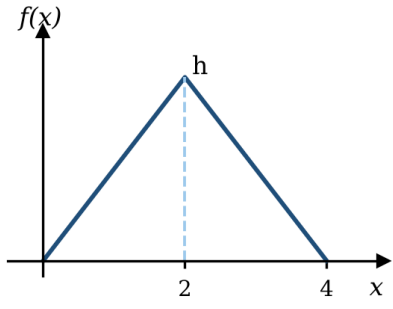


Continuous Random Variables

- 1** State whether each of the following random variables is discrete or continuous.
- (a) The time, in seconds, taken by an athlete to run 100 m. [1]
 - (b) The number of goals scored in a football match. [1]
 - (c) The mass, in kg, of a randomly chosen parcel. [1]
-
- 2** X is a continuous random variable with $P(X < 2.5) = 0.4$.
- (a) Write down $P(X = 2.5)$. [1]
 - (b) Find $P(X \geq 2.5)$. [1]
-
- 3** The continuous random variable X has probability density function $f(x) = k$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Find the value of k . [2]
-
- 4** The continuous random variable X has probability density function $f(x) = \frac{x}{8}$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Show that $f(x)$ is a valid probability density function. [3]
-
- 5** The continuous random variable X has probability density function $f(x) = \frac{x}{2}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. Find $P(X < 1)$. [2]
-
- 6** The continuous random variable X has probability density function $f(x) = 0.25$ for $1 \leq x \leq 5$, and $f(x) = 0$ otherwise. Find $P(2 < X < 4)$. [2]
-
- 7** The continuous random variable X has probability density function $f(x) = kx^2$ for $0 \leq x \leq 3$, and $f(x) = 0$ otherwise.
- (a) Find the value of k . [2]
 - (b) Find $P(X > 2)$. [2]
-
- 8** The diagram shows the probability density function $f(x)$ of a continuous random variable X . The graph is made of two straight line segments forming a triangle with the x -axis between $x = 0$ and $x = 4$, with the peak at $x = 2$ and height h .
- 
- (a) Find the value of h . [2]
 - (b) Find $P(X < 1)$. [2]
-
- 9** The continuous random variable X has probability density function $f(x) = \frac{3}{32}(4x - x^2)$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Find $P(1 < X < 3)$. [4]
-
- 10** For each of the following functions, decide whether it could be the probability density function of a continuous random variable on the given range. Give a reason in each case.
- (a) $f(x) = \frac{x}{3} - \frac{1}{3}$, $0 \leq x \leq 3$ [2]
 - (b) $g(x) = \frac{x^2}{12}$, $0 \leq x \leq 3$ [2]

- 11** The waiting time, T minutes, for a bus has probability density function $f(t) = \frac{(20 - t)}{200}$ for $0 \leq t \leq 20$, and $f(t) = 0$ otherwise. Find the probability that a passenger waits for more than 5 minutes. [3]
-
- 12** The continuous random variable X has probability density function $f(x) = \frac{3x^2}{8}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. Find the value a such that $P(X < a) = \frac{1}{4}$. [3]
-
- 13** The continuous random variable X has probability density function $f(x) = ke^{-2x}$ for $x \geq 0$, and $f(x) = 0$ otherwise.
 (a) Show that $k = 2$. [2]
 (b) Find the exact value of $P(X > 1)$. [2]
-
- 14** The continuous random variable X has probability density function $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \leq x \leq 1$, and $f(x) = 0$ otherwise.
 (a) Show that f is a valid probability density function. [3]
 (b) Find $P(X > \frac{1}{2})$. [3]
-
- 15** The continuous random variable X has probability density function $f(x) = kx^2$ for $1 \leq x \leq 2$, and $f(x) = 0$ otherwise. Find the value of k and hence find $P(X < 1.5)$. [5]
-
- 16** The continuous random variable X has probability density function $f(x) = \frac{x}{18}$ for $0 \leq x \leq a$, and $f(x) = 0$ otherwise.
 Find the value of a and hence find $P(X > 4)$. [5]
-
- 17** The continuous random variable X has probability density function $f(x) = \frac{3x^2}{8}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. Find $P(X > 1.5 | X > 1)$. [4]
-
- 18** The length, L cm, of a metal rod produced by a machine is modelled by a continuous random variable with probability density function $f(l) = k(l - 9)(11 - l)$ for $9 \leq l \leq 11$, and $f(l) = 0$ otherwise.
 (a) Show that $k = \frac{3}{4}$. [3]
 (b) A rod is acceptable if its length is within 0.5 cm of 10 cm. Find the probability that a randomly chosen rod is acceptable. [3]
-
- 19** A student is asked to find the value of k so that $f(x) = kx$ for $0 \leq x \leq 3$ is a probability density function. The student writes:
 " $f(3) = 1$, so $3k = 1$ and $k = \frac{1}{3}$."
 Explain the error in the student's method, and find the correct value of k . Hence find $P(X \geq 2)$. [5]
-
- 20** The continuous random variable X has probability density function $f(x) = kx$ for $0 \leq x \leq \frac{1}{2}$, and $f(x) = 0$ otherwise.
 (a) Find k . [2]
 (b) Hence find $f(\frac{1}{2})$.
 A student claims that this value is impossible because a probability cannot exceed 1.
 Explain why the student is wrong. [3]
-

- 21** The continuous random variable X has probability density function $f(x) = kx(5 - x)$ for $0 \leq x \leq 5$, and $f(x) = 0$ otherwise.
- (a) Show that $k = \frac{6}{125}$. [3]
- (b) Write down the mode of X , giving a reason. [1]
- (c) Find $P(1 < X < 2)$. [2]
-
- 22** The continuous random variable X has probability density function $f(x) = kx$ for $0 \leq x \leq 2$, $f(x) = 2k$ for $2 < x \leq 4$, $f(x) = 0$ otherwise.
- (a) Find the value of k . [3]
- (b) Find $P(1 < X < 3)$. [3]
-
- 23** The continuous random variable X has probability density function $f(x) = \frac{k}{x^2}$ for $x \geq 1$, and $f(x) = 0$ otherwise.
- (a) Show that $k = 1$. [2]
- (b) Find $P(2 < X < 5)$. [2]
- (c) Find the value of m such that $P(X > m) = 0.05$. [3]
-
- 24** The continuous random variable X has probability density function $f(x) = ax$ for $0 \leq x \leq 2$, $f(x) = b$ for $2 < x \leq 4$, $f(x) = 0$ otherwise, where a and b are constants. The function f is continuous at $x = 2$.
- (a) Find the values of a and b . [5]
- (b) Find $P(X > 1.5)$. [3]
-
- 25** The continuous random variable X has probability density function $f(x) = \frac{3x^2}{8}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise.
- Two independent observations of X are made.
- (a) Find the probability that exactly one of them exceeds 1. [4]
- (b) Ten independent observations of X are made. Find the probability that at least 9 of them exceed 1, giving your answer to 3 significant figures. [3]
-
- 26** The continuous random variable X has probability density function $f(x) = k \sin x$ for $0 \leq x \leq \pi$, and $f(x) = 0$ otherwise.
- (a) Show that $k = \frac{1}{2}$. [3]
- (b) Find $P(X > \frac{\pi}{3})$. [3]
-
- 27** The continuous random variable X has probability density function $f(x) = kx^2$ for $0 \leq x \leq a$, and $f(x) = 0$ otherwise, where k and a are positive constants. It is given that $P(X < 1) = \frac{1}{8}$.
- (a) Find the value of k and the value of a . [6]
- (b) Find $P(X > 1.5)$. [3]
-
- 28** The continuous random variable X has probability density function $f(x) = k(2 - |x|)$ for $-2 \leq x \leq 2$, and $f(x) = 0$ otherwise.
- (a) Sketch the graph of $f(x)$, and show that $k = \frac{1}{4}$. [4]
- (b) Find $P(|X| > 1)$. [2]
- (c) Find $P(X > -1 \mid X < 1)$. [4]

