{4}, \text{ so } e^{-q/2} = \frac{3}{4}$$

M1

$$q = -2 \ln \frac{3}{4} = 2 \ln \frac{4}{3}$$

A1

$$\text{Answer: } 2 \ln \frac{4}{3} \approx 0.575$$

3 marks

11 (5 marks)

(a) Triangle with vertices (0, 0), (1, 1) and (2, 0)

B1

$f(x) = 0$ outside $0 \leq x \leq 2$

B1

Answer: Triangle with peak (1, 1)

2 marks

(b) The graph is symmetrical about $x = 1$, so half the area lies either side: median = 1

B1

Answer: 1

1 mark

(c) Triangle with base 0.5 and height 0.5: area = $\frac{1}{2} \times 0.5 \times 0.5$

M1

$$= \frac{1}{8}$$

A1

$$\text{Answer: } \frac{1}{8}$$

2 marks

12 (4 marks)

$$\int_1^e \frac{k}{x} \, dx = [k \ln x]_1^e$$

M1

$$= k(1 - 0) = k, \text{ so } k = 1$$

A1

$$P(X < 2) = [\ln x]_1^2$$

M1

$$= \ln 2 \approx 0.693$$

A1

$$\text{Answer: } k = 1, P(X < 2) = \ln 2$$

4 marks

13 (3 marks)

$$P(X < c) = 1 - e^{-c/2} = 0.9$$

M1

$$e^{-c/2} = 0.1, \text{ so } c = -2 \ln 0.1 = 2 \ln 10$$

M1

$$c = 4.61$$

A1

$$\text{Answer: } 4.61$$

3 marks

14 (6 marks)

$$(a) \int_0^{10} k(10x^2 - x^3) dx = k\left[\frac{10x^3}{3} - \frac{x^4}{4}\right]_0^{10} \quad \text{M1}$$

$$= k\left(\frac{10000}{3} - 2500\right) = \frac{2500k}{3} \quad \text{A1}$$

$$\frac{2500k}{3} = 1, \text{ so } k = \frac{3}{2500} \quad \text{A1}$$

$$\text{Answer: } k = \frac{3}{2500} \quad \text{3 marks}$$

$$(b) f'(x) = k(20x - 3x^2) = 0 \quad \text{M1}$$

$$x(20 - 3x) = 0, \text{ so } x = 0 \text{ or } x = \frac{20}{3} \quad \text{A1}$$

$$f(0) = 0, \text{ so the maximum is at } x = \frac{20}{3} \approx 6.67 \quad \text{A1}$$

$$\text{Answer: } \frac{20}{3} \approx 6.67 \quad \text{3 marks}$$

15 (6 marks)

$$(a) \int_0^4 \frac{(x+1)}{12} dx = \frac{1}{12}\left[\frac{x^2}{2} + x\right]_0^4 \quad \text{M1}$$

$$= \frac{1}{12}(8 + 4) = 1, \text{ and } f(x) > 0 \text{ on the range} \quad \text{A1}$$

$$\text{Answer: Valid} \quad \text{2 marks}$$

$$(b) \frac{1}{12}\left[\frac{x^2}{2} + x\right]_0^m = \frac{1}{2} \quad \text{M1}$$

$$\frac{m^2}{2} + m = 6, \text{ so } m^2 + 2m - 12 = 0 \quad \text{A1}$$

$$m = \frac{-2 + \sqrt{4 + 48}}{2} = -1 + \sqrt{13} \quad \text{M1}$$

$$m = 2.61 \quad \text{A1}$$

$$\text{Answer: 2.61} \quad \text{4 marks}$$

16 (7 marks)

$$(a) f(x) = 12x^2 - 12x^3, \text{ so } f'(x) = 24x - 36x^2 = 12x(2 - 3x) \quad \text{M1}$$

$$f'(x) = 0 \text{ gives } x = 0 \text{ or } x = \frac{2}{3} \quad \text{A1}$$

$$f(0) = 0, \text{ so the mode is } \frac{2}{3} \quad \text{A1}$$

$$\text{Answer: } \frac{2}{3} \quad \text{3 marks}$$

$$(b) P(X < m) = [4x^3 - 3x^4]_0^m = 4m^3 - 3m^4 = \frac{1}{2} \quad \text{M1}$$

$$\text{So } 3m^4 - 4m^3 + \frac{1}{2} = 0 \quad \text{A1}$$

$$g(m) = 4m^3 - 3m^4: g(0.6) = 0.8640 - 0.3888 = 0.4752 < 0.5 \quad \text{M1}$$

$$g(0.65) = 1.0985 - 0.5355 = 0.5630 > 0.5, \text{ so } 0.6 < m < 0.65 \quad \text{A1}$$

$$\text{Answer: } 0.6 < m < 0.65 \text{ (} m \approx 0.614 \text{)} \quad \text{4 marks}$$

17 (4 marks)

$$P(1 < X < 1.5) = \left[\frac{x^4}{16}\right]_1^{1.5} = \frac{1}{16}\left(\frac{81}{16} - 1\right) = \frac{65}{256} \quad \text{M1 A1}$$

$$P(X > 1) = 1 - \frac{1}{16} = \frac{15}{16} \quad \text{B1}$$

$$P = \frac{65}{256} \div \frac{15}{16} = \frac{13}{48} \quad \text{A1}$$

$$\text{Answer: } \frac{13}{48} \quad \text{4 marks}$$

18 (6 marks)

$$(a) \int_0^1 kx^2 dx = \frac{k}{3} \quad \text{M1}$$

$$\int_1^2 k(2-x) dx = \frac{k}{2}, \text{ so } \frac{k}{3} + \frac{k}{2} = \frac{5k}{6} = 1 \quad \text{M1}$$

$$k = \frac{6}{5} \quad \text{A1}$$

$$\text{Answer: } k = \frac{6}{5} \quad \text{3 marks}$$

$$(b) \int_{1/2}^1 \frac{6x^2}{5} dx = \frac{2}{5} \left(1 - \frac{1}{8}\right) = \frac{7}{20} \quad \text{M1}$$

$$\int_1^{3/2} \frac{6}{5}(2-x) dx = \frac{6}{5} \times \frac{3}{8} = \frac{9}{20} \quad \text{M1}$$

$$P = \frac{7}{20} + \frac{9}{20} = \frac{4}{5} \quad \text{A1}$$

$$\text{Answer: } \frac{4}{5} \quad \text{3 marks}$$

19 (5 marks)

$$\int_1^b \frac{k}{x^2} dx = k \left(1 - \frac{1}{b}\right) = 1 \quad \text{M1 A1}$$

$$P(X > 2) = k \left(\frac{1}{2} - \frac{1}{b}\right) = \frac{1}{4} \quad \text{M1}$$

$$\text{Dividing: } \frac{1}{2} - \frac{1}{b} = \frac{1}{4} \left(1 - \frac{1}{b}\right), \text{ so } \frac{1}{4} = \frac{3}{4b} \quad \text{M1}$$

$$b = 3, \text{ then } k = \frac{3}{2} \quad \text{A1}$$

$$\text{Answer: } k = \frac{3}{2}, b = 3 \quad \text{5 marks}$$

20 (7 marks)

$$(a) \int_1^\infty \frac{3}{x^4} dx = \left[-\frac{1}{x^3}\right]_1^\infty \quad \text{M1}$$

$$= 0 - (-1) = 1, \text{ and } f(x) > 0 \text{ for } x \geq 1 \quad \text{A1}$$

$$\text{Answer: Valid} \quad \text{2 marks}$$

$$(b) P(X > 2) = \left[-\frac{1}{x^3}\right]_2^\infty = \frac{1}{8} \quad \text{M1 A1}$$

$$\text{Answer: } \frac{1}{8} \quad \text{2 marks}$$

$$(c) P(\text{none exceeds } 2) = \left(\frac{7}{8}\right)^3 \quad \text{M1 A1}$$

$$P(\text{at least one}) = 1 - \frac{343}{512} = \frac{169}{512} \quad \text{A1}$$

$$\text{Answer: } \frac{169}{512} \quad \text{3 marks}$$

21 (10 marks)

$$(a) \text{ Triangle with base 3 and height } a: \frac{1}{2} \times 3 \times a = 1 \quad \text{M1}$$

$$a = \frac{2}{3} \quad \text{A1}$$

$$\text{Answer: } a = \frac{2}{3} \quad \text{2 marks}$$

$$(b) \text{ Gradient} = -\frac{a}{2} = -\frac{1}{3}, \text{ so } f(x) = \frac{2}{3} - \frac{1}{3}(x-1) = \frac{1}{3}(3-x) \quad \text{M1 A1}$$

$$P(X > 2) = \frac{1}{2} \times 1 \times f(2) = \frac{1}{2} \times 1 \times \frac{1}{3} \quad \text{M1}$$

$$= \frac{1}{6} \quad \text{A1}$$

$$\text{Answer: } f(x) = \frac{(3-x)}{3}, P = \frac{1}{6} \quad \text{4 marks}$$

$$(c) P(X < 1) = \frac{1}{2} \times 1 \times \frac{2}{3} = \frac{1}{3} < \frac{1}{2}, \text{ so the median exceeds } 1$$

M1 A1

$$P(X > m) = \frac{1}{2}(3 - m) \times \frac{(3 - m)}{3} = \frac{(3 - m)^2}{6} = \frac{1}{2}$$

M1

$$(3 - m)^2 = 3, m = 3 - \sqrt{3} \approx 1.27$$

A1

$$\text{Answer: } m = 3 - \sqrt{3} \approx 1.27$$

4 marks

22 (10 marks)

$$(a) \int_0^{\infty} x e^{-x/2} dx = [-2x e^{-x/2}]_0^{\infty} + \int_0^{\infty} 2 e^{-x/2} dx$$

M1 A1

$$= 0 + [-4e^{-x/2}]_0^{\infty} = 4$$

A1

$$4k = 1, \text{ so } k = \frac{1}{4}$$

A1

$$\text{Answer: } k = \frac{1}{4}$$

4 marks

$$(b) f'(x) = k(1 - \frac{x}{2})e^{-x/2}$$

M1 A1

$f'(x) = 0$ at $x = 2$, which is the maximum (f increases then decreases), so the mode is 2

A1

Answer: 2

3 marks

$$(c) \int_4^{\infty} \frac{1}{4} x e^{-x/2} dx = \frac{1}{4} [-2x e^{-x/2} - 4e^{-x/2}]_4^{\infty}$$

M1

$$= \frac{1}{4} (8e^{-2} + 4e^{-2})$$

M1

$$= 3e^{-2}$$

A1

$$\text{Answer: } 3e^{-2}$$

3 marks

23 (7 marks)

$$(a) \int_0^2 \frac{x}{10} dx = [\frac{x^2}{20}]_0^2 = \frac{1}{5}$$

M1

$$\frac{1}{5} + \frac{1}{5}(b - 2) = 1$$

M1

$$b = 6$$

A1

$$\text{Answer: } b = 6$$

3 marks

$$(b) P(X < 2) = \frac{1}{5} < \frac{1}{2}, \text{ so the median } m \text{ lies in the second piece}$$

M1 A1

$$\frac{1}{5} + \frac{1}{5}(m - 2) = \frac{1}{2}$$

M1

$$m - 2 = \frac{3}{2}, \text{ so } m = \frac{7}{2}$$

A1

$$\text{Answer: } \frac{7}{2}$$

4 marks

24 (7 marks)

$$(a) P(X < x) = \int_1^x \frac{2}{t^3} dt = [-\frac{1}{t^2}]_1^x$$

M1

$$= 1 - \frac{1}{x^2}$$

A1

$$\text{Answer: } 1 - \frac{1}{x^2}$$

2 marks

$$(b) \text{ Lower quartile: } 1 - \frac{1}{q^2} = \frac{1}{4}, \text{ so } q^2 = \frac{4}{3}$$

M1

$$q_1 = \frac{2}{\sqrt{3}} \approx 1.155$$

A1

$$\text{Upper quartile: } 1 - \frac{1}{q^2} = \frac{3}{4}, \text{ so } q^2 = 4$$

M1

$$q_3 = 2$$

A1

$$\text{IQR} = 2 - \frac{2}{\sqrt{3}} \approx 0.845$$

A1

Answer: $\text{IQR} = 2 - \frac{2}{\sqrt{3}} \approx 0.845$

5 marks

25 (7 marks)

(a) $\int_0^1 (a + bx^2) dx = a + \frac{b}{3} = 1$

M1 A1

$\int_0^{1/2} (a + bx^2) dx = \frac{a}{2} + \frac{b}{24} = 0.2$

M1 A1

Solving simultaneously: $a = 0.2$, $b = 2.4$

A1

Answer: $a = 0.2$, $b = 2.4$

5 marks

(b) $f(x) = 0.2 + 2.4x^2 > 0$ for $0 \leq x \leq 1$ and the area is 1, so f is valid
 f is increasing on $[0, 1]$, so the mode is at $x = 1$

B1

B1

Answer: Valid; mode = 1

2 marks

26 (7 marks)

(a) $\int_2^3 (2x + k) dx = [x^2 + kx]_2^3 = (9 + 3k) - (4 + 2k)$

M1

$= 5 + k = 1$, so $k = -4$

A1

Answer: $k = -4$

2 marks

(b) $f(2) = 0$ and $f(3) = 2$, with a straight line between them and $f = 0$ elsewhere
 f is increasing on the range, so the mode is $x = 3$

B1

B1

Answer: Straight line from (2, 0) to (3, 2); mode = 3

2 marks

(c) $E(X) = \int_2^3 x(2x - 4) dx = [\frac{2x^3}{3} - 2x^2]_2^3$

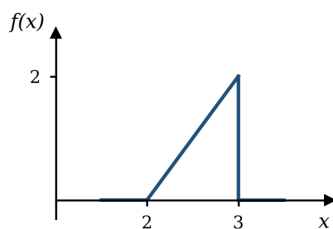
M1 A1

$= (18 - 18) - (\frac{16}{3} - 8) = \frac{8}{3}$

A1

Answer: $\frac{8}{3}$

3 marks



27 (10 marks)

(a) $\int_0^1 (n + 1)x^n dx = [x^{n+1}]_0^1$

M1

$= 1$, and $f(x) > 0$ on the range

A1

Answer: Valid

2 marks

(b) $P(X > \frac{1}{2}) = 1 - (\frac{1}{2})^{n+1}$

M1

$(\frac{1}{2})^{n+1} = \frac{1}{8}$

M1

$n + 1 = 3$, so $n = 2$

A1

Answer: $n = 2$

3 marks

(c) $P(X < c) = c^3$

B1

Median: $m^3 = \frac{1}{2}$, $m = \sqrt[3]{\frac{1}{2}} \approx 0.794$

M1 A1

$q_1^3 = \frac{1}{4}$, $q_3^3 = \frac{3}{4}$

M1

$\text{IQR} = \sqrt[3]{\frac{3}{4}} - \sqrt[3]{\frac{1}{4}} \approx 0.909 - 0.630 = 0.279$

A1

Answer: $m = 2^{-1/3} \approx 0.794$, $\text{IQR} \approx 0.279$

5 marks

28 (8 marks)

(a) The curve rises up to P and the line then falls, so the maximum of f is at P

B1

Answer: 3

1 mark

(b) The median is 3, so $P(X < 3) = \frac{1}{2}$

M1

$$\int_0^3 kx^2 dx = 9k = \frac{1}{2}, \text{ so } k = \frac{1}{18}$$

A1

The triangle for $3 < x \leq a$ has height $f(3) = 9k = \frac{1}{2}$ and area $\frac{1}{2}$

M1

$$\frac{1}{2} \times (a - 3) \times \frac{1}{2} = \frac{1}{2}$$

M1

$$a - 3 = 2, \text{ so } a = 5$$

A1

Answer: $k = \frac{1}{18}$, $a = 5$

5 marks

(c) $E(X) = \frac{71}{24} \approx 2.96$, which is less than the median and mode (both 3)

M1

mean < median = mode, so the distribution has negative skew

A1

Answer: Negative skew

2 marks

29 (12 marks)