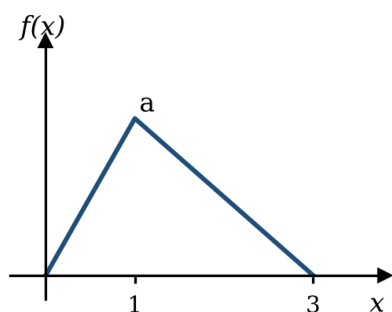
- 29** The continuous random variable X has probability density function $f(x) = kxe^{-x^2}$ for $x \geq 0$, and $f(x) = 0$ otherwise.
- (a) Show that $k = 2$. [3]
 (b) Find $P(X > 1)$, giving your answer in terms of e . [2]
 (c) Find the exact value of a such that $P(X < a) = 0.5$. [4]
- 30** The continuous random variable X has probability density function $f(x) = ax + b$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. It is given that $P(X < 1) = 0.3$.
- (a) Find the values of a and b . [5]
 (b) Show that $f(x) \geq 0$ over the whole range, and find $P(X > 1.5)$. [4]

Probability Density Functions

- 1** The continuous random variable X has probability density function $f(x) = \frac{x}{8}$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Find $P(1 < X < 3)$. [2]
- 2** The continuous random variable X has probability density function $f(x) = k(3 - x)$ for $0 \leq x \leq 3$, and $f(x) = 0$ otherwise. Find the value of k . [2]
- 3** The continuous random variable X has probability density function $f(x) = \frac{3}{32}(4x - x^2)$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Find the mode of X . [2]
- 4** The continuous random variable X has probability density function $f(x) = \frac{x}{2}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. Sketch the graph of $y = f(x)$, labelling the key values. [2]
- 5** The continuous random variable X has probability density function $f(x) = \frac{1}{2}e^{-x/2}$ for $x \geq 0$, and $f(x) = 0$ otherwise. Find $P(X > 2)$. [2]
- 6** The continuous random variable X has probability density function $f(x) = 0.4$ for $0 \leq x \leq 1$, $f(x) = 0.6$ for $1 < x \leq 2$, and $f(x) = 0$ otherwise. Find $P(X > 0.5)$. [2]
- 7** The continuous random variable X has probability density function $f(x) = k(4 - x^2)$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise.
- (a) Find the value of k . [3]
 (b) Find $P(X < 1)$. [2]
- 8** The continuous random variable X has probability density function $f(x) = kx^3$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. Find k and hence $P(X > 1)$. [4]
- 9** The continuous random variable X has probability density function $f(x) = \frac{x}{8}$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Find the median of X . [3]
- 10** The continuous random variable X has probability density function $f(x) = \frac{1}{2}e^{-x/2}$ for $x \geq 0$, and $f(x) = 0$ otherwise. Find the lower quartile of X , giving your answer in exact form. [3]
- 11** The continuous random variable X has probability density function $f(x) = x$ for $0 \leq x \leq 1$, $f(x) = 2 - x$ for $1 < x \leq 2$, and $f(x) = 0$ otherwise.
- (a) Sketch the graph of $y = f(x)$. [2]
 (b) Write down the median of X , giving a reason. [1]
 (c) Find $P(X > 1.5)$. [2]

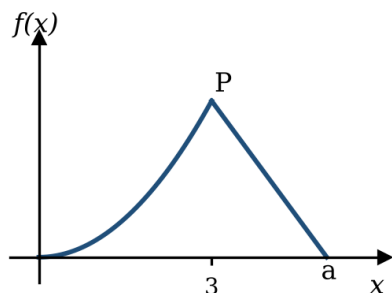
- 12** The continuous random variable X has probability density function $f(x) = \frac{k}{x}$ for $1 \leq x \leq e$, and $f(x) = 0$ otherwise. Show that $k = 1$ and find $P(X < 2)$. [4]
-
- 13** The continuous random variable X has probability density function $f(x) = \frac{1}{2}e^{-x/2}$ for $x \geq 0$, and $f(x) = 0$ otherwise. Find the 90th percentile of X , giving your answer to 3 significant figures. [3]
-
- 14** The continuous random variable X has probability density function $f(x) = kx^2(10 - x)$ for $0 \leq x \leq 10$, and $f(x) = 0$ otherwise.
- (a) Show that $k = \frac{3}{2500}$. [3]
- (b) Find the mode of X . [3]
-
- 15** The continuous random variable X has probability density function $f(x) = \frac{(x+1)}{12}$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise.
- (a) Verify that f is a valid probability density function. [2]
- (b) Show that the median m satisfies $m^2 + 2m - 12 = 0$, and hence find the median to 3 significant figures. [4]
-
- 16** The continuous random variable X has probability density function $f(x) = 12x^2(1 - x)$ for $0 \leq x \leq 1$, and $f(x) = 0$ otherwise.
- (a) Find the mode of X . [3]
- (b) Show that the median m satisfies $3m^4 - 4m^3 + \frac{1}{2} = 0$, and show that $0.6 < m < 0.65$. [4]
-
- 17** The continuous random variable X has probability density function $f(x) = \frac{x^3}{4}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. Find $P(X < 1.5 | X > 1)$. [4]
-
- 18** The continuous random variable X has probability density function $f(x) = kx^2$ for $0 \leq x \leq 1$, $f(x) = k(2 - x)$ for $1 < x \leq 2$, and $f(x) = 0$ otherwise.
- (a) Find the value of k . [3]
- (b) Find $P(\frac{1}{2} < X < \frac{3}{2})$. [3]
-
- 19** The continuous random variable X has probability density function $f(x) = \frac{k}{x^2}$ for $1 \leq x \leq b$, and $f(x) = 0$ otherwise, where k and b are constants. It is given that $P(X > 2) = \frac{1}{4}$. Find the values of k and b . [5]
-
- 20** The continuous random variable X has probability density function $f(x) = \frac{3}{x^4}$ for $x \geq 1$, and $f(x) = 0$ otherwise.
- (a) Verify that f is a valid probability density function. [2]
- (b) Find $P(X > 2)$. [2]
- (c) Three independent observations of X are made. Find the probability that at least one of them exceeds 2. [3]