(a) $\int_0^3 k(3x^2 - x^3) dx = k[x^3 - \frac{x^4}{4}]_0^3 = \frac{27k}{4}$

M1 A1

$$\frac{27k}{4} = 1, \text{ so } k = \frac{4}{27}$$

A1

$f'(x) = k(6x - 3x^2) = 0$ gives $x = 0$ or $x = 2$; $f(0) = 0$, so the mode is 2

M1 A1

Answer: $k = \frac{4}{27}$, mode = 2

5 marks

(b) $P(X < m) = \frac{4}{27}[x^3 - \frac{x^4}{4}]_0^m = \frac{4m^3 - m^4}{27} = \frac{1}{2}$

M1 A1

$$g(m) = 4m^3 - m^4 - 13.5: g(1.8) = 23.328 - 10.4976 - 13.5 = -0.669$$

M1

$$g(2) = 32 - 16 - 13.5 = 2.5; \text{ sign change, so } 1.8 < m < 2$$

A1

Answer: $1.8 < m < 2$

4 marks

(c) $g'(m) = 12m^2 - 4m^3$; $g'(1.8) = 38.88 - 23.328 = 15.552$

M1

$$m_1 = 1.8 - \frac{-0.6696}{15.552}$$

M1

$$m_1 = 1.843$$

A1

Answer: 1.843

3 marks

30 (9 marks)

(a) Trapezium with parallel sides 5 and 2 and height h

M1

$$\text{Area} = \frac{1}{2}(5 + 2)h = \frac{7h}{2}$$

A1

$$\frac{7h}{2} = 1, \text{ so } h = \frac{2}{7}$$

A1

Answer: $h = \frac{2}{7}$

3 marks

(b) $P(X < 1) = \frac{1}{2} \times 1 \times \frac{2}{7} = \frac{1}{7} < \frac{1}{2}$; $P(X < 3) = \frac{1}{7} + \frac{4}{7} = \frac{5}{7} > \frac{1}{2}$, so the median m is between 1 and 3

M1 A1

$$\frac{1}{7} + \frac{2}{7}(m - 1) = \frac{1}{2}$$

M1

$$m = \frac{9}{4}$$

A1

Answer: $\frac{9}{4}$

4 marks

(c) For $3 \leq x \leq 5$, $f(x) = \frac{(5-x)}{7}$; triangle with base 1 and height $f(4) = \frac{1}{7}$

M1

$$P = \frac{1}{2} \times 1 \times \frac{1}{7} = \frac{1}{14}$$

A1

Answer: $\frac{1}{14}$

2 marks

Cumulative Distribution Functions

1 (3 marks)

$$(a) F(2) = \frac{4}{16} = \frac{1}{4}$$

B1

$$\text{Answer: } \frac{1}{4}$$

1 mark

$$(b) F(3) - F(1) = \frac{9}{16} - \frac{1}{16} \\ = \frac{1}{2}$$

M1

A1

$$\text{Answer: } \frac{1}{2}$$

2 marks

2 (2 marks)

$$F(m) = \frac{1}{2} \cdot \frac{m-1}{4} = \frac{1}{2}$$

M1

$$m-1=2, \text{ so } m=3$$

A1

$$\text{Answer: } 3$$

2 marks

3 (2 marks)

$$F(x) = \int_0^x \frac{1}{4} dt$$

M1

$$= \frac{x}{4}$$

A1

$$\text{Answer: } F(x) = \frac{x}{4}$$

2 marks

4 (2 marks)

$$f(x) = F'(x)$$

M1

$$f(x) = \frac{x^2}{9} \text{ for } 0 \leq x \leq 3, \text{ and } 0 \text{ otherwise}$$

A1

$$\text{Answer: } f(x) = \frac{x^2}{9} \text{ for } 0 \leq x \leq 3$$

2 marks

5 (2 marks)

$$P(X > 3) = 1 - F(3)$$

M1

$$= e^{-1} \approx 0.368$$

A1

$$\text{Answer: } e^{-1} \approx 0.368$$

2 marks

6 (2 marks)

F is non-decreasing (and continuous)

B1

$F(x) \rightarrow 0$ as $x \rightarrow -\infty$ and $F(x) \rightarrow 1$ as $x \rightarrow \infty$

B1

Answer: Non-decreasing; limits 0 and 1

2 marks

7 (6 marks)

$$(a) \text{ For } 0 \leq x \leq 2: F(x) = \int_0^x \frac{3}{4}(2t - t^2) dt$$

M1

$$= \frac{3}{4} \left[t^2 - \frac{t^3}{3} \right]_0^x = \frac{3x^2}{4} - \frac{x^3}{4}$$

A1

$$F(x) = 0 \text{ for } x < 0$$

B1

$$F(x) = 1 \text{ for } x > 2$$

B1

$$\text{Answer: } F(x) = \frac{3x^2}{4} - \frac{x^3}{4} \text{ on } [0, 2]; 0 \text{ below; } 1 \text{ above}$$

4 marks

$$(b) P(X > 1.5) = 1 - F(1.5) = 1 - \left(\frac{27}{16} - \frac{27}{32} \right)$$

M1

$$= \frac{5}{32}$$

A1

$$\text{Answer: } \frac{5}{32}$$

2 marks

8 (5 marks)

$$\text{For } 0 \leq x \leq 2: F(x) = \int_0^x \frac{t}{6} dt = \frac{x^2}{12}$$

M1 A1

$$F(2) = \frac{4}{12} = \frac{1}{3}$$

B1

$$\text{For } 2 < x \leq 4: F(x) = \frac{1}{3} + \int_2^x \frac{1}{3} dt$$

M1

$$= \frac{1}{3} + \frac{x-2}{3} = \frac{x-1}{3}$$

A1

$$\text{Answer: } F(x) = \frac{x^2}{12} \text{ for } 0 \leq x \leq 2; F(x) = \frac{x-1}{3} \text{ for } 2 < x \leq 4$$

5 marks

9 (4 marks)

$$\text{For } 0 \leq x \leq 1: f(x) = F'(x) = x$$

M1 A1

$$\text{For } 1 < x \leq 2: f(x) = F'(x) = (2 - x)$$

A1

$f(x) = 0$ otherwise; the graph is a triangle with vertices (0, 0), (1, 1) and (2, 0)

A1

Answer: $f(x) = x$ ($0 \leq x \leq 1$), $2 - x$ ($1 < x \leq 2$); triangle

4 marks

10 (4 marks)

$$F(m) = \frac{1}{2}: \frac{8}{m^3} = \frac{1}{2}, \text{ so } m^3 = 16$$

M1

$$m = 2^3\sqrt{2} \approx 2.52$$

A1

$$P(3 < X < 4) = F(4) - F(3) = (1 - \frac{8}{64}) - (1 - \frac{8}{27})$$

M1

$$= \frac{8}{27} - \frac{1}{8} = \frac{37}{216}$$

A1

$$\text{Answer: } m = 2^3\sqrt{2} \approx 2.52, P = \frac{37}{216}$$

4 marks

11 (5 marks)

$$\text{(a) } F(2) - F(1) = \frac{6}{12} - \frac{2}{12}$$

M1

$$= \frac{1}{3}$$

A1

$$\text{Answer: } \frac{1}{3}$$

2 marks

$$\text{(b) } f(x) = F'(x) = \frac{2x+1}{12} \text{ for } 0 \leq x \leq 3$$

M1 A1

f is increasing, so the mode is 3

B1

$$\text{Answer: } f(x) = \frac{2x+1}{12}; \text{ mode} = 3$$

3 marks

12 (5 marks)

$$\text{(a) } F(0) = 0 \text{ and } F(1) = \frac{2}{2} = 1, \text{ so } F \text{ is continuous}$$

B1

$$F'(x) = \frac{3-3x^2}{2} \geq 0 \text{ on } [0, 1], \text{ so } F \text{ is non-decreasing}$$

M1

Valid

A1

Answer: Valid

3 marks

$$\text{(b) } G(1) = 1 \text{ but } G(2) = 0 \neq 1, \text{ so } G \text{ decreases for } 1 < x < 2 \text{ and jumps at } x = 2$$

M1

So G is not non-decreasing (and not continuous): not valid

A1

Answer: Not valid

2 marks

13 (4 marks)

$$P(T > 2) = 1 - F(2)$$

M1

$$= 1 - \frac{4}{9} = \frac{5}{9}$$

A1

$$P(1 < T < 2) = F(2) - F(1)$$

M1

$$= \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

A1

$$\text{Answer: } \frac{5}{9} \text{ and } \frac{1}{3}$$

4 marks

14 (7 marks)

$$(a) F(2) = 1: 4a + 2b = 1$$

M1

$$P(X > 1) = 0.4 \text{ so } F(1) = 0.6: a + b = 0.6$$

M1 A1

$$\text{Solving: } a = -0.1, b = 0.7$$

A1

$$\text{Answer: } a = -0.1, b = 0.7$$

4 marks

$$(b) f(x) = F'(x) = 2ax + b = 0.7 - 0.2x$$

M1 A1

$$\text{For } 0 \leq x \leq 2, f(x) \geq 0.7 - 0.4 = 0.3 > 0$$

A1

$$\text{Answer: } f(x) = 0.7 - 0.2x$$

3 marks

15 (7 marks)

$$(a) F(x) = \int_0^x te^{-t^2/2} dt = [-e^{-t^2/2}]_0^x$$

M1 A1

$$= 1 - e^{-x^2/2}$$

A1

$$\text{Answer: } F(x) = 1 - e^{-x^2/2}$$

3 marks

$$(b) P = F(2) - F(1) = e^{-1/2} - e^{-2}$$

M1

$$\approx 0.606 - 0.135 = 0.471$$

A1

$$e^{-m^2/2} = \frac{1}{2}, \text{ so } m^2 = 2 \ln 2$$

M1

$$m = \sqrt{2 \ln 2} \approx 1.18$$

A1

$$\text{Answer: } 0.471; m = \sqrt{2 \ln 2}$$

4 marks

16 (6 marks)

$$\text{For } 0 \leq x \leq 1: F(x) = \int_0^x \frac{6}{5}t^2 dt = \frac{2x^3}{5}$$

M1 A1

$$F(1) = \frac{2}{5}$$

B1

$$\text{For } 1 < x \leq 2: F(x) = \frac{2}{5} + \int_1^x \frac{6}{5}(2 - t) dt$$

M1

$$= \frac{2}{5} + \frac{6}{5} \left[2t - \frac{t^2}{2} \right]_1^x = -\frac{3x^2}{5} + \frac{12x}{5} - \frac{7}{5}$$

A1

$$F(2) = -\frac{12}{5} + \frac{24}{5} - \frac{7}{5} = 1$$

A1

$$\text{Answer: } F(x) = \frac{2x^3}{5} \text{ (} 0 \leq x \leq 1 \text{); } -\frac{3x^2}{5} + \frac{12x}{5} - \frac{7}{5} \text{ (} 1 < x \leq 2 \text{)}$$

6 marks

17 (5 marks)

$$P(X > 3) = 1 - \frac{9}{16} = \frac{7}{16}, P(X > 2) = 1 - \frac{4}{16} = \frac{3}{4}$$

M1 A1

$$P(X > 3 | X > 2) = \frac{7}{16} \div \frac{3}{4} = \frac{7}{12}$$

A1

$$\frac{c^2}{16} = 0.81$$

M1

$$c = 4 \times 0.9 = 3.6$$

A1

$$\text{Answer: } \frac{7}{12}, c = 3.6$$

5 marks

18 (7 marks)

$$(a) F'(x) = -e^{-x} + (1+x)e^{-x} \\ = xe^{-x}$$

M1 A1

A1

$$\text{Answer: } f(x) = xe^{-x}$$

3 marks

$$(b) f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

M1

$$f'(x) = 0 \text{ at } x = 1, \text{ and } f \text{ increases then decreases}$$

A1

$$\text{Mode} = 1$$

A1

$$\text{Answer: } 1$$

3 marks

$$(c) 1 - F(2) = 3e^{-2}$$

B1

$$\text{Answer: } 3e^{-2}$$

1 mark

19 (5 marks)

$$P(X > q) = 1 - F(q) = e^{-q/4}$$

M1

$$e^{-q/4} = 0.2$$

M1

$$q = -4 \ln 0.2 = 4 \ln 5 = 6.44$$

A1

$$P(X < 2) = F(2) = 1 - e^{-1/2}$$

M1

$$= 0.393$$

A1

$$\text{Answer: } q = 6.44, P = 0.393$$

5 marks

20 (6 marks)

$$(a) F(x) = \int_{-1}^x \frac{3}{4}(1-t^2) dt = \frac{3}{4} \left[t - \frac{t^3}{3} \right]_{-1}^x$$

M1

$$= \frac{3}{4} \left(x - \frac{x^3}{3} + \frac{2}{3} \right)$$

A1

$$= \frac{1}{2} + \frac{3x}{4} - \frac{x^3}{4}$$

A1

$$\text{Answer: } F(x) = \frac{1}{2} + \frac{3x}{4} - \frac{x^3}{4}$$

3 marks

$$(b) P\left(-\frac{1}{2} < X < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right)$$

M1

$$F\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{3}{8} - \frac{1}{32} = \frac{27}{32}, F\left(-\frac{1}{2}\right) = \frac{5}{32}$$

A1

$$P = \frac{27}{32} - \frac{5}{32} = \frac{11}{16}$$

A1

$$\text{Answer: } \frac{11}{16}$$

3 marks

21 (10 marks)

$$(a) \text{Continuity at } x = 2: F(2) = \frac{1}{2}, \text{ so } -\frac{1}{4} + 2a + b = \frac{1}{2}$$

M1 A1

$$F(4) = 1: -1 + 4a + b = 1$$

M1

$$\text{Solving: } a = \frac{5}{8}, b = -\frac{1}{2}$$

A1

$$\text{Answer: } a = \frac{5}{8}, b = -\frac{1}{2}$$

4 marks

$$(b) \text{For } 0 \leq x \leq 2: f(x) = F'(x) = \frac{x}{4}$$

M1 A1

$$\text{For } 2 < x \leq 4: f(x) = F'(x) = -\frac{x}{8} + \frac{5}{8} = \frac{5-x}{8}$$

A1

$$\text{Answer: } f(x) = \frac{x}{4} \text{ (} 0 \leq x \leq 2 \text{); } \frac{5-x}{8} \text{ (} 2 < x \leq 4 \text{)}$$

3 marks

$$(c) F(3) = -\frac{9}{16} + \frac{15}{8} - \frac{1}{2} = \frac{13}{16}$$

M1 A1

$$P = F(3) - F(1) = \frac{13}{16} - \frac{1}{8} = \frac{11}{16}$$

A1

$$\text{Answer: } \frac{11}{16}$$

3 marks

22 (8 marks)

$$(a) P(T > 2) = 1 - \frac{4}{9} = \frac{5}{9}$$

M1 A1

$$P(\text{both}) = \left(\frac{5}{9}\right)^2 = \frac{25}{81}$$

A1

$$\text{Answer: } \frac{25}{81}$$

3 marks

$$(b) P(\text{device works after 1 year}) = \left(\frac{8}{9}\right)^2 = \frac{64}{81}$$

M1

$$P(\text{fails within 1 year}) = 1 - \frac{64}{81} = \frac{17}{81}$$

A1

$$\text{Answer: } \frac{17}{81}$$

2 marks

$$(c) P(T > 2 | T > 1) = \frac{P(T > 2)}{P(T > 1)}$$

M1

$$= \frac{5}{9} \div \frac{8}{9}$$

M1

$$= \frac{5}{8}$$

A1

$$\text{Answer: } \frac{5}{8}$$

3 marks

23 (8 marks)

$$(a) \text{Median: } 1 - \frac{(6 - m)^2}{16} = \frac{1}{2}, \text{ so } (6 - m)^2 = 8 \text{ and } m = 6 - 2\sqrt{2} = 3.17$$

M1 A1

$$\text{Lower quartile: } (6 - q_1)^2 = 12, \text{ so } q_1 = 6 - 2\sqrt{3} = 2.54$$

M1

$$\text{Upper quartile: } (6 - q_3)^2 = 4, \text{ so } q_3 = 4.00$$

A1

Answer: Median 3.17, lower quartile 2.54, upper quartile 4.00

4 marks

$$(b) q_2 - q_1 = 0.636 \text{ and } q_3 - q_2 = 0.828$$

M1

$$q_3 - q_2 > q_2 - q_1, \text{ so the distribution has positive skew}$$

A1

Answer: Positive skew

2 marks

$$(c) f(x) = F'(x)$$

M1

$$f(x) = \frac{6 - x}{8} \text{ for } 2 \leq x \leq 6, \text{ and } f(x) = 0 \text{ otherwise}$$

A1

$$\text{Answer: } f(x) = \frac{6 - x}{8} \text{ for } 2 \leq x \leq 6$$

2 marks

24 (7 marks)

$$(a) G(1) = \frac{1}{4} + 0 = \frac{1}{4}, \text{ but } G(x) = 0 \text{ for } x < 1, \text{ so } G \text{ jumps by } \frac{1}{4} \text{ at } x = 1$$

M1 A1

The cumulative distribution function of a continuous random variable must be continuous, so G is not valid

E1

Answer: Jump at x = 1

3 marks

$$(b) P(X = 1) = G(1) - (\text{limit of } G \text{ from the left}) = \frac{1}{4}$$

B1

For a continuous random variable every single value has probability 0

E1

$$\text{Answer: } \frac{1}{4}$$

2 marks

$$(c) F(1) = 0 \text{ and } F(3) = 1, \text{ gradient } \frac{1}{2}$$

M1

$$F(x) = \frac{x - 1}{2}$$

A1

$$\text{Answer: } F(x) = \frac{x - 1}{2}$$

2 marks

25 (10 marks)

$$(a) F(m) = \frac{1}{2}: \ln m = \frac{1}{2} \ln 4 = \ln 2$$

M1

$$m = 2$$

A1

Answer: 2

2 marks

$$(b) \frac{\ln q_1}{\ln 4} = \frac{1}{4}, \text{ so } \ln q_1 = \frac{1}{4} \ln 4 = \frac{1}{2} \ln 2$$

M1 A1

$$q_1 = \sqrt{2}$$

A1

$$\frac{\ln q_3}{\ln 4} = \frac{3}{4}, \text{ so } q_3 = 4^{3/4} = 2\sqrt{2}$$

A1

Answer: $\sqrt{2}$ and $2\sqrt{2}$

4 marks

$$(c) f(x) = F'(x) = \frac{1}{x \ln 4} \text{ for } 1 \leq x \leq 4$$

M1 A1

$$P(X > 3) = 1 - \frac{\ln 3}{\ln 4}$$

M1

$$= 0.208$$

A1

Answer: $f(x) = \frac{1}{x \ln 4}$; 0.208

4 marks

26 (8 marks)

$$(a) P(X > 2) = 1 - F(2) = e^{-4/c}$$

M1

$$e^{-4/c} = e^{-1}, \text{ so } c = 4$$

A1

Answer: $c = 4$

2 marks

$$(b) 1 - e^{-m^2/4} = \frac{1}{2}, \text{ so } e^{-m^2/4} = \frac{1}{2}$$

M1

$$m^2 = 4 \ln 2$$

A1

$$m = 2\sqrt{\ln 2} \approx 1.67$$

A1

Answer: $2\sqrt{\ln 2} \approx 1.67$

3 marks

$$(c) P(X > 3 | X > 2) = \frac{1 - F(3)}{1 - F(2)} = \frac{e^{-9/4}}{e^{-1}}$$

M1 A1

$$= e^{-5/4}$$

A1

Answer: $e^{-5/4} \approx 0.287$

3 marks

27 (10 marks)

$$(a) \sin^2\left(\frac{\pi m}{8}\right) = \frac{1}{2}, \text{ so } \sin\left(\frac{\pi m}{8}\right) = \frac{1}{\sqrt{2}}$$

M1

$$\frac{\pi m}{8} = \frac{\pi}{4}, \text{ so } m = 2$$

A1

Answer: 2

2 marks

$$(b) F'(x) = 2 \sin\left(\frac{\pi x}{8}\right) \cos\left(\frac{\pi x}{8}\right) \times \frac{\pi}{8}$$

M1 A1

$$= \frac{\pi}{8} \sin\left(\frac{\pi x}{4}\right) \text{ using } \sin 2A = 2 \sin A \cos A$$

A1

$$\text{Maximum when } \sin\left(\frac{\pi x}{4}\right) = 1, x = 2$$

A1

Answer: $f(x) = \frac{\pi}{8} \sin\left(\frac{\pi x}{4}\right)$; mode = 2

4 marks

$$(c) F(3) - F(1) = \sin^2\left(\frac{3\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right)$$

M1

$$\text{Using } \sin^2 \theta = \frac{1 - \cos 2\theta}{2}; = \frac{1}{2} \left(\cos \frac{\pi}{4} - \cos \frac{3\pi}{4} \right)$$

M1 A1

$$= \frac{1}{2} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{2}$$

A1

Answer: $\frac{\sqrt{2}}{2}$

4 marks

28 (12 marks)

$$(a) F(2) = \frac{4}{8} = \frac{1}{2}, \text{ using the first piece}$$

M1 A1

$$\text{Hence } P(X \leq 2) = \frac{1}{2}, \text{ so the median is 2}$$

A1

Answer: 2

3 marks

$$(b) F(q) = \frac{3}{4} \cdot -\frac{q^2}{16} + \frac{5q}{8} - \frac{1}{2} = \frac{3}{4}$$

M1

Multiply by -16 : $q^2 - 10q + 20 = 0$

A1

$$q = \frac{10 \pm \sqrt{100 - 80}}{2} = 5 \pm \sqrt{5}$$

M1

$q = 5 + \sqrt{5} \approx 7.24$ is outside the range $2 < q \leq 4$

E1

$q = 5 - \sqrt{5} \approx 2.76$

A1

Answer: $5 - \sqrt{5} \approx 2.76$

5 marks

(c) $F(3) = \frac{13}{16}$, so $P(X > 3) = \frac{3}{16}$

M1 A1

$P(X > 1) = 1 - \frac{1}{8} = \frac{7}{8}$

B1

$P = \frac{3}{16} \div \frac{7}{8} = \frac{3}{14}$

A1

Answer: $\frac{3}{14}$

4 marks

29 (9 marks)

(a) $Y \leq y$ means $1 - e^{-2X} \leq y$

M1

$e^{-2X} \geq 1 - y$, so $X \leq -\frac{1}{2} \ln(1 - y)$

M1 A1

$P(Y \leq y) = F(-\frac{1}{2} \ln(1 - y)) = 1 - e^{\ln(1 - y)} = 1 - (1 - y) = y$

M1

This is the cdf of a continuous uniform distribution on $[0, 1]$

A1

Answer: Y is uniform on $[0, 1]$

5 marks

(b) Set $u = 1 - e^{-2x}$

M1

$e^{-2x} = 1 - u$, so $x = -\frac{1}{2} \ln(1 - u)$

A1

$u = 0.75$: $x = -\frac{1}{2} \ln 0.25 = \ln 2$

M1

$x = 0.693$

A1

Answer: $x = -\frac{1}{2} \ln(1 - u)$; 0.693

4 marks

30 (11 marks)

(a) For $-1 \leq x \leq 0$: $f(x) = F'(x) = 1 + x$

M1 A1

For $0 < x \leq 1$: $f(x) = F'(x) = 1 - x$

A1

$f(-x) = 1 - x = f(x)$ on the range, so f is an even function and X is symmetrical about 0

E1

Answer: $f(x) = 1 - |x|$ for $-1 \leq x \leq 1$

4 marks

(b) $F(q) = 0.9$ lies in the second piece since $F(0) = 0.5$: $1 - \frac{(1 - q)^2}{2} = 0.9$

M1

$(1 - q)^2 = 0.2$

A1

$q = 1 - \sqrt{0.2} = 0.553$

A1

Answer: 0.553

3 marks

(c) $P(X > \frac{1}{2}) = 1 - F(\frac{1}{2}) = \frac{(1 - \frac{1}{2})^2}{2} = \frac{1}{8}$

M1 A1

$P(|X| > \frac{1}{2}) = 2 \times \frac{1}{8} = \frac{1}{4}$

M1

$P = \frac{1}{8} \div \frac{1}{4} = \frac{1}{2}$; by symmetry the two tails are equally likely

A1

Answer: $\frac{1}{2}$

4 marks

Continuous Uniform Distribution

1 (2 marks)

(a) Width 3 out of 10: $P = \frac{3}{10}$

B1

Answer: 0.3

1 mark

(b) Width 5 out of 10: $P = \frac{5}{10} = 0.5$

B1

Answer: 0.5

1 mark

2 (2 marks)

$f(x) = \frac{1}{6-2} = \frac{1}{4}$ for $2 \leq x \leq 6$, and 0 otherwise

B1

$E(X) = \frac{2+6}{2} = 4$

B1

Answer: $f(x) = \frac{1}{4}$ on $[2, 6]$; $E(X) = 4$

2 marks

3 (2 marks)

$E(X) = \frac{1+7}{2} = 4$

B1

$\text{Var}(X) = \frac{(7-1)^2}{12} = 3$

B1

Answer: $E(X) = 4$, $\text{Var}(X) = 3$

2 marks

4 (2 marks)

The median equals the mean: $\frac{3+9}{2} = 6$

B1

$P(X > 8) = \frac{1}{6}$

B1

Answer: 6; $\frac{1}{6}$

2 marks

5 (2 marks)

$f(t) = \frac{1}{30}$

B1

$P(T > 20) = 10 \times \frac{1}{30} = \frac{1}{3}$

B1

Answer: $\frac{1}{3}$

2 marks

6 (2 marks)

$f(x) = \frac{1}{6}$ on $[-2, 4]$

B1

$P(X < 0) = 2 \times \frac{1}{6} = \frac{1}{3}$

B1

Answer: $\frac{1}{3}$

2 marks

7 (5 marks)

(a) $f(x) = \frac{1}{6}$

B1

$P = (8 - 5.5) \times \frac{1}{6} = \frac{5}{12}$

A1

Answer: $\frac{5}{12}$

2 marks

(b) $E(X) = 5$

B1

$\text{Var}(X) = \frac{(8-2)^2}{12} = 3$

M1

Standard deviation = $\sqrt{3} \approx 1.73$

A1

Answer: 5, 3, $\sqrt{3} \approx 1.73$

3 marks

8 (4 marks)

$$f(x) = \frac{1}{6} \text{ on } [-1, 5]$$

B1

$$\text{For } -1 \leq x \leq 5: F(x) = \int_{-1}^x \frac{1}{6} dt = \frac{x+1}{6}$$

M1 A1

$$F(x) = 0 \text{ for } x < -1 \text{ and } F(x) = 1 \text{ for } x > 5; P(X \leq 2) = F(2) = \frac{3}{6} = \frac{1}{2}$$

A1

$$\text{Answer: } F(x) = \frac{x+1}{6} \text{ on } [-1, 5]; P = \frac{1}{2}$$

4 marks

9 (3 marks)

$$\frac{a+b}{2} = 6 \text{ and } b - a = 8$$

M1

$$a + b = 12, b - a = 8$$

M1

$$a = 2, b = 10$$

A1

$$\text{Answer: } a = 2, b = 10$$

3 marks

10 (4 marks)

$$\text{(a) } f(e) = 1, \text{ so } P(-0.1 < E < 0.1) = 0.2 \times 1 = 0.2$$

M1

A1

$$\text{Answer: } 0.2$$

2 marks

$$\text{(b) } \text{Var}(E) = \frac{1^2}{12} = \frac{1}{12}$$

M1

$$\text{sd} = \frac{1}{\sqrt{12}} \approx 0.289$$

A1

$$\text{Answer: } 0.289$$

2 marks

11 (4 marks)

$$E(X) = 3$$

B1

$$\text{Var}(X) = \frac{36}{12} = 3, \text{ so } \text{sd} = \sqrt{3}$$

M1

$$P(3 - \sqrt{3} < X < 3 + \sqrt{3}) = \frac{2\sqrt{3}}{6}$$

M1

$$= \frac{\sqrt{3}}{3} \approx 0.577$$

A1

$$\text{Answer: } \frac{\sqrt{3}}{3} \approx 0.577$$

4 marks

12 (3 marks)

$$f(\theta) = \frac{1}{360}$$

B1

$$P(30 < \theta < 75) = \frac{45}{360} = \frac{1}{8}$$

B1

$$P(\theta > 250) = \frac{110}{360} = \frac{11}{36}$$

B1

$$\text{Answer: } \frac{1}{8} \text{ and } \frac{11}{36}$$

3 marks

13 (3 marks)

The midpoint is at $AP = 6$, so we need $3 < AP < 9$

M1

Width 6 out of 12

M1

$$P = \frac{1}{2}$$

A1

$$\text{Answer: } \frac{1}{2}$$

3 marks

14 (4 marks)

$$\frac{a+b}{2} = 10, \text{ so } a+b = 20$$

B1

$$\frac{(b-a)^2}{12} = 12, \text{ so } (b-a)^2 = 144$$

M1

$$b-a = 12 \text{ (since } b > a)$$

A1

$$a = 4, b = 16$$

A1

Answer: a = 4, b = 16

4 marks

15 (4 marks)

$$\text{Real roots when } X^2 - 4 \geq 0$$

M1

$$\text{Since } X \geq 0, \text{ this means } X \geq 2$$

A1

$$P(X \geq 2) = \frac{5-2}{5}$$

M1

$$= \frac{3}{5}$$

A1

$$\text{Answer: } \frac{3}{5}$$

4 marks

16 (4 marks)

$$P(X > 9 | X > 4) = \frac{P(X > 9)}{P(X > 4)} = \frac{3}{12} \div \frac{8}{12}$$

M1 A1

$$= \frac{3}{8}$$

A1

$$\text{Given } X > 4, \text{ the density is } \frac{1}{12} \div \frac{8}{12} = \frac{1}{8}, \text{ constant on } [4, 12], \text{ so } X | X > 4 \sim U[4, 12]$$

E1

$$\text{Answer: } \frac{3}{8}$$

4 marks

17 (5 marks)

$$\frac{10-a}{b-a} = 0.25 \text{ and } \frac{30-a}{b-a} = 0.75$$

M1 A1

$$\text{Subtracting: } \frac{20}{b-a} = 0.5, \text{ so } b-a = 40$$

M1

$$10-a = 0.25 \times 40 = 10, \text{ so } a = 0$$

M1

$$b = 40$$

A1

Answer: a = 0, b = 40

5 marks

18 (4 marks)

$$\text{Pieces are } X \text{ and } 1-X. \text{ The longer piece exceeds twice the shorter when } X < \frac{1}{3} \text{ or } X > \frac{2}{3}$$

M1 A1

$$\text{Check: if } X < \frac{1}{2}, \text{ longer} = 1-X > 2X \text{ gives } X < \frac{1}{3}$$

M1

$$P = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

A1

$$\text{Answer: } \frac{2}{3}$$

4 marks

19 (4 marks)

$$E(R) = 2, \text{ Var}(R) = \frac{4}{12} = \frac{1}{3}$$

B1 M1

$$E(R^2) = \text{Var}(R) + [E(R)]^2 = \frac{1}{3} + 4 = \frac{13}{3}$$

M1

$$E(\text{Area}) = \pi E(R^2) = \frac{13\pi}{3} \text{ cm}^2 \approx 13.6 \text{ cm}^2$$

A1

$$\text{Answer: } \frac{13\pi}{3} \text{ cm}^2 \approx 13.6 \text{ cm}^2$$

4 marks

20 (4 marks)

The formula for the variance is $\frac{(b-a)^2}{12}$; the student has not squared $(b-a)$

E1

$$\text{Var}(X) = \frac{36}{12} = 3$$

A1

$$\frac{c^2}{12} = 3$$

M1

$$c^2 = 36, \text{ so } c = 6$$

A1

Answer: Var(X) = 3, c = 6

4 marks

21 (6 marks)