- 21** The diagram shows the graph of the probability density function $f(x)$ of the continuous random variable X . The graph consists of two straight line segments from $(0, 0)$ to $(1, a)$ and from $(1, a)$ to $(3, 0)$, and $f(x) = 0$ otherwise.



- (a) Find the value of a . [2]
 (b) Write down $f(x)$ for $1 \leq x \leq 3$, and find $P(X > 2)$. [4]
 (c) Show that the median is greater than 1 and find its value to 3 significant figures. [4]
-
- 22** The continuous random variable X has probability density function $f(x) = kxe^{-x/2}$ for $x \geq 0$, and $f(x) = 0$ otherwise.
- (a) Use integration by parts to show that $k = \frac{1}{4}$. [4]
 (b) Find the mode of X . [3]
 (c) Find $P(X > 4)$, giving your answer in terms of e . [3]
-
- 23** The continuous random variable X has probability density function $f(x) = \frac{x}{10}$ for $0 \leq x \leq 2$, $f(x) = \frac{1}{5}$ for $2 < x \leq b$, and $f(x) = 0$ otherwise, where b is a constant.
- (a) Find the value of b . [3]
 (b) Find the median of X . [4]
-
- 24** The continuous random variable X has probability density function $f(x) = \frac{2}{x^3}$ for $x \geq 1$, and $f(x) = 0$ otherwise.
- (a) Show that $P(X < x) = 1 - \frac{1}{x^2}$ for $x \geq 1$. [2]
 (b) Find the lower quartile and the upper quartile of X , and hence the interquartile range. [5]
-
- 25** The continuous random variable X has probability density function $f(x) = a + bx^2$ for $0 \leq x \leq 1$, and $f(x) = 0$ otherwise. It is given that $P(X < \frac{1}{2}) = 0.2$.
- (a) Find the values of a and b . [5]
 (b) Hence explain why f is a valid probability density function and state its mode. [2]
-
- 26** The continuous random variable X has probability density function $f(x) = 2x + k$ for $2 \leq x \leq 3$, and $f(x) = 0$ otherwise.
- (a) Show that $k = -4$. [2]
 (b) Sketch the graph of $y = f(x)$ and write down the mode of X . [2]
 (c) Find $E(X)$. [3]
-
- 27** The continuous random variable X has probability density function $f(x) = (n + 1)x^n$ for $0 \leq x \leq 1$, and $f(x) = 0$ otherwise, where n is a positive integer. It is given that $P(X > \frac{1}{2}) = \frac{7}{8}$.
- (a) Show that f is a valid probability density function for any positive integer n . [2]
 (b) Find the value of n . [3]
 (c) Find the median and the interquartile range of X , in exact form. [5]

- 28** The diagram shows the graph of the probability density function $f(x)$ of a continuous random variable X . For $0 \leq x \leq 3$ the graph is the curve $y = kx^2$, where k is a positive constant. For $3 < x \leq a$ the graph is a straight line from the point $P(3, 9k)$ to the point $(a, 0)$, where $a > 3$, and $f(x) = 0$ otherwise. It is given that the mode of X is equal to the median of X .

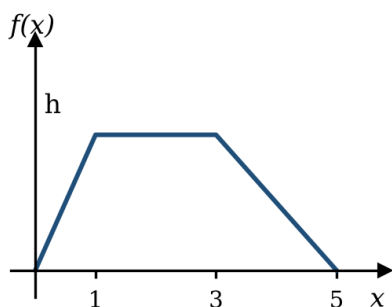


- (a) State the mode of X . [1]
- (b) Show that $k = \frac{1}{18}$ and find the value of a . [5]
- (c) Given that $E(X) = \frac{71}{24}$, determine with justification the skewness of the distribution. [2]

- 29** The continuous random variable X has probability density function $f(x) = kx^2(3 - x)$ for $0 \leq x \leq 3$, and $f(x) = 0$ otherwise.

- (a) Show that $k = \frac{4}{27}$ and find the mode of X . [5]
- (b) Show that the median m satisfies $4m^3 - m^4 = 13.5$, and show that m lies between 1.8 and 2. [4]
- (c) Using one application of the Newton-Raphson method with $m_0 = 1.8$, find an improved estimate of m , giving your answer to 3 decimal places. [3]

- 30** The diagram shows the graph of the probability density function $f(x)$ of a continuous random variable X . The graph is made of three straight line segments joining $(0, 0)$, $(1, h)$, $(3, h)$ and $(5, 0)$, and $f(x) = 0$ otherwise.



- (a) Show that $h = \frac{2}{7}$. [3]
- (b) Find the median of X . [4]
- (c) Find $P(X > 4)$. [2]

Cumulative Distribution Functions

- 1** The continuous random variable X has cumulative distribution function $F(x) = \frac{x^2}{16}$ for $0 \leq x \leq 4$, with $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 4$.

- (a) Find $P(X < 2)$. [1]
- (b) Find $P(1 < X < 3)$. [2]

- 2** The continuous random variable X has cumulative distribution function $F(x) = \frac{x-1}{4}$ for $1 \leq x \leq 5$, with $F(x) = 0$ for $x < 1$ and $F(x) = 1$ for $x > 5$. Find the median of X . [2]
- 3** The continuous random variable X has probability density function $f(x) = \frac{1}{4}$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Find the cumulative distribution function $F(x)$ for $0 \leq x \leq 4$. [2]
- 4** The continuous random variable X has cumulative distribution function $F(x) = \frac{x^3}{27}$ for $0 \leq x \leq 3$, with $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 3$. Find the probability density function $f(x)$. [2]
- 5** The continuous random variable X has cumulative distribution function $F(x) = 1 - e^{-x/3}$ for $x \geq 0$, and $F(x) = 0$ for $x < 0$. Find $P(X > 3)$. [2]
- 6** State two properties that the cumulative distribution function $F(x)$ of any continuous random variable must satisfy. [2]
- 7** The continuous random variable X has probability density function $f(x) = \frac{3}{4}x(2-x)$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise.
 (a) Find the cumulative distribution function $F(x)$ for all x . [4]
 (b) Hence find $P(X > 1.5)$. [2]
- 8** The continuous random variable X has probability density function $f(x) = \frac{x}{6}$ for $0 \leq x \leq 2$, $f(x) = \frac{1}{3}$ for $2 < x \leq 4$, and $f(x) = 0$ otherwise. Find the cumulative distribution function $F(x)$ for $0 \leq x \leq 4$. [5]
- 9** The continuous random variable X has cumulative distribution function $F(x) = 0$ for $x < 0$, $F(x) = \frac{x^2}{2}$ for $0 \leq x \leq 1$, $F(x) = 1 - \frac{(2-x)^2}{2}$ for $1 < x \leq 2$, $F(x) = 1$ for $x > 2$.
 Find the probability density function $f(x)$, and describe the shape of its graph. [4]
- 10** The continuous random variable X has cumulative distribution function $F(x) = 1 - \frac{8}{x^3}$ for $x \geq 2$, and $F(x) = 0$ for $x < 2$. Find the median of X and the probability that X lies between 3 and 4. [4]
- 11** The continuous random variable X has cumulative distribution function $F(x) = \frac{x^2+x}{12}$ for $0 \leq x \leq 3$, with $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 3$.
 (a) Find $P(1 < X < 2)$. [2]
 (b) Find the probability density function and state the mode of X . [3]
- 12** Decide whether each of the following could be the cumulative distribution function of a continuous random variable. In each case give a reason.
 (a) $F(x) = 0$ for $x < 0$; $F(x) = \frac{3x-x^3}{2}$ for $0 \leq x \leq 1$; $F(x) = 1$ for $x > 1$. [3]
 (b) $G(x) = 0$ for $x < 0$; $G(x) = x(2-x)$ for $0 \leq x \leq 2$; $G(x) = 1$ for $x > 2$. [2]
- 13** The time, T hours, taken to repair a machine has cumulative distribution function $F(t) = \frac{t^2}{9}$ for $0 \leq t \leq 3$, with $F(t) = 0$ for $t < 0$ and $F(t) = 1$ for $t > 3$. Find the probability that a repair takes longer than 2 hours, and the probability that it takes between 1 and 2 hours. [4]

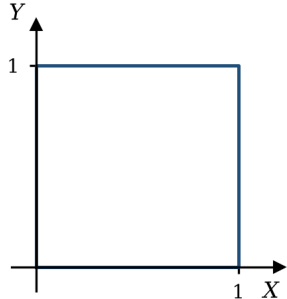
- 14** The continuous random variable X has cumulative distribution function $F(x) = ax^2 + bx$ for $0 \leq x \leq 2$, with $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 2$. It is given that $P(X > 1) = 0.4$.
- (a) Find the values of a and b . [4]
- (b) Find the probability density function and show that it is non-negative on the range. [3]
-
- 15** The continuous random variable X has probability density function $f(x) = xe^{-x^2/2}$ for $x \geq 0$, and $f(x) = 0$ otherwise.
- (a) Show that the cumulative distribution function is $F(x) = 1 - e^{-x^2/2}$ for $x \geq 0$. [3]
- (b) Find $P(1 < X < 2)$, and the exact value of the median. [4]
-
- 16** The continuous random variable X has probability density function $f(x) = \frac{6}{5}x^2$ for $0 \leq x \leq 1$, $f(x) = \frac{6}{5}(2 - x)$ for $1 < x \leq 2$, and $f(x) = 0$ otherwise. Find the cumulative distribution function $F(x)$, and check that $F(2) = 1$. [6]
-
- 17** The continuous random variable X has cumulative distribution function $F(x) = \frac{x^2}{16}$ for $0 \leq x \leq 4$, with $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 4$. Find $P(X > 3 | X > 2)$, and the value of c such that $P(X < c) = 0.81$. [5]
-
- 18** The continuous random variable X has cumulative distribution function $F(x) = 1 - (1 + x)e^{-x}$ for $x \geq 0$, and $F(x) = 0$ for $x < 0$.
- (a) Show that the probability density function is $f(x) = xe^{-x}$ for $x \geq 0$. [3]
- (b) Find the mode of X . [3]
- (c) Find $P(X > 2)$ in terms of e . [1]
-
- 19** The time, X minutes, that a customer waits to be served has cumulative distribution function $F(x) = 1 - e^{-x/4}$ for $x \geq 0$, and $F(x) = 0$ for $x < 0$. Find the value of q such that 20% of customers wait longer than q minutes, giving your answer to 3 significant figures, and find the probability that a customer waits less than 2 minutes. [5]
-
- 20** The continuous random variable X has probability density function $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \leq x \leq 1$, and $f(x) = 0$ otherwise.
- (a) Show that the cumulative distribution function is $F(x) = \frac{1}{2} + \frac{3x}{4} - \frac{x^3}{4}$ for $-1 \leq x \leq 1$. [3]
- (b) Hence find $P(|X| < \frac{1}{2})$. [3]
-
- 21** The continuous random variable X has cumulative distribution function $F(x) = 0$ for $x < 0$, $F(x) = \frac{x^2}{8}$ for $0 \leq x \leq 2$, $F(x) = -\frac{x^2}{16} + ax + b$ for $2 < x \leq 4$, $F(x) = 1$ for $x > 4$, where a and b are constants, and F is continuous.
- (a) Find the values of a and b . [4]
- (b) Find the probability density function $f(x)$. [3]
- (c) Find $P(1 < X < 3)$. [3]
-
- 22** The lifetime, T years, of a certain type of component has cumulative distribution function $F(t) = \frac{t^2}{9}$ for $0 \leq t \leq 3$, with $F(t) = 0$ for $t < 0$ and $F(t) = 1$ for $t > 3$. Two such components, whose lifetimes are independent, are fitted in a device. The device works only while both components are working.
- (a) Find the probability that the device is still working after 2 years. [3]
- (b) Find the probability that the device fails within the first year. [2]
- (c) Given that a single component has lasted 1 year, find the probability that it lasts for at least 2 years. [3]