(a) $E(X) = 2, \text{ Var}(X) = \frac{16}{12} = \frac{4}{3}$

B1

$$E(Y) = 3 \times 2 - 2 = 4$$

M1

$$\text{Var}(Y) = 9 \times \frac{4}{3} = 12$$

A1

Answer: E(Y) = 4, Var(Y) = 12

3 marks

(b) $Y > 5$ means $3X - 2 > 5$, so $X > \frac{7}{3}$

M1 A1

$$P = (4 - \frac{7}{3}) \times \frac{1}{4} = \frac{5}{12}$$

A1

Answer: $\frac{5}{12}$

3 marks

22 (8 marks)

(a) $a + 0.3(b-a) = 12$ and $a + 0.8(b-a) = 27$

M1 M1

Subtracting: $0.5(b-a) = 15$, so $b-a = 30$

A1

$a = 12 - 9 = 3$ and $b = 33$

A1

Answer: a = 3, b = 33

4 marks

(b) $E(X) = 18, \text{ Var}(X) = \frac{30^2}{12} = 75$

B1

$S \approx N(1800, 7500)$ by the central limit theorem

M1

$$P(S > 1850) = P(Z > \frac{1850 - 1800}{\sqrt{7500}}) = P(Z > 0.577)$$

M1

$$= 0.282$$

A1

Answer: 0.282

4 marks

23 (6 marks)

$f(x) = \frac{1}{b-a}$ on $[a, b]$

B1

$$E(X) = \int_a^b \frac{x}{b-a} dx = \frac{1}{b-a} \times \frac{b^2 - a^2}{2}$$

M1

$$= \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$

A1

$$E(X^2) = \frac{1}{b-a} \times \frac{b^3 - a^3}{3} = \frac{b^2 + ab + a^2}{3}$$

M1

$$\text{Var}(X) = \frac{b^2 + ab + a^2}{3} - \frac{(a+b)^2}{4} = \frac{4b^2 + 4ab + 4a^2 - 3a^2 - 6ab - 3b^2}{12}$$

M1

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

A1

Answer: $E(X) = \frac{a+b}{2}, \text{ Var}(X) = \frac{(b-a)^2}{12}$

6 marks

24 (7 marks)

(a) The region $X + Y > 1.5$ is the triangle with vertices (0.5, 1), (1, 0.5) and (1, 1)

M1

$$\text{Area} = \frac{1}{2} \times 0.5 \times 0.5 = \frac{1}{8}$$

M1

$$P = \frac{1}{8} \text{ (the square has area 1)}$$

A1

Answer: $\frac{1}{8}$

3 marks

(b) The region $|X - Y| \geq 0.2$ is two right-angled triangles with legs 0.8

M1 A1

$$\text{Total area} = 2 \times \frac{1}{2} \times 0.8 \times 0.8 = 0.64$$

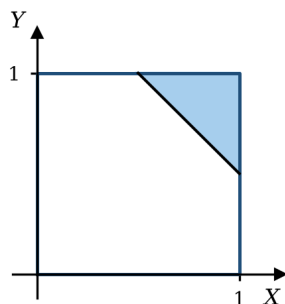
M1

$$P(|X - Y| < 0.2) = 1 - 0.64 = 0.36$$

A1

Answer: 0.36

4 marks

**25 (6 marks)**

(a) $P(W > 10) = \frac{5}{15} = \frac{1}{3}$

B1

Answer: $\frac{1}{3}$

1 mark

(b) $Y \sim B(6, \frac{1}{3})$

B1

$$P(Y = 2) = 15 \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^4$$

M1

$$= \frac{240}{729} = 0.329$$

A1

Answer: 0.329

3 marks

(c) $P(W > t) = \frac{15 - t}{15} = 0.05$

M1

$$t = 15 - 0.75 = 14.25$$

A1

Answer: 14.25

2 marks

26 (7 marks)

(a) $P(Y \leq y) = P(X^2 \leq y)$

M1

$$= P(X \leq \sqrt{y}) \text{ since } X \geq 0$$

M1

$$= \frac{\sqrt{y}}{3}$$

A1

Answer: $\frac{\sqrt{y}}{3}$

3 marks

(b) Differentiating: $g(y) = \frac{1}{6\sqrt{y}}$ for $0 < y \leq 9$

M1 A1

$$P(Y > 4) = 1 - \frac{\sqrt{4}}{3}$$

M1

$$= \frac{1}{3}$$

A1

Answer: $g(y) = \frac{1}{6\sqrt{y}}; \frac{1}{3}$

4 marks

27 (8 marks)

(X, Y) is uniformly distributed over a square of area 100

B1

$$XY < 10 \text{ means } Y < \frac{10}{X}$$

M1

For $0 \leq x \leq 1$, $\frac{10}{x} \geq 10$, so every y in $[0, 10]$ works: area = 10

M1 A1

For $1 < x \leq 10$, the area under $y = \frac{10}{x}$ is $\int_1^{10} \frac{10}{x} dx$

M1

$$= [10 \ln x]_1^{10} = 10 \ln 10$$

A1

Total area = $10 + 10 \ln 10$

M1

$$P = \frac{10 + 10 \ln 10}{100} = \frac{1 + \ln 10}{10} \approx 0.330$$

A1

$$\text{Answer: } \frac{1 + \ln 10}{10} \approx 0.330$$

8 marks

28 (8 marks)

$$(a) P(Y \leq y) = P(X \leq y \mid X < c) = \frac{P(X \leq y)}{P(X < c)}$$

M1

$$= \frac{y}{a} \div \frac{c}{a}$$

M1 A1

$$= \frac{y}{c}, \text{ which is the distribution function of } U[0, c]$$

A1

Answer: $Y \sim U[0, c]$

4 marks

(b) Given $X < 8$, $X \sim U[0, 8]$

M1

$$P(X > 5 \mid X < 8) = \frac{3}{8}$$

A1

$$E(X \mid X < 8) = 4$$

M1 A1

$$\text{Answer: } \frac{3}{8}; 4$$

4 marks

29 (9 marks)

(a) $M \leq m$ only if every $X_i \leq m$

M1

$$P(X_i \leq m) = \frac{m}{\theta}$$

B1

$$\text{Independence gives } P(M \leq m) = \left(\frac{m}{\theta}\right)^n$$

A1

$$\text{Answer: } \left(\frac{m}{\theta}\right)^n$$

3 marks

$$(b) P(M > 0.9\theta) = 1 - (0.9)^5$$

M1

$$= 0.410$$

A1

Answer: 0.410

2 marks

$$(c) 1 - (0.9)^n > 0.99, \text{ so } (0.9)^n < 0.01$$

M1

$$n \ln 0.9 < \ln 0.01$$

M1

$$n > \frac{\ln 0.01}{\ln 0.9} \approx 43.7 \text{ (dividing by a negative reverses the inequality)}$$

A1

$$n = 44$$

A1

Answer: $n = 44$

4 marks

30 (10 marks)

$$(a) P(T > 4) = P(A > 4)P(B > 4) = \frac{6}{10} \times \frac{11}{15}$$

M1 A1

$$P(T < 4) = 1 - \frac{11}{25} = \frac{14}{25}$$

A1

$$\text{Answer: } \frac{14}{25} = 0.56$$

3 marks

(b) $T > t$ only if $A > t$ and $B > t$, which are independent

M1

$$P(T > t) = \left(1 - \frac{t}{10}\right)\left(1 - \frac{t}{15}\right) = 1 - \frac{t}{6} + \frac{t^2}{150}$$

A1

$$F(t) = \frac{t}{6} - \frac{t^2}{150}$$

A1

$$f(t) = \frac{1}{6} - \frac{t}{75} \text{ for } 0 \leq t \leq 10$$

A1

$$\text{Answer: } F(t) = \frac{t}{6} - \frac{t^2}{150}$$

4 marks

$$(c) E(T) = \int_0^{10} \left(1 - \frac{t}{6} + \frac{t^2}{150}\right) dt$$

M1

$$= \left[t - \frac{t^2}{12} + \frac{t^3}{450}\right]_0^{10}$$

M1

$$= 10 - \frac{25}{3} + \frac{20}{9} = \frac{35}{9}$$

A1

$$\text{Answer: } \frac{35}{9} \approx 3.89 \text{ minutes}$$

3 marks

Exponential Distribution

1 (2 marks)

$$E(X) = \frac{1}{\lambda} = \frac{1}{0.5} = 2$$

B1

$$\text{Var}(X) = \frac{1}{\lambda^2} = 4$$

B1

$$\text{Answer: } E(X) = 2, \text{Var}(X) = 4$$

2 marks

2 (2 marks)

$$P(X < 1) = 1 - e^{-2 \times 1}$$

M1

$$= 0.865$$

A1

$$\text{Answer: } 0.865$$

2 marks

3 (2 marks)

$$f(x) = 3e^{-3x} \text{ for } x \geq 0 \text{ (0 otherwise)}$$

B1

$$F(x) = 1 - e^{-3x} \text{ for } x \geq 0 \text{ (0 otherwise)}$$

B1

$$\text{Answer: } f(x) = 3e^{-3x}, F(x) = 1 - e^{-3x}$$

2 marks

4 (3 marks)

$$\text{Mean} = \frac{1}{\lambda} = 4, \text{ so } \lambda = \frac{1}{4}$$

B1

$$P(X > 4) = e^{-\lambda \times 4}$$

M1

$$= e^{-1}$$

A1

$$\text{Answer: } \lambda = \frac{1}{4}; e^{-1}$$

3 marks

5 (2 marks)

$$T \sim \text{Exp}(3)$$

B1

$$E(T) = \frac{1}{3} \text{ hour} = 20 \text{ minutes}$$

B1

$$\text{Answer: } T \sim \text{Exp}(3); 20 \text{ minutes}$$

2 marks

6 (2 marks)

$$P(X > 10) = e^{-0.1 \times 10}$$

M1

$$= e^{-1}$$

A1

$$\text{Answer: } e^{-1}$$

2 marks

7 (3 marks)

$$e^{-3\lambda} = 0.2$$

M1

$$-3\lambda = \ln 0.2, \text{ so } \lambda = \frac{\ln 5}{3}$$

M1

$$\lambda = 0.536$$

A1

$$\text{Answer: } 0.536$$

3 marks

8 (4 marks)

$$\text{Median: } 1 - e^{-0.2m} = \frac{1}{2}, \text{ so } e^{-0.2m} = \frac{1}{2}$$

M1

$$m = 5 \ln 2 \approx 3.47$$

A1

$$\text{Upper quartile: } e^{-0.2q} = \frac{1}{4}$$

M1

$$q = 5 \ln 4 = 10 \ln 2 \approx 6.93$$

A1

Answer: $5 \ln 2$ and $5 \ln 4$

4 marks

9 (4 marks)

$$\text{(a) } \lambda = \frac{1}{800}; P(L > 1000) = e^{-1000/800}$$

M1

$$= e^{-1.25} = 0.287$$

A1

Answer: 0.287

2 marks

$$\text{(b) } P(L < 400) = 1 - e^{-400/800}$$

M1

$$= 0.393$$

A1

Answer: 0.393

2 marks

10 (3 marks)

$$P(X > 15 | X > 5) = \frac{e^{-1.5}}{e^{-0.5}}$$

M1

$$= e^{-1} \approx 0.368$$

A1

This is the memoryless property: $P(X > 5 + 10 | X > 5) = P(X > 10)$

E1

Answer: $e^{-1} \approx 0.368$

3 marks

11 (4 marks)**(a)** Number of arrivals $N \sim \text{Po}(2 \times 1.5) = \text{Po}(3)$

M1

$$P(N = 0) = e^{-3} \approx 0.0498$$

A1

Answer: $e^{-3} \approx 0.0498$

2 marks

(b) $T \sim \text{Exp}(2)$: $P(T > 1.5) = e^{-2 \times 1.5}$

M1

$$= e^{-3} \approx 0.0498, \text{ the same value as in part (a)}$$

A1

Answer: $e^{-3} \approx 0.0498$

2 marks

12 (3 marks)

$$E(X) = 3, \text{ Var}(X) = 9$$

B1

$$E(3X + 2) = 3 \times 3 + 2 = 11$$

A1

$$\text{Var}(3X + 2) = 3^2 \times 9 = 81$$

A1

Answer: $E = 11$, $\text{Var} = 81$

3 marks

13 (4 marks)

$$E(X) = 2, \text{ Var}(X) = 4, \text{ sd} = 2$$

B1

Interval: $2 - 2 < X < 2 + 2$, i.e. $0 < X < 4$ (X cannot be negative)

M1

$$P = F(4) = 1 - e^{-0.5 \times 4}$$

M1

$$= 1 - e^{-2} \approx 0.865$$

A1

Answer: $1 - e^{-2} \approx 0.865$

4 marks

14 (4 marks)Memoryless: $P(X > 5 | X > 2) = P(X > 3)$

M1

$$e^{-2\lambda} = 0.3, \text{ so } e^{-\lambda} = 0.3^{1/2}$$

M1

$$P(X > 3) = e^{-3\lambda} = 0.3^{3/2}$$

M1

$$= 0.164$$

A1

Answer: 0.164

4 marks

15 (4 marks)

$$\lambda = \frac{1}{6}; \text{ by the memoryless property } P(T > 10 | T > 4) = P(T > 6)$$

M1

$$= e^{-1} \approx 0.368$$

A1

$$P(T > 10) = e^{-10/6} = e^{-5/3}$$

M1

$$\approx 0.189$$

A1

$$\text{Answer: } e^{-1} \approx 0.368; e^{-5/3} \approx 0.189$$

4 marks

16 (5 marks)

$$(a) P(2 < X < 6) = F(6) - F(2) = e^{-0.25 \times 2} - e^{-0.25 \times 6}$$

M1 A1

$$= e^{-0.5} - e^{-1.5} = 0.383$$

A1

$$\text{Answer: } 0.383$$

3 marks

$$(b) 1 - e^{-0.25m} = 0.5, \text{ so } e^{-0.25m} = 0.5$$

M1

$$m = 4 \ln 2 \approx 2.77$$

A1

$$\text{Answer: } 4 \ln 2 \approx 2.77$$

2 marks

17 (5 marks)

$$P(\text{a component lasts longer than 100 h}) = e^{-100/100} = e^{-1} \approx 0.3679$$

M1 A1

$$Y \sim B(5, e^{-1})$$

M1

$$P(Y \geq 4) = 5p^4(1 - p) + p^5 \text{ with } p = 0.3679$$

M1

$$= 0.0646$$

A1

$$\text{Answer: } 0.0646$$

5 marks

18 (5 marks)

$$(a) \lambda = \frac{1}{12}; P(T > 30) = e^{-30/12}$$

M1

$$= e^{-2.5} \approx 0.0821$$

A1

$$\text{Answer: } e^{-2.5} \approx 0.0821$$

2 marks

$$(b) 1 - e^{-t/12} = 0.9, \text{ so } e^{-t/12} = 0.1$$

M1

$$t = -12 \ln 0.1 = 12 \ln 10$$

M1

$$t = 27.6 \text{ minutes}$$

A1

$$\text{Answer: } 27.6 \text{ minutes}$$

3 marks

19 (5 marks)

$$1 - e^{-\lambda} = 0.6, \text{ so } e^{-\lambda} = 0.4$$

M1

$$\lambda = -\ln 0.4 = \ln 2.5 \approx 0.916$$

A1

$$\text{Mean} = \frac{1}{\lambda} = \frac{1}{\ln 2.5} \approx 1.09$$

A1

$$P(X > 2) = e^{-2\lambda} = (e^{-\lambda})^2$$

M1

$$= 0.4^2 = 0.16$$

A1

$$\text{Answer: } \lambda = \ln 2.5; \text{ mean} \approx 1.09; 0.16$$

5 marks

20 (4 marks)

$$P(X > x) = 1 - F(x) = e^{-\lambda x} \text{ for } x \geq 0$$

B1

$$P(X > s + t | X > s) = \frac{P(X > s + t)}{P(X > s)}$$

M1

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

M1

$$= e^{-\lambda t} = P(X > t)$$

A1

$$\text{Answer: } P(X > s + t | X > s) = e^{-\lambda t} = P(X > t)$$

4 marks

21 (5 marks)