- 23** The continuous random variable X has cumulative distribution function $F(x) = 1 - \frac{(6-x)^2}{16}$ for $2 \leq x \leq 6$, with $F(x) = 0$ for $x < 2$ and $F(x) = 1$ for $x > 6$.
- Find the median, the lower quartile and the upper quartile of X , giving your answers to 3 significant figures. [4]
 - Determine, with justification, the skewness of the distribution. [2]
 - Find the probability density function of X . [2]
-
- 24** A function G is defined by $G(x) = 0$ for $x < 1$, $G(x) = \frac{1}{4} + \frac{x-1}{4}$ for $1 \leq x \leq 3$, and $G(x) = 1$ for $x > 3$.
- Show that G is not the cumulative distribution function of a continuous random variable. [3]
 - Write down the value of $P(X = 1)$ that this function would imply, and explain why it contradicts a continuous random variable. [2]
 - Write down a linear function $F(x)$ for $1 \leq x \leq 3$ that would give a valid cumulative distribution function with $F(3) = 1$. [2]
-
- 25** The continuous random variable X has cumulative distribution function $F(x) = \frac{\ln x}{\ln 4}$ for $1 \leq x \leq 4$, with $F(x) = 0$ for $x < 1$ and $F(x) = 1$ for $x > 4$.
- Show that the median of X is 2. [2]
 - Find the exact values of the lower and upper quartiles of X . [4]
 - Find the probability density function and $P(X > 3)$, giving your answer to 3 significant figures. [4]
-
- 26** The lifetime, X years, of a type of sensor has cumulative distribution function $F(x) = 1 - e^{-x^2/c}$ for $x \geq 0$, and $F(x) = 0$ for $x < 0$, where c is a positive constant. It is given that the probability that a sensor lasts longer than 2 years is e^{-1} .
- Show that $c = 4$. [2]
 - Find the median lifetime. [3]
 - A sensor has already lasted 2 years. Find the probability that it lasts for more than 3 years.
Give your answer in terms of e . [3]
-
- 27** The continuous random variable X has cumulative distribution function $F(x) = \sin^2(\frac{\pi x}{8})$ for $0 \leq x \leq 4$, with $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 4$.
- Find the median of X . [2]
 - Show that the probability density function is $f(x) = \frac{\pi}{8} \sin(\frac{\pi x}{4})$, and find the mode. [4]
 - Show that $P(1 < X < 3) = \frac{\sqrt{2}}{2}$. [4]
-
- 28** The continuous random variable X has cumulative distribution function $F(x) = 0$ for $x < 0$, $F(x) = \frac{x^2}{8}$ for $0 \leq x \leq 2$, $F(x) = -\frac{x^2}{16} + \frac{5x}{8} - \frac{1}{2}$ for $2 < x \leq 4$, $F(x) = 1$ for $x > 4$.
- Find the median of X . [3]
 - Find the upper quartile of X , giving your answer to 3 significant figures. [5]
 - Find $P(X > 3 | X > 1)$. [4]
-
- 29** The random variable X has cumulative distribution function $F(x) = 1 - e^{-2x}$ for $x \geq 0$, and $F(x) = 0$ for $x < 0$. The random variable Y is defined by $Y = F(X)$.
- Show that, for $0 \leq y < 1$, $P(Y \leq y) = y$, and hence state the distribution of Y . [5]
 - A random number generator produces values u from the continuous uniform distribution on $[0, 1]$. Use part (a) to find a formula that transforms u into an observation x of X , and evaluate x when $u = 0.75$, giving your answer to 3 significant figures. [4]

- 30** The continuous random variable X has cumulative distribution function
- $$F(x) = 0 \text{ for } x < -1, F(x) = \frac{(1+x)^2}{2} \text{ for } -1 \leq x \leq 0, F(x) = 1 - \frac{(1-x)^2}{2} \text{ for } 0 < x \leq 1, F(x) = 1 \text{ for } x > 1.$$
- (a) Find the probability density function $f(x)$, and show that the distribution of X is symmetrical about 0. [4]
- (b) Find the 90th percentile of X , giving your answer to 3 significant figures. [3]
- (c) Find $P(X > \frac{1}{2} \mid |X| > \frac{1}{2})$ and comment on the result. [4]

Continuous Uniform Distribution

- 1** The random variable X is uniformly distributed on the interval $[0, 10]$. Find
- (a) $P(X < 3)$, [1]
- (b) $P(4 < X < 9)$. [1]
- 2** The random variable X has a continuous uniform distribution on $[2, 6]$. Write down the probability density function of X and the value of $E(X)$. [2]
- 3** $X \sim U[1, 7]$. Find $E(X)$ and $\text{Var}(X)$. [2]
- 4** $X \sim U[3, 9]$. Write down the median of X and find $P(X > 8)$. [2]
- 5** The waiting time, T minutes, of a passenger at a station is modelled by a continuous uniform distribution on $[0, 30]$. Find the probability that a passenger waits longer than 20 minutes. [2]
- 6** $X \sim U[-2, 4]$. Find $P(X < 0)$. [2]
- 7** $X \sim U[2, 8]$.
- (a) Find $P(X > 5.5)$. [2]
- (b) Find $E(X)$, $\text{Var}(X)$ and the standard deviation of X . [3]
- 8** The random variable X is uniformly distributed on $[-1, 5]$. Find the cumulative distribution function $F(x)$ for all x , and hence find $P(X \leq 2)$. [4]
- 9** The random variable X is uniformly distributed on $[a, b]$. It is given that $E(X) = 6$ and that the length of the interval is 8. Find a and b . [3]
- 10** A length is measured to the nearest centimetre. The rounding error, E cm, is modelled by a continuous uniform distribution on $[-0.5, 0.5]$.
- (a) Find the probability that the magnitude of the error is less than 0.1 cm. [2]
- (b) Find the standard deviation of E . [2]
- 11** $X \sim U[0, 6]$. Find the probability that X lies within one standard deviation of its mean. [4]
- 12** A spinner stops at an angle θ degrees, which is uniformly distributed on $[0, 360]$. Find the probability that θ is between 30 and 75, and the probability that θ is greater than 250. [3]
- 13** A point P is chosen at random on a line segment AB of length 12 cm, so that the distance AP is uniformly distributed on $[0, 12]$. Find the probability that P is within 3 cm of the midpoint of AB . [3]
- 14** The random variable X has a continuous uniform distribution on $[a, b]$, where $a < b$. It is given that $E(X) = 10$ and $\text{Var}(X) = 12$. Find a and b . [4]
- 15** The random variable $X \sim U[0, 5]$. The quadratic equation $t^2 + Xt + 1 = 0$ is formed. Find the probability that this equation has real roots. [4]

- 16** $X \sim U[0, 12]$. Find $P(X > 9 \mid X > 4)$, and show that the conditional distribution of X given $X > 4$ is itself uniform. [4]
- 17** The random variable X is uniformly distributed on $[a, b]$. It is given that $P(X < 10) = 0.25$ and $P(X < 30) = 0.75$. Find a and b . [5]
- 18** A piece of string of length 1 m is cut at a random point, so that the distance X m from one end to the cut is uniformly distributed on $[0, 1]$. Find the probability that the longer piece is more than twice as long as the shorter piece. [4]
- 19** The radius R cm of a circular disc is modelled as a continuous uniform random variable on $[1, 3]$. Find the expected area of a disc, giving your answer in terms of π . [4]
- 20** A student states that for $X \sim U[2, 8]$ the variance is $\frac{8-2}{12} = 0.5$. Explain the error, find the correct variance, and find the value of c such that $X \sim U[0, c]$ has variance 3. [4]
- 21** The random variable $X \sim U[0, 4]$, and $Y = 3X - 2$.
 (a) Find $E(Y)$ and $\text{Var}(Y)$. [3]
 (b) Find $P(Y > 5)$. [3]
- 22** The random variable X is uniformly distributed on $[a, b]$. The 30th and 80th percentiles of X are 12 and 27 respectively.
 (a) Find the values of a and b . [4]
 (b) The sum S of 100 independent observations of X is calculated. Use the central limit theorem to estimate $P(S > 1850)$. [4]
- 23** The random variable X is uniformly distributed on $[a, b]$. Prove that $E(X) = \frac{a+b}{2}$ and $\text{Var}(X) = \frac{(b-a)^2}{12}$. [6]
- 24** X and Y are independent random variables, each uniformly distributed on $[0, 1]$. The point (X, Y) is therefore equally likely to lie anywhere in the unit square shown.
- 
- (a) Find $P(X + Y > 1.5)$. [3]
 (b) Find $P(|X - Y| < 0.2)$. [4]
- 25** The time W minutes that a commuter waits for a train is uniformly distributed on $[0, 15]$. Find
 (a) the probability that the commuter waits for more than 10 minutes on a given day, [1]
 (b) the probability that over 6 independent days the commuter waits for more than 10 minutes on exactly 2 days, [3]
 (c) the value of t such that $P(W > t) = 0.05$. [2]
- 26** The random variable X is uniformly distributed on $[0, 3]$ and $Y = X^2$.
 (a) Show that $P(Y \leq y) = \frac{\sqrt{y}}{3}$ for $0 \leq y \leq 9$. [3]
 (b) Find the probability density function of Y and find $P(Y > 4)$. [4]

- 27** X and Y are independent random variables, each uniformly distributed on $[0, 10]$. Find, in exact form, the probability that the product XY is less than 10. [8]
- 28** The random variable $X \sim U[0, a]$. For $0 < c < a$, the random variable Y has the distribution of X conditional on $X < c$.
 (a) Show that $P(Y \leq y) = \frac{y}{c}$ for $0 \leq y \leq c$, and hence identify the distribution of Y. [4]
 (b) A random number X is chosen from $U[0, 20]$. Given that it is less than 8, find the probability that it is greater than 5, and the expected value of X. [4]
- 29** The random variables $X_1, X_2, \dots, X_n$ are independent and each is uniformly distributed on $[0, \theta]$. The random variable M is the largest of the n observations.
 (a) Show that $P(M \leq m) = \left(\frac{m}{\theta}\right)^n$ for $0 \leq m \leq \theta$. [3]
 (b) For $n = 5$, find the probability that M exceeds 0.9θ , to 3 significant figures. [2]
 (c) Find the smallest value of n for which $P(M > 0.9\theta) > 0.99$. [4]
- 30** Two bus routes serve a stop. Route A buses are every 10 minutes and route B buses every 15 minutes. A passenger arrives at a random time, so that the waiting time A for route A is $U[0, 10]$ and the waiting time B for route B is $U[0, 15]$, independently. The passenger takes whichever bus comes first. Let $T = \min(A, B)$.
 (a) Find the probability that the passenger waits less than 4 minutes. [3]
 (b) Show that $P(T > t) = \left(1 - \frac{t}{10}\right)\left(1 - \frac{t}{15}\right)$ for $0 \leq t \leq 10$, and hence find the cumulative distribution function and the probability density function of T. [4]
 (c) Show that $E(T) = \frac{35}{9}$ minutes, using $E(T) = \int_0^{10} P(T > t) dt$. [3]

Exponential Distribution

- 1** $X \sim \text{Exp}(0.5)$. Write down $E(X)$ and $\text{Var}(X)$. [2]
- 2** $X \sim \text{Exp}(2)$. Find $P(X < 1)$, giving your answer to 3 significant figures. [2]
- 3** The random variable X has an exponential distribution with parameter $\lambda = 3$. Write down the probability density function and the cumulative distribution function of X. [2]
- 4** X has an exponential distribution with mean 4. Write down the value of λ and find $P(X > 4)$ in terms of e. [3]
- 5** Telephone calls arrive at a helpline according to a Poisson process with rate 3 per hour. The time T hours between consecutive calls has an exponential distribution. Write down the distribution of T and find the mean time between calls in minutes. [2]
- 6** $X \sim \text{Exp}(0.1)$. Find $P(X > 10)$, giving your answer in terms of e. [2]
- 7** $X \sim \text{Exp}(\lambda)$ and $P(X > 3) = 0.2$. Find the value of λ , giving your answer to 3 significant figures. [3]
- 8** $X \sim \text{Exp}(0.2)$. Find the exact values of the median and the upper quartile of X. [4]
- 9** The lifetime, L hours, of a type of light bulb has an exponential distribution with mean 800 hours. Find the probability that a bulb
 (a) lasts for more than 1000 hours, [2]
 (b) fails before 400 hours. [2]