The mean of an exponential distribution is $\frac{1}{\lambda}$, not λ , and the variance is $\frac{1}{\lambda^2}$

E1

$$E(X) = \frac{1}{0.5} = 2$$

A1

$$\text{Var}(X) = \frac{1}{0.25} = 4$$

A1

$$P(X > 2) = e^{-0.5 \times 2}$$

M1

$$= e^{-1} \approx 0.368$$

A1

Answer: $E(X) = 2$, $\text{Var}(X) = 4$; e^{-1}

5 marks

22 (5 marks)

(a) Sample mean = $\frac{2.1 + 5.4 + 1.2 + 6.8 + 1.5}{5} = \frac{17}{5} = 3.4$

M1

$$\lambda = \frac{1}{3.4} \approx 0.294$$

A1

Answer: $\lambda = \frac{5}{17} \approx 0.294$

2 marks

(b) $P(T > 5) = e^{-5/3.4}$

M1

$$= 0.230$$

A1

Observed: 2 of 5 values exceed 5 (0.4), compared with 0.23, which is reasonably close given the small sample

E1

Answer: 0.230

3 marks

23 (6 marks)

(a) $P(A > 10) = e^{-0.2}$, $P(B > 10) = e^{-0.3}$

M1 A1

$$P(\text{both}) = e^{-0.5} \approx 0.607$$

A1

Answer: $e^{-0.5} \approx 0.607$

3 marks

(b) $P(\text{system} > t) = P(A > t)P(B > t) = e^{-0.02t}e^{-0.03t}$

M1

$$= e^{-0.05t}$$
, so the system lifetime is $\text{Exp}(0.05)$

A1

$$\text{Mean} = \frac{1}{0.05} = 20 \text{ hours}$$

A1

Answer: $\text{Exp}(0.05)$; mean 20 hours

3 marks

24 (8 marks)

$$E(X) = \int_0^{\infty} x\lambda e^{-\lambda x} dx = [-xe^{-\lambda x}]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx$$

M1 A1

The first term is 0 ($x e^{-\lambda x} \rightarrow 0$), so $E(X) = [-\frac{1}{\lambda} e^{-\lambda x}]_0^{\infty} = \frac{1}{\lambda}$

A1

$$E(X^2) = \int_0^{\infty} x^2\lambda e^{-\lambda x} dx = [-x^2 e^{-\lambda x}]_0^{\infty} + \int_0^{\infty} 2xe^{-\lambda x} dx$$

M1 A1

$$= 0 + \frac{2}{\lambda} \int_0^{\infty} x\lambda e^{-\lambda x} dx$$

M1

$$= \frac{2}{\lambda} \times \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

A1

$$\text{Var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

A1

Answer: $E(X) = \frac{1}{\lambda}$, $E(X^2) = \frac{2}{\lambda^2}$, $\text{Var}(X) = \frac{1}{\lambda^2}$

8 marks

25 (7 marks)

(a) Time T hours to the first bus $\sim \text{Exp}(4)$

B1

$$20 \text{ minutes} = \frac{1}{3} \text{ hour}$$

M1

$$P(T > \frac{1}{3}) = e^{-4/3} \approx 0.264$$

A1

Answer: $e^{-4/3} \approx 0.264$

3 marks

(b) $N \sim \text{Po}(4 \times 0.5) = \text{Po}(2)$

M1

$$P(N \geq 2) = 1 - e^{-2}(1 + 2)$$

$$= 1 - 3e^{-2} \approx 0.594$$

$$P(T < \frac{1}{2}) = P(\text{at least 1 bus}) \text{ which is larger } (\approx 0.865); \text{ at least 2 is a stricter condition}$$

Answer: $1 - 3e^{-2} \approx 0.594$

M1

A1

E1

4 marks

26 (7 marks)

(a) $P(\text{a component fails before 10 h}) = 1 - e^{-1} \approx 0.6321$ M1 A1

$P(\text{all three fail before 10 h}) = (1 - e^{-1})^3 \approx 0.2526$ M1

$P(\text{system works}) = 1 - 0.2526 = 0.747$ A1

Answer: 0.747 4 marks

(b) Series: all three last longer than 10 h: $(e^{-1})^3 = e^{-3}$ M1

$= 0.0498$ A1

Parallel is far more reliable than series since only one component needs to survive E1

Answer: $e^{-3} \approx 0.0498$ 3 marks

27 (7 marks)

By the memoryless property, the remaining life $X - s$ given $X > s$ has the same distribution $\text{Exp}(\lambda)$ M1

So $E(X - s | X > s) = \frac{1}{\lambda}$ A1

$E(X | X > s) = s + \frac{1}{\lambda}$ A1

Direct check: $\int_s^\infty x\lambda e^{-\lambda x} dx = [-xe^{-\lambda x}]_s^\infty + \int_s^\infty e^{-\lambda x} dx = se^{-\lambda s} + \frac{e^{-\lambda s}}{\lambda}$ M1 A1

Dividing by $P(X > s) = e^{-\lambda s}$ gives $s + \frac{1}{\lambda}$ M1

$\lambda = \frac{1}{50}$: expected total lifetime = $80 + 50 = 130$ hours A1

Answer: 130 hours 7 marks

28 (8 marks)

(a) $P(X < Y) = \int_0^\infty \lambda e^{-\lambda x} P(Y > x) dx$ M1

$= \int_0^\infty \lambda e^{-\lambda x} e^{-\mu x} dx$ A1

$= \lambda \left[-\frac{e^{-(\lambda+\mu)x}}{\lambda+\mu} \right]_0^\infty$ M1

$= \frac{\lambda}{\lambda+\mu}$ A1

Answer: $\frac{\lambda}{\lambda+\mu}$ 4 marks

(b) $\lambda = \frac{1}{4}, \mu = \frac{1}{6}$ B1

$P(A \text{ before } B) = \frac{1}{4/1/4 + 1/6} = \frac{3}{5}$ M1 A1

$P(\text{both busy after 3}) = e^{-3/4} e^{-3/6} = e^{-5/4} \approx 0.287$ A1

Answer: $\frac{3}{5}; e^{-5/4} \approx 0.287$ 4 marks

29 (9 marks)

(a) $P(K = k) = F(k + 1) - F(k)$ M1

$= e^{-\lambda k} - e^{-\lambda(k+1)}$ A1

$= e^{-\lambda k}(1 - e^{-\lambda})$ A1

Writing $r = e^{-\lambda}$: $P(K + 1 = j) = (1 - r)r^{j-1}, j = 1, 2, \dots$ M1

So $K + 1 \sim \text{Geo}(p)$ with $p = 1 - e^{-\lambda}$ A1

Answer: $K + 1 \sim \text{Geo}(1 - e^{-\lambda})$ 5 marks

(b) $P(K = 2) = e^{-1}(1 - e^{-0.5})$ M1

$$= 0.3679 \times 0.3935 = 0.145$$

A1

$$E(K + 1) = \frac{1}{p} = \frac{1}{1 - e^{-0.5}} \approx 2.541$$

M1

$$E(K) = 1.54$$

A1

Answer: 0.145; 1.54

4 marks

30 (9 marks)(a) $M \leq m$ only if both $X_1 \leq m$ and $X_2 \leq m$

M1

$$P(X_i \leq m) = 1 - e^{-\lambda m}$$

B1

$$\text{Independence: } G(m) = (1 - e^{-\lambda m})^2$$

A1

Answer: $G(m) = (1 - e^{-\lambda m})^2$

3 marks

$$(b) g(m) = G'(m) = 2(1 - e^{-\lambda m}) \times \lambda e^{-\lambda m}$$

M1

$$= 2\lambda e^{-\lambda m} - 2\lambda e^{-2\lambda m}$$

A1

Answer: $g(m) = 2\lambda(e^{-\lambda m} - e^{-2\lambda m})$

2 marks

$$(c) E(M) = \int_0^\infty m(2\lambda e^{-\lambda m} - 2\lambda e^{-2\lambda m}) dm$$

M1

$$\int_0^\infty m\lambda e^{-\lambda m} dm = \frac{1}{\lambda} \text{ and } \int_0^\infty m\lambda e^{-2\lambda m} dm = \frac{1}{4\lambda}$$

M1 A1

$$E(M) = 2 \times \frac{1}{\lambda} - 2 \times \frac{1}{4\lambda} = \frac{3}{2\lambda}$$

A1

Answer: $E(M) = \frac{3}{2\lambda}$

4 marks

Normal Distribution**1 (3 marks)**

$$P(Z < 1.2) = 0.8849$$

B1

$$P(-0.5 < Z < 1.5) = \Phi(1.5) - \Phi(-0.5) = 0.9332 - 0.3085$$

M1

$$= 0.6247$$

A1

Answer: 0.8849; 0.6247

3 marks

2 (2 marks)

$$z = \frac{130 - 100}{15} = 2$$

M1

$$P = 1 - \Phi(2) = 0.0228$$

A1

Answer: 0.0228

2 marks

3 (3 marks)

$$a = 1.2816 \approx 1.28$$

B1

$$P(Z < z) = 0.975 \text{ gives } z = 1.96$$

B1

$$b = 10 + 1.96 \times 2 = 13.92$$

B1

Answer: $a \approx 1.28$, $b = 13.92$

3 marks

4 (2 marks)

$$E(X + Y) = 5 + 8 = 13$$

B1

$$\text{Var}(X + Y) = 4 + 9 = 13, \text{ so } X + Y \sim N(13, 13)$$

B1

Answer: $N(13, 13)$

2 marks

5 (2 marks)

$$z = \frac{245 - 250}{5} = -1$$

M1

$$P = \Phi(-1) = 0.1587$$

A1

Answer: 0.1587

2 marks

6 (4 marks)

$$z_1 = \frac{160 - 172}{6} = -2, z_2 = \frac{180 - 172}{6} = 1.333$$

M1 A1

$$P = \Phi(1.333) - \Phi(-2)$$

M1

$$= 0.9088 - 0.0228 = 0.886$$

A1

Answer: 0.886

4 marks

7 (5 marks)

$$P(X < a) = 0.9, \text{ so } z = 1.2816$$

M1 A1

$$a = 80 + 1.2816 \times 10 = 92.8$$

A1

$$\text{Interval: } 80 \pm 1.96 \times 10$$

M1

$$= (60.4, 99.6)$$

A1

Answer: a = 92.8; (60.4, 99.6)

5 marks

8 (3 marks)

$$P(X < 58) = 0.9332, \text{ so } \frac{58 - 50}{\sigma} = 1.5$$

M1 A1

$$\sigma = \frac{8}{1.5} = 5.33$$

A1

Answer: $\sigma \approx 5.33$

3 marks

9 (3 marks)

$$\frac{30 - \mu}{4} = -0.8416$$

M1 A1

$$\mu = 30 + 0.8416 \times 4 = 33.4$$

A1

Answer: $\mu \approx 33.4$

3 marks

10 (4 marks)

$$E(Y) = 4 \times 10 - 1 = 39$$

B1

$$\text{Var}(Y) = 16 \times 9 = 144, \text{ so } Y \sim N(39, 12^2)$$

B1

$$P(Y > 45) = P(Z > \frac{45 - 39}{12}) = 1 - \Phi(0.5)$$

M1

$$= 0.3085$$

A1

Answer: $N(39, 12^2)$; 0.3085

4 marks

11 (4 marks)

$$T = A + B: E(T) = 12$$

B1

$$\text{Var}(T) = 0.16 + 0.09 = 0.25, \text{ so } T \sim N(12, 0.5^2)$$

M1

$$P(T > 13) = P(Z > \frac{13 - 12}{0.5}) = 1 - \Phi(2)$$

M1

$$= 0.0228$$

A1

Answer: 0.0228

4 marks

12 (5 marks)

$$\text{Let } S = X_1 + X_2 + X_3 + X_4: E(S) = 100$$

B1

$$\text{Var}(S) = 4 \times 4 = 16$$

M1

$$S \sim N(100, 4^2)$$

A1

$$P(S > 105) = 1 - \Phi(\frac{105 - 100}{4}) = 1 - \Phi(1.25)$$

M1

$$= 0.1056$$

A1

Answer: 0.1056

5 marks

13 (5 marks)

$$\text{Let } D = X - 3Y: E(D) = 40 - 36 = 4$$

B1

$$\text{Var}(D) = 25 + 9 \times 4 = 61$$

M1 A1

$$P(D > 0) = P(Z > \frac{0 - 4}{\sqrt{61}}) = P(Z > -0.512)$$

M1

$$= 0.6957$$

A1

Answer: 0.6957

5 marks

14 (5 marks)

$np = 40$ and $np(1 - p) = 24$, so X is approximately $N(40, 24)$

B1 M1

$$P(X \leq 35) \approx P(Y < 35.5)$$

M1

$$z = \frac{35.5 - 40}{\sqrt{24}} = -0.9186$$

A1

$$P = \Phi(-0.9186) \approx 0.179$$

A1

Answer: ≈ 0.179

5 marks

15 (5 marks)

The mean is large ($30 > 10$), so X is approximately $N(30, 30)$

E1 M1

$$P(X > 35) = P(X \geq 36) \approx P(Y > 35.5)$$

M1

$$z = \frac{35.5 - 30}{\sqrt{30}} = 1.004$$

A1

$$P = 1 - \Phi(1.004) \approx 0.158$$

A1

Answer: ≈ 0.158

5 marks

16 (5 marks)

$$P(H > 180) = P(Z > 1.25) = 0.1056$$

M1 A1

$$P(H > 160) = P(Z > -1.25) = 0.8944$$

A1

$$P(H > 180 | H > 160) = \frac{P(H > 180)}{P(H > 160)}$$

M1

$$= 0.118$$

A1

Answer: 0.118

5 marks

17 (5 marks)

The total of two independent customers is $X_1 + X_2$, not $2X$; $2X$ would mean one customer's time doubled

E1

$$E(X_1 + X_2) = 20$$

B1

$$\text{Var}(X_1 + X_2) = 4 + 4 = 8, \text{ so the total } \sim N(20, 8)$$

M1

$$P(\text{total} > 25) = P(Z > \frac{25 - 20}{\sqrt{8}}) = P(Z > 1.768)$$

M1

$$= 0.0385$$

A1

Answer: ≈ 0.0385

5 marks

18 (5 marks)

$$D = A - B \sim N(0, 3^2 + 4^2) = N(0, 5^2)$$

M1 A1

$$P(|D| < 2) = P(-2 < D < 2)$$

M1

$$= P(-0.4 < Z < 0.4) = 2\Phi(0.4) - 1$$

M1

$$= 0.311$$

A1

Answer: 0.311

5 marks

19 (5 marks)

$$C = 3X + 2Y: E(C) = 24 + 30 = 54$$

M1 A1

$$\text{Var}(C) = 9 \times 4 + 4 \times 9 = 72$$

M1

$$P(C > 70) = P(Z > \frac{70 - 54}{\sqrt{72}}) = P(Z > 1.886)$$

M1

$$= 0.0297$$

A1

Answer: 0.0297

5 marks

20 (6 marks)

$$(a) \text{Var}(W) = 4^2 \text{Var}(Y) + \text{Var}(X)$$

M1

$$= 16 \times 6 + \sigma^2 = 96 + \sigma^2$$

M1

$$96 + \sigma^2 = 121, \text{ so } \sigma = 5$$

A1

Answer: $\sigma = 5$

3 marks

$$(b) E(W) = 4 \times 6 - 40 + 50 = 34$$

B1

$$P(X < 34) = P(Z < \frac{34 - 40}{5}) = \Phi(-1.2)$$

M1

$$= 0.1151$$

Answer: 0.1151

A1

3 marks

21 (6 marks)

$$P(X < 42) = 0.1: \frac{42 - \mu}{\sigma} = -1.2816$$

M1 A1

$$P(X > 60) = 0.05: \frac{60 - \mu}{\sigma} = 1.6449$$

M1 A1

$$\text{Subtracting: } \frac{18}{\sigma} = 2.9265, \text{ so } \sigma = 6.15$$

A1

$$\mu = 42 + 1.2816\sigma = 49.9$$

A1

Answer: $\sigma \approx 6.15$, $\mu \approx 49.9$

6 marks

22 (6 marks)

$$\text{(a) } z = \frac{19.85 - 20}{0.1} = -1.5 \text{ and } z = \frac{20.2 - 20}{0.1} = 2$$