- 10** $X \sim \text{Exp}(0.1)$. Find $P(X > 15 \mid X > 5)$, and state the property of the exponential distribution that this illustrates. [3]
- 11** Customers arrive at a shop according to a Poisson process at a rate of 2 per minute. The probability that there is no arrival in a given period of 1.5 minutes is to be found in two ways.
 (a) Use the Poisson distribution. [2]
 (b) Use the time T until the next arrival, and show that the answer agrees with part (a). [2]
- 12** $X \sim \text{Exp}(\frac{1}{3})$. Find $E(3X + 2)$ and $\text{Var}(3X + 2)$. [3]
- 13** $X \sim \text{Exp}(0.5)$. Find the probability that X lies within one standard deviation of its mean. [4]
- 14** $X \sim \text{Exp}(\lambda)$ and $P(X > 2) = 0.3$. Find $P(X > 5 \mid X > 2)$, giving your answer to 3 significant figures. [4]
- 15** The length of a telephone call, in minutes, to a service centre is modelled by an exponential distribution with mean 6 minutes. A call has already lasted 4 minutes. Find the probability that it lasts at least 6 more minutes, and find the probability that a call lasts longer than 10 minutes. [4]
- 16** $X \sim \text{Exp}(0.25)$.
 (a) Find $P(2 < X < 6)$. [3]
 (b) Find the median of X . [2]
- 17** Five components operate independently. The lifetime of each is exponentially distributed with mean 100 hours. Find the probability that at least 4 of the components are still working after 100 hours. [5]
- 18** Emails arrive in an inbox such that the time between consecutive emails is exponentially distributed with mean 12 minutes.
 (a) Find the probability that no email arrives in a period of 30 minutes. [2]
 (b) Find the time t such that there is a probability of 0.9 that the next email arrives within t minutes. [3]
- 19** $X \sim \text{Exp}(\lambda)$ and $P(X < 1) = 0.6$. Find the exact value of λ , the mean of X , and $P(X > 2)$. [5]
- 20** The random variable X has an exponential distribution with parameter λ . Prove the memoryless property: for $s, t > 0$, $P(X > s + t \mid X > s) = P(X > t)$. [4]
- 21** A student is told that $X \sim \text{Exp}(0.5)$ and writes: " $E(X) = 0.5$ and $\text{Var}(X) = 0.25$." Explain the error, state the correct values, and find the probability that X exceeds its mean. [5]
- 22** The times, in minutes, between five consecutive arrivals at a clinic were 2.1, 5.4, 1.2, 6.8 and 1.5. Assuming that the time between arrivals is exponentially distributed with mean equal to the sample mean, find
 (a) the value of λ , [2]
 (b) the probability that the time between two arrivals exceeds 5 minutes, and compare with the data. [3]
- 23** A system consists of two components A and B connected in series, so the system fails as soon as either component fails. The lifetimes, in hours, of A and B are independent and exponentially distributed with parameters 0.02 and 0.03 respectively.
 (a) Find the probability that the system is still working after 10 hours. [3]
 (b) Show that the lifetime of the system has an exponential distribution, and find its mean. [3]

- 24** $X \sim \text{Exp}(\lambda)$. Use integration by parts to show that $E(X) = \frac{1}{\lambda}$ and $E(X^2) = \frac{2}{\lambda^2}$, and hence show that $\text{Var}(X) = \frac{1}{\lambda^2}$. [8]
-
- 25** Buses arrive at a stop according to a Poisson process with rate 4 per hour.
- (a) Find the probability that the first bus arrives more than 20 minutes after a passenger arrives. [3]
- (b) Find the probability that at least 2 buses arrive in a 30-minute period, and explain why this is not the same as $P(T < \frac{1}{2})$. [4]
-
- 26** Three identical components, each with an independent exponentially distributed lifetime with mean 10 hours, are connected in parallel, so the system works as long as at least one component is working.
- (a) Find the probability that the system is still working after 10 hours. [4]
- (b) Find the probability that the system works after 10 hours if the three components were instead connected in series. [3]
- Comment on your answers.
-
- 27** $X \sim \text{Exp}(\lambda)$ and $s > 0$. Show that $E(X | X > s) = s + \frac{1}{\lambda}$. A tool has an exponentially distributed lifetime with mean 50 hours. Given that it has already lasted 80 hours, find its expected total lifetime. [7]
-
- 28** X and Y are independent random variables with $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$.
- (a) By considering $\int_0^\infty f_X(x) P(Y > x) dx$, show that $P(X < Y) = \frac{\lambda}{\lambda + \mu}$. [4]
- (b) Two servers A and B each deal with a single customer. The service times, in minutes, are independent exponentials with means 4 and 6 respectively. Find the probability that A finishes before B, and the probability that both are still busy after 3 minutes. [4]
-
- 29** The time X , in minutes, spent by a user on a website is exponentially distributed with parameter $\lambda = 0.5$. The random variable K is the number of complete minutes spent, i.e. $K = k$ when $k \leq X < k + 1$, for $k = 0, 1, 2, \dots$
- (a) Show that $P(K = k) = (1 - e^{-\lambda})e^{-\lambda k}$, and deduce that $K + 1$ has a geometric distribution, stating its parameter. [5]
- (b) For $\lambda = 0.5$ find $P(K = 2)$, and $E(K)$, giving your answers to 3 significant figures. [4]
-
- 30** X_1 and X_2 are independent random variables, each exponentially distributed with parameter λ . The random variable M is the larger of X_1 and X_2 .
- (a) Show that the cumulative distribution function of M is $G(m) = (1 - e^{-\lambda m})^2$ for $m \geq 0$. [3]
- (b) Find the probability density function of M . [2]
- (c) Show that $E(M) = \frac{3}{2\lambda}$. [4]

Normal Distribution

- 1** $Z \sim N(0, 1)$. Find $P(Z < 1.2)$ and $P(-0.5 < Z < 1.5)$. [3]
-
- 2** The IQ scores of a population are modelled by $X \sim N(100, 15^2)$. Find the probability that a randomly chosen person has an IQ above 130. [2]
-
- 3** Find the value a such that $P(Z < a) = 0.9$, where $Z \sim N(0, 1)$, and find the value b such that $X \sim N(10, 2^2)$ satisfies $P(X < b) = 0.975$. [3]