M1 A1

$$P = \Phi(2) - \Phi(-1.5) = 0.9772 - 0.0668 = 0.910$$

A1

Answer: 0.910

3 marks

$$\text{(b) New mean} = \frac{19.85 + 20.2}{2} = 20.025$$

B1

$$\text{Half-width } 0.175: z = \pm 1.75$$

M1

$$P = 2\Phi(1.75) - 1 = 0.920$$

A1

Answer: 0.920

3 marks

23 (6 marks)

$$E(2A - B) = 40 - 25 = 15$$

B1

$$\text{Var}(2A - B) = 4k + 2k = 6k$$

M1 A1

$$P(2A - B < 6) = \Phi\left(\frac{6 - 15}{\sqrt{6k}}\right) = 0.0668$$

M1

$$\Phi(-1.5) = 0.0668, \text{ so } \frac{9}{\sqrt{6k}} = 1.5 \text{ and } \sqrt{6k} = 6$$

M1

$$k = 6$$

A1

Answer: $k = 6$

6 marks

24 (6 marks)

$$\text{(a) } P(X < 500) = 0.02: \frac{500 - \mu}{6} = -2.0537$$

M1 A1

$$\mu = 500 + 2.0537 \times 6 = 512.3$$

A1

Answer: $\mu \approx 512.3$

3 marks

$$\text{(b) } z = \frac{520 - 512.32}{6} = 1.28$$

M1 A1

$$P = 1 - \Phi(1.28) \approx 0.100$$

A1

Answer: 0.100

3 marks

25 (6 marks)

$$\text{(a) } P(Y \leq 8) = e^{-12}(1 + 12 + 72 + 288 + 864 + 2073.6 + 4147.2 + 7109.5 + 10664.3) = 0.155$$

M1

A1

Answer: 0.155

2 marks

$$\text{(b) } Y \approx N(12, 12)$$

B1

$$P(Y \leq 8) \approx P(X < 8.5) = \Phi\left(\frac{8.5 - 12}{\sqrt{12}}\right) = \Phi(-1.010)$$

M1

$$= 0.156$$

A1

Answer: 0.156

3 marks

$$\text{(c) Error} = \frac{(0.156 - 0.155)}{0.155} \approx 0.73\%$$

A1

Answer: 0.73%

1 mark

26 (9 marks)

(a) $S \sim N(8 \times 75, 8 \times 144) = N(600, 1152)$

M1 A1

$$P(S > 650) = P(Z > \frac{50}{\sqrt{1152}}) = P(Z > 1.473)$$

M1

$$= 0.0704$$

A1

Answer: 0.0704

4 marks

(b) For n passengers $S \sim N(75n, 144n)$; need $P(S > 650) < 0.01$

M1

Require $\frac{650 - 75n}{12\sqrt{n}} > 2.3263$

M1 A1

$$n = 7: \frac{650 - 525}{12\sqrt{7}} = 3.94 > 2.33, \text{ so satisfied}$$

A1

$$n = 8: \frac{650 - 600}{12\sqrt{8}} = 1.47 < 2.33, \text{ so not satisfied; answer 7}$$

A1

Answer: 7 passengers

5 marks

27 (9 marks)

(a) $\frac{a - \mu}{\sigma} = -0.5244$

M1 A1

$$\frac{2a - \mu}{\sigma} = 1.2816$$

M1 A1

Subtracting: $\frac{a}{\sigma} = 1.8060$, so $\sigma = 0.554a$

A1

$$\mu = a + 0.5244\sigma = 1.29a$$

A1

Answer: $\sigma \approx 0.554a$, $\mu \approx 1.29a$

6 marks

(b) $z = \frac{1.5a - \mu}{\sigma} = \frac{1.5 - 1.29}{0.554} = 0.379$

M1 A1

$$P = 1 - \Phi(0.379) \approx 0.353$$

A1

Answer: 0.353

3 marks

28 (9 marks)

(a) $\bar{X} = \frac{1}{n}(X_1 + \dots + X_n)$, a linear combination of independent normal variables, so normal

B1

$$E(\bar{X}) = \mu$$

B1

$$\text{Var}(\bar{X}) = \frac{1}{n^2} \times n \times 4 = \frac{4}{n}$$

M1

Answer: $\bar{X} \sim N(\mu, \frac{4}{n})$

3 marks

(b) $P(|\bar{X} - \mu| < 0.5) = P(|Z| < \frac{0.5}{2/\sqrt{n}}) = P(|Z| < 0.25\sqrt{n})$

M1 A1

Require $2\Phi(0.25\sqrt{n}) - 1 \geq 0.95$, so $\Phi(0.25\sqrt{n}) \geq 0.975$

M1

$$0.25\sqrt{n} \geq 1.96$$

A1

$$\sqrt{n} \geq 7.84, n \geq 61.5$$

M1

$$n = 62$$

A1

Answer: $n = 62$

6 marks

29 (11 marks)

(a) $T \sim N(960, 600)$

M1 A1

$$P(T < 1000) = \Phi(\frac{1000 - 960}{\sqrt{600}}) = \Phi(1.633)$$

M1

$$= 0.949$$

A1

Answer: 0.949

4 marks

(b) $P(T < 900) = \Phi(-2.449) = 0.00716$

M1 A1

$$P = 0.949 - 0.00715 = 0.942$$

A1

Answer: 0.942

3 marks

(c) Pack mass $\sim N(6\mu, 600)$

B1

$$P(T < 1000) \leq 0.02: \frac{1000 - 6\mu}{\sqrt{600}} \leq -2.0537$$

M1 A1

$$6\mu \geq 1000 + 2.0537 \times 24.49 = 1050.3, \text{ so } \mu \geq 175$$

A1

Answer: $\mu \approx 175$ **4 marks****30 (9 marks)**

$$(a) P(Y \leq y) = P(aX + b \leq y) = P(X \leq \frac{y-b}{a}) \text{ since } a > 0$$

M1

$$= \Phi\left(\frac{y-b}{a - \mu/\sigma}\right) = \Phi\left(\frac{y - (a\mu + b)}{a\sigma}\right)$$

M1 A1

This is the distribution function of a normal variable with mean $a\mu + b$ and standard deviation $a\sigma$, so $Y \sim N(a\mu + b, a^2\sigma^2)$

M1**A1****Answer: $Y \sim N(a\mu + b, a^2\sigma^2)$** **5 marks**

$$(b) E(F) = 1.8 \times 20 + 32 = 68$$

B1

$$\text{Var}(F) = 1.8^2 \times 9 = 29.16, \text{ so } F \sim N(68, 5.4^2)$$

M1

$$P(F > 75) = P(Z > \frac{7}{5.4}) = P(Z > 1.296)$$

M1

$$= 0.0974$$

A1**Answer: $N(68, 5.4^2)$; 0.0974****4 marks**

Expectation and Variance of Continuous Variables

1 (2 marks)

$$E(X) = \int_0^2 x \times \frac{x}{2} dx = \left[\frac{x^3}{6}\right]_0^2$$

M1

$$= \frac{8}{6} = \frac{4}{3}$$

A1**Answer: $\frac{4}{3}$** **2 marks****2 (2 marks)**

$$E(X) = \int_0^4 \frac{x}{4} dx = \left[\frac{x^2}{8}\right]_0^4 = 2$$

M1

The graph is symmetrical about $x = 2$, so the mean is 2

A1**Answer: 2****2 marks****3 (3 marks)**

$$E(2X + 1) = 2 \times 3 + 1 = 7$$

B1

$$\text{Var}(2X + 1) = 2^2 \times 2 = 8$$

B1

$$E(X^2) = \text{Var}(X) + [E(X)]^2 = 2 + 9 = 11$$

B1**Answer: 7, 8, 11****3 marks****4 (2 marks)**

For a symmetrical distribution the mean is the line of symmetry: $E(X) = 3$

B1

The median is also 3

B1**Answer: $E(X) = 3$, median = 3****2 marks****5 (2 marks)**

$$E(X^2) = 4 + 5^2 = 29$$

B1

$$\text{Standard deviation} = \sqrt{4} = 2$$

B1**Answer: 29, 2****2 marks****6 (2 marks)**

$$E(X) = \int_0^2 \frac{3x^3}{8} dx = \left[\frac{3x^4}{32}\right]_0^2$$

M1

$$= \frac{3}{2}$$

A1**Answer: $\frac{3}{2}$** **2 marks**

7 (6 marks)

$$E(X) = \int_0^4 \frac{3}{32}(4x^2 - x^3) dx = \frac{3}{32} \left[\frac{4x^3}{3} - \frac{x^4}{4} \right]_0^4$$

$$= \frac{3}{32} \left(\frac{256}{3} - 64 \right) = 2$$

M1

A1

$$E(X^2) = \int_0^4 \frac{3}{32}(4x^3 - x^4) dx = \frac{3}{32} \left[x^4 - \frac{x^5}{5} \right]_0^4$$

$$= \frac{3}{32} \left(256 - \frac{1024}{5} \right) = \frac{24}{5}$$

M1 A1

A1

$$\text{Var}(X) = \frac{24}{5} - 4 = \frac{4}{5}$$

A1

Answer: $E(X) = 2$, $\text{Var}(X) = \frac{4}{5}$

6 marks

8 (5 marks)

The graph of f is symmetrical about $x = \frac{1}{2}$, so $E(X) = \frac{1}{2}$

B1

$$E(X^2) = \int_0^1 6x^3(1 - x) dx = 6 \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1$$

$$= 6 \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{3}{10}$$

M1 A1

A1

$$\text{Var}(X) = \frac{3}{10} - \left(\frac{1}{2} \right)^2 = \frac{1}{20}$$

A1

Answer: $E(X) = \frac{1}{2}$, $\text{Var}(X) = \frac{1}{20}$

5 marks

9 (4 marks)

$$E\left(\frac{1}{X}\right) = \int_0^2 \frac{1}{x} \times \frac{x}{2} dx = \int_0^2 \frac{1}{2} dx$$

$$= 1$$

M1

A1

$$E(X^3) = \int_0^2 x^3 \times \frac{x}{2} dx = \left[\frac{x^5}{10} \right]_0^2$$

$$= \frac{32}{10} = \frac{16}{5}$$

M1

A1

Answer: $E(1/X) = 1$, $E(X^3) = \frac{16}{5}$

4 marks

10 (6 marks)

$$E(X) = \int_0^4 \frac{x^2}{8} dx = \left[\frac{x^3}{24} \right]_0^4 = \frac{8}{3}$$

M1 A1

$$E(X^2) = \int_0^4 \frac{x^3}{8} dx = \left[\frac{x^4}{32} \right]_0^4 = 8$$

M1

$$\text{Var}(X) = 8 - \frac{64}{9} = \frac{8}{9}$$

A1

$$E(P) = 5 \times \frac{8}{3} - 2 = \frac{34}{3}$$

A1

$$\text{Var}(P) = 25 \times \frac{8}{9} = \frac{200}{9}$$

A1

Answer: $E(P) = \frac{34}{3}$, $\text{Var}(P) = \frac{200}{9}$

6 marks

11 (4 marks)

$$E(3X - 2Y) = 3 \times 4 - 2 \times 2 = 8$$

M1 A1

$$\text{Var}(3X - 2Y) = 9 \times 3 + 4 \times 5$$

M1

$$= 47$$

A1

Answer: $E = 8$, $\text{Var} = 47$

4 marks

12 (5 marks)

(a) The graph decreases from $x = 0$, so the mode is 0

B1

$$1 - e^{-m} = \frac{1}{2}, \text{ so } e^{-m} = \frac{1}{2}$$

M1

$$m = \ln 2 \approx 0.693$$

A1

Answer: mode 0, median $\ln 2$

3 marks

(b) Mode < median < mean ($0 < 0.693 < 1$)

M1

So the distribution is positively skewed

A1

Answer: Positive skew

2 marks

13 (4 marks)

$$\int_0^2 x \times \frac{x}{6} dx = \left[\frac{x^3}{18} \right]_0^2 = \frac{4}{9}$$

M1 A1

$$\int_2^6 x \times \frac{6-x}{12} dx = \frac{1}{12} \left[3x^2 - \frac{x^3}{3} \right]_2^6$$

M1

$$= \frac{1}{12} \left(36 - \frac{28}{3} \right) = \frac{20}{9}, \text{ so } E(X) = \frac{4}{9} + \frac{20}{9} = \frac{8}{3}$$

A1

Answer: $\frac{8}{3}$

4 marks

14 (6 marks)

$$E(T) = \int_0^{12} \frac{t^2}{72} dt = \left[\frac{t^3}{216} \right]_0^{12} = 8$$

M1 A1

$$E(T^2) = \int_0^{12} \frac{t^3}{72} dt = \left[\frac{t^4}{288} \right]_0^{12}$$

M1

$$= \frac{20736}{288} = 72$$

A1

$$\text{Var}(T) = 72 - 8^2 = 8$$

M1

$$\text{So } E(T) = \text{Var}(T) = 8$$

A1

Answer: $E(T) = \text{Var}(T) = 8$

6 marks

15 (6 marks)

$$\int_0^2 (ax + b) dx = 2a + 2b = 1$$

M1 A1

$$E(X) = \int_0^2 (ax^2 + bx) dx = \frac{8a}{3} + 2b = \frac{7}{6}$$

M1 A1

$$\text{Substituting } b = \frac{1}{2} - a: \frac{8a}{3} + 1 - 2a = \frac{7}{6}, \text{ so } \frac{2a}{3} = \frac{1}{6}$$

M1

$$a = \frac{1}{4}, b = \frac{1}{4}$$

A1

Answer: $a = \frac{1}{4}, b = \frac{1}{4}$

6 marks

16 (7 marks)

$$(a) E(X) = \int_0^2 x \times \frac{5x^4}{32} dx = \left[\frac{5x^6}{192} \right]_0^2 = \frac{5}{3}$$

M1 A1

$$E(X^2) = \int_0^2 \frac{5x^6}{32} dx = \left[\frac{5x^7}{224} \right]_0^2 = \frac{20}{7}$$

M1

$$\text{Var}(X) = \frac{20}{7} - \left(\frac{5}{3} \right)^2 = \frac{5}{63}$$

A1

Answer: $E(X) = \frac{5}{3}, \text{Var}(X) = \frac{5}{63}$

4 marks

(b) f is increasing on the range, so the mode is 2

B1

$$\text{Measure} = \frac{5}{3 - 2\sqrt{\frac{5}{63}}}$$

M1

$= -1.18$, which is negative, so the distribution has negative skew

A1

Answer: Mode 2; -1.18 (negative skew)

3 marks

17 (6 marks)

$$E(X - Y) = 12 - 8 = 4$$

B1

$$\text{Var}(X - Y) = 4 + 9 = 13$$

M1 A1

$$\text{Var}\left(\frac{X+Y}{2}\right) = \frac{1}{4} \times \text{Var}(X + Y) = \frac{1}{4}(4 + 9)$$

M1 A1

$$\text{sd} = \frac{\sqrt{13}}{2} \approx 1.80$$

A1

$$\text{Answer: } 4, 13, \frac{\sqrt{13}}{2} \approx 1.80$$

6 marks

18 (7 marks)

$$E(X) = \int_1^e x \times \frac{1}{x} dx = [x]_1^e = e - 1$$

M1 A1

$$E(\ln X) = \int_1^e \frac{\ln x}{x} dx = \left[\frac{(\ln x)^2}{2}\right]_1^e = \frac{1}{2}$$

M1 A1

$$E((\ln X)^2) = \int_1^e \frac{(\ln x)^2}{x} dx = \left[\frac{(\ln x)^3}{3}\right]_1^e = \frac{1}{3}$$

M1 A1

$$\text{Var}(\ln X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

A1

$$\text{Answer: } e - 1, \frac{1}{2}, \frac{1}{12}$$

7 marks

19 (8 marks)

$$(a) \int_0^1 \frac{1}{2}x^{-1/2} dx = [x^{1/2}]_0^1$$

M1

= 1, and $f(x) > 0$ on the range

A1

Answer: Valid

2 marks

$$(b) E(X) = \int_0^1 \frac{1}{2}x^{1/2} dx = \left[\frac{x^{3/2}}{3}\right]_0^1 = \frac{1}{3}$$

M1 A1

$$E(X^2) = \int_0^1 \frac{1}{2}x^{3/2} dx = \left[\frac{x^{5/2}}{5}\right]_0^1 = \frac{1}{5}$$

M1

$$\text{Var}(X) = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}$$

A1

$$\text{Answer: } E(X) = \frac{1}{3}, \text{Var}(X) = \frac{4}{45}$$

4 marks

$$(c) E(\sqrt{X}) = \int_0^1 \sqrt{x} \times \frac{1}{2\sqrt{x}} dx = \int_0^1 \frac{1}{2} dx$$

M1

$$= \frac{1}{2}$$

A1

$$\text{Answer: } \frac{1}{2}$$

2 marks

20 (5 marks)

The formula is $\text{Var}(X) = E(X^2) - [E(X)]^2$; the student has not squared the mean

E1

$$\text{Var}(X) = 8 - \left(\frac{8}{3}\right)^2 = 8 - \frac{64}{9} = \frac{8}{9}$$

M1 A1

$$\text{Var}(Y) = 9 \times \frac{8}{9} = 8$$

M1

$$\text{Standard deviation} = \sqrt{8} = 2\sqrt{2} \approx 2.83$$

A1

$$\text{Answer: } \text{Var}(X) = \frac{8}{9}; \text{sd}(Y) = 2\sqrt{2}$$

5 marks

21 (8 marks)

$$\int_0^{\infty} x^2 e^{-x} dx = [-x^2 e^{-x}]_0^{\infty} + \int_0^{\infty} 2xe^{-x} dx$$

M1 A1

$$= 0 + 2 \int_0^{\infty} xe^{-x} dx$$

A1

$$\int_0^{\infty} xe^{-x} dx = [-xe^{-x}]_0^{\infty} + \int_0^{\infty} e^{-x} dx = 0 + 1 = 1$$

M1 A1

$$\text{So } E(X) = 2 \times 1 = 2$$

A1

$$E(X^2) = \int_0^{\infty} x^3 e^{-x} dx = [-x^3 e^{-x}]_0^{\infty} + 3 \int_0^{\infty} x^2 e^{-x} dx = 3 \times 2 = 6$$

M1

$$\text{Var}(X) = 6 - 4 = 2$$

A1

$$\text{Answer: } E(X) = 2, E(X^2) = 6, \text{Var}(X) = 2$$

8 marks

22 (8 marks)

$$E(X) = \frac{6}{5} \int_0^1 x^3 dx + \frac{6}{5} \int_1^2 (2x - x^2) dx$$

M1

$$= \frac{6}{5} \times \frac{1}{4} + \frac{6}{5} \left[x^2 - \frac{x^3}{3} \right]_1^2$$

M1 A1

$$= \frac{3}{10} + \frac{6}{5} \times \frac{2}{3} = \frac{3}{10} + \frac{4}{5} = \frac{11}{10}$$

A1

$$E(X^2) = \frac{6}{5} \int_0^1 x^4 dx + \frac{6}{5} \int_1^2 (2x^2 - x^3) dx$$

M1

$$= \frac{6}{5} \times \frac{1}{5} + \frac{6}{5} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_1^2$$

A1

$$= \frac{6}{25} + \frac{6}{5} \times \frac{11}{12} = \frac{6}{25} + \frac{11}{10}$$

A1

$$= \frac{67}{50}, \text{ so } \text{Var}(X) = \frac{67}{50} - \frac{121}{100} = \frac{13}{100}$$

A1

$$\text{Answer: } E(X) = \frac{11}{10}, \text{Var}(X) = \frac{13}{100}$$

8 marks

23 (7 marks)

$$(a) E(X) = 12 \int_0^1 (x^3 - x^4) dx = 12 \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{3}{5}$$

M1 A1

$$E(X^2) = 12 \int_0^1 (x^4 - x^5) dx = 12 \left(\frac{1}{5} - \frac{1}{6} \right) = \frac{2}{5}$$

M1 A1

$$\text{Var}(X) = \frac{2}{5} - \frac{9}{25} = \frac{1}{25}$$

A1

$$\text{Answer: } E(X) = \frac{3}{5}, \text{Var}(X) = \frac{1}{25}$$

5 marks

(b) Mean $0.6 <$ median $0.614 <$ mode 0.667

M1

So the distribution is negatively skewed

A1

Answer: Negative skew

2 marks

24 (8 marks)

$$(a) E(C) = 50 + 20E(X^2)$$

M1

$$E(X^2) = \int_0^2 \frac{3x^4}{8} dx = \frac{12}{5}$$

M1 A1

$$E(C) = 50 + 48 = 98$$

A1

Answer: £98

4 marks

$$(b) 50 + 20X^2 > 120, \text{ so } X^2 > 3.5$$

M1

$$X > \sqrt{3.5} \approx 1.871$$

A1

$$P(X > 1.871) = 1 - \frac{x^3}{8} \text{ evaluated at } 1.871$$

M1

$$= 1 - \frac{6.549}{8} \approx 0.182$$

A1

Answer: 0.182

4 marks

25 (8 marks)

(a) $f(x) = F'(x)$

M1

$$f(x) = \frac{24}{x^4} \text{ for } x \geq 2$$

A1

Answer: $f(x) = \frac{24}{x^4}$ for $x \geq 2$

2 marks

(b) $E(X) = \int_2^{\infty} x \times \frac{24}{x^4} dx = \left[-\frac{12}{x^2}\right]_2^{\infty} = 3$

M1 A1

$$E(X^2) = \int_2^{\infty} \frac{24}{x^2} dx = \left[-\frac{24}{x}\right]_2^{\infty} = 12$$

M1

$$\text{Var}(X) = 12 - 9 = 3$$

A1

Answer: $E(X) = 3$, $\text{Var}(X) = 3$

4 marks

(c) $1 - \frac{8}{m^3} = \frac{1}{2}$, so $m^3 = 16$ and $m = 2.52$

M1

Mean $3 >$ median 2.52 , which suggests positive skew

A1

Answer: Median 2.52; positive skew

2 marks

26 (6 marks)

$$E(aX + b) = \int_{-\infty}^{\infty} (ax + b)f(x) dx = a \int_{-\infty}^{\infty} xf(x) dx + b \int_{-\infty}^{\infty} f(x) dx$$

M1 A1

$$= a\mu + b \times 1 = a\mu + b$$

A1

$$\text{Var}(aX + b) = E[(aX + b - (a\mu + b))^2]$$

M1

$$= E[a^2(X - \mu)^2] = a^2 \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

M1

$$= a^2 \sigma^2$$

A1

Answer: $E(aX + b) = a\mu + b$, $\text{Var}(aX + b) = a^2 \sigma^2$

6 marks

27 (8 marks)

The graph is symmetrical about $x = 2.5$, so $E(X) = \frac{5}{2}$

B1

$$E(X^2) = \frac{6}{125} \int_0^5 (5x^3 - x^4) dx = \frac{6}{125} \left[\frac{5x^4}{4} - \frac{x^5}{5} \right]_0^5$$

M1 A1

$$= \frac{6}{125} \left(\frac{3125}{4} - 625 \right) = \frac{6}{125} \times \frac{625}{4} = \frac{15}{2}$$

A1

$$\text{Var}(X) = \frac{15}{2} - \frac{25}{4} = \frac{5}{4}$$

A1

$$sd = \frac{\sqrt{5}}{2} \approx 1.118, \text{ so the interval is } (1.382, 3.618)$$

M1

$$P = F(3.618) - F(1.382) \text{ with } F(x) = \frac{6}{125} \left(\frac{5x^2}{2} - \frac{x^3}{3} \right)$$

M1

$$\approx 0.813 - 0.187 = 0.626$$

A1

Answer: $E(X) = \frac{5}{2}$, $\text{Var}(X) = \frac{5}{4}$, probability ≈ 0.626

8 marks

28 (10 marks)

(a) $E(X) = \int_1^{\infty} \frac{3}{x^3} dx = \left[-\frac{3}{2x^2}\right]_1^{\infty} = \frac{3}{2}$

M1 A1

$$E(X^2) = \int_1^{\infty} \frac{3}{x^2} dx = \left[-\frac{3}{x}\right]_1^{\infty} = 3$$

M1 A1

$$\text{Var}(X) = 3 - \frac{9}{4} = \frac{3}{4}$$

A1

Answer: $E(X) = \frac{3}{2}$, $\text{Var}(X) = \frac{3}{4}$

5 marks

(b) $E(Y) = \int_1^{\infty} \frac{2}{y^2} dy = \left[-\frac{2}{y}\right]_1^{\infty} = 2$

M1 A1

$$E(Y^2) = \int_1^{\infty} \frac{2}{y} dy = [2 \ln y]_1^{\infty}$$

M1 A1

This integral diverges ($2 \ln y \rightarrow \infty$), so $E(Y^2)$ and hence $\text{Var}(Y)$ do not exist

E1

Answer: $E(Y) = 2$; $\text{Var}(Y)$ does not exist**5 marks****29 (12 marks)**

(a) $\int_0^1 (ax^2 + bx) dx = \frac{a}{3} + \frac{b}{2} = 1$

M1 A1

$E(X) = \int_0^1 (ax^3 + bx^2) dx = \frac{a}{4} + \frac{b}{3} = 0.6$

M1 A1Solving: $a = -2.4$, $b = 3.6$ **A1****Answer: $a = -2.4$, $b = 3.6$** **5 marks****(b)** $f(x) = 1.2x(3 - 2x)$, and $3 - 2x \geq 1 > 0$ for $0 \leq x \leq 1$, so $f(x) \geq 0$ **B1**

$f'(x) = 3.6 - 4.8x = 0$

M1 $x = 0.75$, a maximum since $f'' < 0$ **A1 A1****Answer: Mode = 0.75****4 marks**

(c) $E(X^2) = \int_0^1 (-2.4x^4 + 3.6x^3) dx = -0.48 + 0.9 = 0.42$

M1 A1

$\text{Var}(X) = 0.42 - 0.36 = 0.06$

A1**Answer: 0.06****3 marks****30 (11 marks)**

(a) Integrate by parts: $\int_0^\infty [1 - F(x)] dx = [x(1 - F(x))]_0^\infty - \int_0^\infty x(-f(x)) dx$

M1 A1The first term is 0 (at 0 because of the factor x , at ∞ by the given condition)**B1**

$= \int_0^\infty xf(x) dx = E(X)$

A1

Answer: $E(X) = \int_0^\infty [1 - F(x)] dx$

4 marks

(b) $P(M \leq x) = x \times x = x^2$ for $0 \leq x \leq 1$

M1 A1

$E(M) = \int_0^1 (1 - x^2) dx$

M1

$= 1 - \frac{1}{3} = \frac{2}{3}$

A1

Answer: $E(M) = \frac{2}{3}$

4 marks

(c) $P(m > x) = (1 - x)^2$

M1

$E(m) = \int_0^1 (1 - x)^2 dx = \frac{1}{3}$

A1

$E(M) + E(m) = 1 = \frac{1}{2} + \frac{1}{2}$

B1

Answer: $E(m) = \frac{1}{3}$

3 marks**Transformations of Random Variables****1 (3 marks)**When $X = 0$, $Y = 1$ and when $X = 2$, $Y = 7$, so $1 \leq Y \leq 7$ **B1**

Width multiplied by 3, so the density is divided by 3

M1

$g(y) = \frac{1}{6}$ for $1 \leq y \leq 7$

A1

Answer: $g(y) = \frac{1}{6}$ on $[1, 7]$

3 marks**2 (2 marks)**

$G(y) = P(2X \leq y) = P(X \leq \frac{y}{2})$

M1

$= \frac{y}{2}$ for $0 \leq y \leq 2$ (0 below, 1 above)

A1

Answer: $G(y) = \frac{y}{2}$ for $0 \leq y \leq 2$

2 marks

3 (2 marks)

$$G(y) = P(X + 1 \leq y) = P(X \leq y - 1) = F(y - 1)$$

M1

$$G(y) = \frac{(y - 1)^2}{4} \text{ for } 1 \leq y \leq 3$$

A1

$$\text{Answer: } G(y) = \frac{(y - 1)^2}{4}, 1 \leq y \leq 3$$

2 marks

4 (3 marks)

$$0 \leq X \leq 3 \text{ gives } 0 \leq Y \leq 9$$

B1

$$P(Y \leq 4) = P(X \leq 2) \text{ since } X \geq 0$$

M1

$$= \int_0^2 \frac{2x}{9} dx = \frac{4}{9}$$

A1

$$\text{Answer: } 0 \leq Y \leq 9; \frac{4}{9}$$

3 marks

5 (2 marks)

$$Y \leq -1 \text{ means } X \geq 1$$

M1

$$P(X \geq 1) = \frac{3}{4}$$

A1

$$\text{Answer: } \frac{3}{4}$$

2 marks

6 (3 marks)

$$1 \leq X \leq 4 \text{ gives } \frac{1}{4} \leq Y \leq 1$$

B1

$$Y > \frac{1}{2} \text{ means } X < 2$$

M1

$$P(X < 2) = \frac{1}{3}$$

A1

$$\text{Answer: } \frac{1}{4} \leq Y \leq 1; \frac{1}{3}$$

3 marks

7 (5 marks)

$$G(y) = P(2X \leq y) = P(X \leq \frac{y}{2})$$

M1

$$= \int_0^{y/2} \frac{3x^2}{8} dx = \left[\frac{x^3}{8} \right]_0^{y/2}$$

M1 A1

$$G(y) = \frac{y^3}{64} \text{ for } 0 \leq y \leq 4$$

A1

$$g(y) = G'(y) = \frac{3y^2}{64} \text{ for } 0 \leq y \leq 4$$

A1

$$\text{Answer: } G(y) = \frac{y^3}{64}, g(y) = \frac{3y^2}{64}, 0 \leq y \leq 4$$

5 marks

8 (5 marks)

$$E(X) = \int_0^2 \frac{x^2}{2} dx = \frac{4}{3}$$

M1 A1

$$E(Y) = \frac{4}{3} + 3 = \frac{13}{3}$$

A1

$$Y = X + 3 \text{ shifts the graph: } g(y) = f(y - 3) = \frac{y - 3}{2} \text{ for } 3 \leq y \leq 5$$

M1 A1

$$\text{Answer: } E(Y) = \frac{13}{3}; g(y) = \frac{y - 3}{2}, 3 \leq y \leq 5$$

5 marks

9 (4 marks)

$$G(y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) \quad \text{M1}$$

$$= \frac{\sqrt{y}}{2} \text{ for } 0 \leq y \leq 4 \quad \text{A1}$$

$$g(y) = G'(y) = \frac{1}{2} \times \frac{1}{2} y^{-1/2} \quad \text{M1}$$

$$= \frac{1}{4\sqrt{y}} \quad \text{A1}$$

Answer: $g(y) = \frac{1}{4\sqrt{y}}$ 4 marks

10 (5 marks)

$$X \leq 1 \text{ so } Y \geq 1 \quad \text{B1}$$

$$G(y) = P\left(\frac{1}{X} \leq y\right) = P(X \geq \frac{1}{y}) \quad \text{M1}$$

$$= 1 - P(X < \frac{1}{y}) = 1 - \left(\frac{1}{y}\right)^2 \text{ since } F(x) = x^2 \quad \text{M1 A1}$$

$$g(y) = G'(y) = \frac{2}{y^3} \text{ for } y \geq 1 \quad \text{A1}$$

Answer: $g(y) = \frac{2}{y^3}, y \geq 1$ 5 marks

11 (4 marks)

$$1 \leq Y \leq e \quad \text{B1}$$

$$G(y) = P(e^X \leq y) = P(X \leq \ln y) = \ln y \text{ for } 1 \leq y \leq e \quad \text{M1 A1}$$

$$g(y) = \frac{1}{y} \quad \text{A1}$$

Answer: $G(y) = \ln y, g(y) = \frac{1}{y}, 1 \leq y \leq e$ 4 marks

12 (4 marks)

$$Y \geq 0 \quad \text{B1}$$

$$G(y) = P(-\ln X \leq y) = P(X \geq e^{-y}) \quad \text{M1}$$

$$= 1 - e^{-y} \text{ for } y \geq 0 \quad \text{A1}$$

This is the cdf of $\text{Exp}(1)$, so $\lambda = 1$ A1

Answer: $Y \sim \text{Exp}(1)$ 4 marks

13 (4 marks)

$$G(y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = \frac{y^2}{9} \text{ for } 0 \leq y \leq 3 \quad \text{M1 A1}$$

$$g(y) = \frac{2y}{9} \quad \text{A1}$$

$$P(Y > 2) = 1 - \frac{4}{9} = \frac{5}{9} \quad \text{A1}$$

Answer: $g(y) = \frac{2y}{9}, 0 \leq y \leq 3; \frac{5}{9}$ 4 marks

14 (6 marks)

$$0 \leq Y \leq 4 \quad \text{B1}$$

$$G(y) = P(4 - X^2 \leq y) = P(X^2 \geq 4 - y) \quad \text{M1}$$

$$= P(X \geq \sqrt{4 - y}) = 1 - \frac{\sqrt{4 - y}}{2} \quad \text{M1 A1}$$

$$g(y) = G'(y) = \frac{1}{4\sqrt{4 - y}} \text{ for } 0 \leq y < 4 \quad \text{M1 A1}$$

Answer: $G(y) = 1 - \frac{\sqrt{4 - y}}{2}, g(y) = \frac{1}{4\sqrt{4 - y}}$ 6 marks

15 (6 marks)

$$\frac{1}{2} \leq Y \leq 1$$

B1

$$G(y) = P(X \geq \frac{1}{y}) = 2 - \frac{1}{y}$$

M1 A1

$$g(y) = \frac{1}{y^2}$$

A1

$$E(Y) = \int_{1/2}^1 y \times \frac{1}{y^2} dy = [\ln y]_{1/2}^1$$

M1

$$= \ln 2 \approx 0.693$$

A1

$$\text{Answer: } g(y) = \frac{1}{y^2}; E(Y) = \ln 2$$

6 marks

16 (6 marks)

$$0 \leq Y \leq 4$$

B1

$$G(y) = P(X \leq \sqrt{y}) = \frac{(\sqrt{y})^3}{8} = \frac{y^{3/2}}{8}$$

M1 A1

$$g(y) = \frac{3}{16} y^{1/2} = \frac{3\sqrt{y}}{16}$$

A1

$$\int_0^4 \frac{3}{16} y^{1/2} dy = \frac{3}{16} \left[\frac{2y^{3/2}}{3} \right]_0^4$$

M1

$$= \frac{3}{16} \times \frac{16}{3} = 1$$

A1

$$\text{Answer: } g(y) = \frac{3\sqrt{y}}{16}, 0 \leq y \leq 4$$

6 marks

17 (4 marks)

$$F(x) = \int_1^x \frac{1}{t} dt = \ln x \text{ for } 1 \leq x \leq e$$

M1 A1

$$G(y) = P(\ln X \leq y) = P(X \leq e^y) = F(e^y) = y$$

M1

$$\text{for } 0 \leq y \leq 1, \text{ so } Y \sim U[0, 1]$$

A1

$$\text{Answer: } Y \sim U[0, 1]$$

4 marks

18 (6 marks)

$$5 \leq C \leq 13$$

B1

$$G(c) = P(2X + 5 \leq c) = P(X \leq \frac{c-5}{2})$$

M1

$$g(c) = \frac{1}{2} f\left(\frac{c-5}{2}\right) = \frac{1}{2} \times \frac{c-5}{2+1/12}$$

M1 A1

$$= \frac{c-3}{48} \text{ for } 5 \leq c \leq 13$$

A1

$$P(C > 11) = P(X > 3) = \int_3^4 \frac{x+1}{12} dx = \frac{1}{12} \left[\frac{x^2}{2} + x \right]_3^4 = \frac{1}{12} \left(12 - \frac{15}{2} \right)$$

A1

$$\text{Answer: } g(c) = \frac{c-3}{48}; \frac{3}{8}$$

6 marks

19 (6 marks)

$$F(x) = \int_1^x \frac{2}{t^3} dt = 1 - \frac{1}{x^2} \text{ for } x \geq 1$$

M1 A1

$$Y \geq 1; G(y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F(\sqrt{y})$$

M1

$$G(y) = 1 - \frac{1}{y} \text{ for } y \geq 1$$

A1

$$g(y) = G'(y) = \frac{1}{y^2}$$

A1

$$P(Y > 9) = 1 - G(9) = \frac{1}{9}$$

A1

$$\text{Answer: } G(y) = 1 - \frac{1}{y}, g(y) = \frac{1}{y^2}; \frac{1}{9}$$

6 marks

20 (5 marks)

With $g(y) = \frac{1}{4}$ on $[0, 16]$ the total area is $16 \times \frac{1}{4} = 4 \neq 1$

E1

$$G(y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = \frac{\sqrt{y}}{4}$$

M1 A1

$$g(y) = G'(y) = \frac{1}{8\sqrt{y}} \text{ for } 0 < y \leq 16$$

M1

$$\text{Check: } \int_0^{16} \frac{1}{8\sqrt{y}} dy = \left[\frac{\sqrt{y}}{4} \right]_0^{16} = 1$$

A1

$$\text{Answer: } g(y) = \frac{1}{8\sqrt{y}}$$

5 marks

21 (8 marks)

$$0 \leq Y \leq 8$$

B1

$$G(y) = P(X^3 \leq y) = P(X \leq y^{1/3}) = \frac{y^{1/3}}{2}$$

M1 A1

$$g(y) = \frac{1}{6}y^{-2/3} \text{ for } 0 < y \leq 8$$

A1

$$E(Y) = \int_0^8 y \times \frac{1}{6}y^{-2/3} dy = \frac{1}{6} \left[\frac{3y^{4/3}}{4} \right]_0^8$$

M1

$$= \frac{1}{6} \times \frac{3 \times 16}{4} = 2$$

A1

$$\text{Alternatively } E(X^3) = \int_0^2 \frac{x^3}{2} dx$$

M1

$$= \left[\frac{x^4}{8} \right]_0^2 = 2$$

A1

$$\text{Answer: } G(y) = \frac{y^{1/3}}{2}, g(y) = \frac{1}{6}y^{-2/3}, E(Y) = 2$$

8 marks

22 (6 marks)

$$0 \leq Y \leq 1$$

B1

$$G(y) = P(|X| \leq y) = P(-y \leq X \leq y)$$

M1

$$= \int_{-y}^y \frac{3}{4}(1 - x^2) dx = 2 \int_0^y \frac{3}{4}(1 - x^2) dx \text{ (symmetry)}$$

M1

$$= \frac{3}{2} \left[x - \frac{x^3}{3} \right]_0^y = \frac{3}{2} \left(y - \frac{y^3}{3} \right)$$

A1

$$g(y) = G'(y) = \frac{3}{2}(1 - y^2)$$

M1 A1

$$\text{Answer: } g(y) = \frac{3}{2}(1 - y^2)$$

6 marks

23 (9 marks)

(a) $X = 2$ gives $Y = 0$ (the minimum)

B1

$X = 0$ gives $Y = 4$, which is the maximum ($X = 3$ gives 1): $0 \leq Y \leq 4$

B1

Answer: $0 \leq Y \leq 4$

2 marks

(b) $Y \leq y$ means $2 - \sqrt{y} \leq X \leq 2 + \sqrt{y}$

M1 A1

For $0 \leq y \leq 1$: both ends lie in $[0, 3]$, so $G(y) = \frac{2\sqrt{y}}{3}$

M1 A1

For $1 < y \leq 4$: $2 + \sqrt{y} > 3$, so only $2 - \sqrt{y} \leq X \leq 3$: $G(y) = \frac{3 - (2 - \sqrt{y})}{3} = \frac{1 + \sqrt{y}}{3}$

M1 A1

$$g(y) = \frac{1}{3\sqrt{y}} \text{ for } 0 < y \leq 1, \text{ and } g(y) = \frac{1}{6\sqrt{y}} \text{ for } 1 < y \leq 4$$

A1

$$\text{Answer: } g(y) = \frac{1}{3\sqrt{y}} \text{ (} 0 < y \leq 1 \text{), } \frac{1}{6\sqrt{y}} \text{ (} 1 < y \leq 4 \text{)}$$

7 marks

24 (7 marks)

$$X \geq 0 \text{ since } 0 < 1 - U \leq 1$$

B1

$$P(X \leq x) = P\left(-\frac{1}{\lambda} \ln(1 - U) \leq x\right) = P(\ln(1 - U) \geq -\lambda x)$$

M1

$$= P(1 - U \geq e^{-\lambda x}) = P(U \leq 1 - e^{-\lambda x})$$

M1

$$= 1 - e^{-\lambda x}, \text{ the cdf of } \text{Exp}(\lambda)$$

A1

$$U = 0.2: X = -2 \ln 0.8 = 0.446$$

M1

$$U = 0.5: X = 1.39$$

A1

$$U = 0.9: X = 4.61$$

A1

Answer: 0.446, 1.39, 4.61

7 marks

25 (7 marks)

$$P(M \leq x) = P(X_1 \leq x)P(X_2 \leq x)P(X_3 \leq x)$$

M1

$$= (1 - e^{-x})^3 \text{ for } x \geq 0$$

A1

$$P(m > x) = P(X_1 > x)P(X_2 > x)P(X_3 > x) = e^{-3x}$$

M1 A1

$$\text{So the cdf of } m \text{ is } 1 - e^{-3x}, \text{ i.e. } m \sim \text{Exp}(3)$$

A1

$$P(m > 0.5) = e^{-1.5}$$

M1

$$\approx 0.223$$

A1

Answer: $(1 - e^{-x})^3$; $1 - e^{-3x}$; $e^{-1.5} \approx 0.223$

7 marks

26 (7 marks)

$$\frac{1}{4} \leq Y \leq 1$$

B1

$$G(y) = P\left(\frac{1}{1+X} \leq y\right) = P\left(1+X \geq \frac{1}{y}\right) = P\left(X \geq \frac{1}{y} - 1\right)$$

M1 A1

$$= 1 - \frac{1}{3}\left(\frac{1}{y} - 1\right) = \frac{4 - 1}{y/3}$$

M1

$$g(y) = G'(y) = \frac{1}{3y^2} \text{ for } \frac{1}{4} \leq y \leq 1$$

A1

$$P(Y > 0.4) = P(1 + X < 2.5) = P(X < 1.5)$$

M1

$$= \frac{1}{2}$$

A1

Answer: $g(y) = \frac{1}{3y^2}$; $\frac{1}{2}$

7 marks

27 (8 marks)

(a) (X_1, X_2) is uniformly distributed over the unit square, so $P(S \leq s) = \text{area of the region } x_1 + x_2 \leq s$

M1

For $0 \leq s \leq 1$ this is a right-angled triangle with legs s : area $\frac{s^2}{2}$

M1 A1

For $1 < s \leq 2$ the region is the square minus the triangle with legs $2 - s$

M1

$$\text{Area} = 1 - \frac{(2-s)^2}{2}$$

A1

Answer: $G(s) = \frac{s^2}{2}$ for $0 \leq s \leq 1$; $G(s) = 1 - \frac{(2-s)^2}{2}$ for $1 < s \leq 2$

5 marks

(b) $g(s) = s$ for $0 \leq s \leq 1$ and $g(s) = 2 - s$ for $1 < s \leq 2$ (a triangle)

M1

$$P(S > 1.5) = 1 - G(1.5) = \frac{(0.5)^2}{2}$$

M1

$$= \frac{1}{8}$$

A1

Answer: $\frac{1}{8}$

3 marks

28 (9 marks)

$$(a) a > 0: G(y) = P(X \leq \frac{y-b}{a}) = F(\frac{y-b}{a}), \text{ so } g(y) = \frac{1}{a}f(\frac{y-b}{a}) \quad \text{M1 A1}$$

$$a < 0: \text{dividing by } a \text{ reverses the inequality, so } G(y) = P(X \geq \frac{y-b}{a}) = 1 - F(\frac{y-b}{a}) \quad \text{M1 A1}$$

$$\text{Differentiating, } g(y) = -\frac{1}{a}f(\frac{y-b}{a}) = \frac{1}{|a|}f(\frac{y-b}{a}) \quad \text{A1}$$

$$\text{Answer: } g(y) = \frac{1}{|a|} f(\frac{y-b}{a}) \quad \text{5 marks}$$

$$(b) \text{ Range: } X = 0 \text{ gives } Y = 5 \text{ and } X = 2 \text{ gives } Y = 1, \text{ so } 1 \leq Y \leq 5 \quad \text{B1}$$

$$g(y) = \frac{1}{2} f(\frac{5-y}{2}) \quad \text{M1}$$

$$= \frac{1}{2} \times \frac{3}{8} (\frac{5-y}{2})^2 = \frac{3(5-y)^2}{64} \quad \text{M1 A1}$$

$$\text{Answer: } g(y) = \frac{3(5-y)^2}{64}, 1 \leq y \leq 5 \quad \text{4 marks}$$

29 (10 marks)

$$(a) G(y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) \quad \text{M1}$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1+x}{2} dx = [\frac{x}{2} + \frac{x^2}{4}]_{-\sqrt{y}}^{\sqrt{y}} \quad \text{M1 A1}$$

$$= \frac{\sqrt{y}}{2} + \frac{\sqrt{y}}{2} = \sqrt{y} \quad \text{A1}$$

$$g(y) = \frac{1}{2\sqrt{y}} \text{ for } 0 < y \leq 1 \quad \text{A1}$$

$$\text{Answer: } g(y) = \frac{1}{2\sqrt{y}} \quad \text{5 marks}$$

$$(b) \text{ Using } g: E(Y) = \int_0^1 y \times \frac{1}{2} y^{-1/2} dy = [\frac{y^{3/2}}{3}]_0^1 = \frac{1}{3} \quad \text{M1 A1}$$

$$\text{Using } f: E(X^2) = \int_{-1}^1 x^2 \times \frac{1+x}{2} dx \quad \text{M1}$$

$$= \frac{1}{2} [\frac{x^3}{3} + \frac{x^4}{4}]_{-1}^1 \quad \text{A1}$$

$$= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}, \text{ agreeing} \quad \text{A1}$$

$$\text{Answer: } \frac{1}{3} \quad \text{5 marks}$$

30 (9 marks)

$$(a) G(y) = P(Z^2 \leq y) = P(-\sqrt{y} \leq Z \leq \sqrt{y}) \quad \text{M1}$$

$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = 2\Phi(\sqrt{y}) - 1 \quad \text{M1 A1}$$

$$g(y) = G'(y) = 2\phi(\sqrt{y}) \times \frac{1}{2\sqrt{y}} \quad \text{M1 A1}$$

$$g(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \text{ for } y > 0 \quad \text{A1}$$

$$\text{Answer: } g(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \quad \text{6 marks}$$

$$(b) P(Y < 1) = 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6827 \quad \text{M1}$$

$$E(Y) = E(Z^2) = \text{Var}(Z) + [E(Z)]^2 = 1 \quad \text{A1}$$

$$\text{Answer: } 0.6827; E(Y) = 1 \quad \text{B1}$$

3 marks