- 4** $X \sim N(5, 2^2)$ and $Y \sim N(8, 3^2)$ are independent. State the distribution of $X + Y$. [2]
- 5** The mass of a bag of flour is modelled by $X \sim N(250, 5^2)$ grams. Find the probability that a bag has mass less than 245 g. [2]
- 6** The heights of adult women in a town are modelled by $H \sim N(172, 6^2)$ cm. Find the probability that a randomly chosen woman has height between 160 cm and 180 cm. [4]
- 7** $X \sim N(80, 10^2)$. Find the value of a such that $P(X > a) = 0.1$, and the symmetric interval about the mean that contains 95% of the distribution. [5]
- 8** $X \sim N(50, \sigma^2)$ and $P(X > 58) = 0.0668$. Find σ . [3]
- 9** $X \sim N(\mu, 4^2)$ and $P(X < 30) = 0.2$. Find μ . [3]
- 10** $X \sim N(10, 3^2)$ and $Y = 4X - 1$. State the distribution of Y and find $P(Y > 45)$. [4]
- 11** The mass of a bag of apples is $A \sim N(5, 0.4^2)$ kg and the mass of a bag of pears is $B \sim N(7, 0.3^2)$ kg, independently. Find the probability that a bag of apples and a bag of pears together weigh more than 13 kg. [4]
- 12** The mass of a bag of sugar is $X \sim N(25, 2^2)$ kg. Four bags, chosen independently, are packed in a box which has negligible mass. Find the probability that the box has mass greater than 105 kg. [5]
- 13** $X \sim N(40, 5^2)$ and $Y \sim N(12, 2^2)$ are independent. Find $P(X > 3Y)$. [5]
- 14** $X \sim B(100, 0.4)$. Use a suitable normal approximation, with a continuity correction, to estimate $P(X \leq 35)$. [5]
- 15** $X \sim \text{Po}(30)$. Use a normal approximation to estimate $P(X > 35)$, explaining why it is suitable. [5]
- 16** The height of adult men is modelled by $H \sim N(170, 8^2)$ cm. Given that a randomly chosen man is taller than 160 cm, find the probability that he is taller than 180 cm. [5]
- 17** The time X taken by a customer to be served is modelled by $X \sim N(10, 2^2)$ minutes. A student models the total time taken for two customers, served one after the other, as $2X$, and writes: " $2X \sim N(20, 4^2)$, so $P(\text{total time} > 25) = P(Z > 1.25) = 0.1056$." Explain the error in the student's method and find the correct probability. [5]
- 18** $A \sim N(30, 3^2)$ and $B \sim N(30, 4^2)$ are independent. Find $P(|A - B| < 2)$. [5]
- 19** A builder buys X metres of timber at £3 per metre and Y metres of pipe at £2 per metre, where $X \sim N(8, 2^2)$ and $Y \sim N(15, 3^2)$ are independent. Find the probability that the total cost of materials is more than £70. [5]
- 20** $X \sim N(40, \sigma^2)$ and $Y \sim \text{Po}(6)$ are independent. The random variable $W = 4Y - X + 50$ has variance 121.
(a) Find the value of σ . [3]
(b) Find $P(X < E(W))$. [3]
- 21** The random variable $X \sim N(\mu, \sigma^2)$ is such that $P(X < 42) = 0.1$ and $P(X > 60) = 0.05$. Find μ and σ . [6]

- 22** The length of a bolt, in mm, is $X \sim N(20, 0.1^2)$. Bolts are acceptable if their length lies between 19.85 mm and 20.2 mm.
- (a) Find the proportion of bolts that are acceptable. [3]
- (b) The mean is adjusted so that the acceptable interval is symmetrical about it, the standard deviation remaining 0.1 mm. Find the new proportion of acceptable bolts. [3]
-
- 23** A and B are independent random variables with $A \sim N(20, k)$ and $B \sim N(25, 2k)$, where k is a positive constant. It is given that $P(2A - B < 6) = 0.0668$. Find the value of k . [6]
-
- 24** A machine fills bags with sugar so that the mass in grams is $N(\mu, 6^2)$. The bags are labelled 500 g. The manufacturer wants only 2% of bags to contain less than 500 g.
- (a) Find the value to which μ must be set. [3]
- (b) With this value of μ , find the probability that a randomly chosen bag has mass greater than 520 g. [3]
-
- 25** $Y \sim \text{Po}(12)$. The probability $P(Y \leq 8)$ is to be found exactly and by a normal approximation.
- (a) Find $P(Y \leq 8)$ exactly. [2]
- (b) Use a normal approximation with continuity correction to estimate $P(Y \leq 8)$. [3]
- (c) Find the percentage error of the approximation. [1]
-
- 26** The mass of an adult passenger is modelled as $N(75, 12^2)$ kg, independently for each person. A lift has a maximum safe load of 650 kg.
- (a) Find the probability that a lift carrying 8 randomly chosen passengers is overloaded. [4]
- (b) Find the largest number of passengers for which the probability of overload is less than 1%. [5]
-
- 27** The random variable $X \sim N(\mu, \sigma^2)$, where μ and σ are positive. For a positive constant a , $P(X < a) = 0.3$ and $P(X > 2a) = 0.1$.
- (a) Find μ and σ in terms of a , giving numerical coefficients to 3 significant figures. [6]
- (b) Find $P(X > 1.5a)$. [3]
-
- 28** The mass of a particular component is $N(\mu, 2^2)$ grams, where μ is unknown. The mean mass of n independent components is $\bar{X}$.
- (a) State the distribution of $\bar{X}$, justifying the variance. [3]
- (b) Find the smallest n such that the probability that $\bar{X}$ lies within 0.5 g of μ is at least 0.95. [6]
-
- 29** The mass of an orange from a farm is $O \sim N(160, 10^2)$ grams. Oranges are sold in packs of 6, selected at random, and the mass of the packaging is negligible.
- (a) Find the probability that a pack has mass less than 1 kg. [4]
- (b) Find the probability that a pack has mass between 900 g and 1000 g. [3]
- (c) The farmer improves the oranges so that the mean mass becomes μ , with standard deviation unchanged. Find the least value of μ , to 3 significant figures, for which at most 2% of packs weigh less than 1 kg. [4]
-
- 30** The random variable $X \sim N(\mu, \sigma^2)$. Let a and b be constants with $a > 0$ and let $Y = aX + b$.
- (a) By considering $P(Y \leq y)$, show that $Y \sim N(a\mu + b, a^2\sigma^2)$. [5]
- (b) Daily maximum temperatures in a city, in $^{\circ}\text{C}$, are modelled by $C \sim N(20, 3^2)$. Using $F = 1.8C + 32$, find the distribution of the temperature in $^{\circ}\text{F}$, and the probability that the temperature exceeds 75°F . [4]

Expectation and Variance of Continuous Variables

- 1 The continuous random variable X has probability density function $f(x) = \frac{x}{2}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. Find $E(X)$. [2]
- 2 The continuous random variable X has probability density function $f(x) = \frac{1}{4}$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Find $E(X)$ and explain how it could be written down by symmetry. [2]
- 3 The random variable X has $E(X) = 3$ and $\text{Var}(X) = 2$. Find $E(2X + 1)$, $\text{Var}(2X + 1)$ and $E(X^2)$. [3]
- 4 The continuous random variable X has a probability density function whose graph is symmetrical about $x = 3$. State the value of $E(X)$, and the value of the median of X . [2]
- 5 The random variable X has mean 5 and variance 4. Find $E(X^2)$ and the standard deviation of X . [2]
- 6 The continuous random variable X has probability density function $f(x) = \frac{3x^2}{8}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. Find $E(X)$. [2]
- 7 The continuous random variable X has probability density function $f(x) = \frac{3}{32}(4x - x^2)$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Find $E(X)$ and $\text{Var}(X)$. [6]
- 8 The continuous random variable X has probability density function $f(x) = 6x(1 - x)$ for $0 \leq x \leq 1$, and $f(x) = 0$ otherwise. Write down $E(X)$, explaining your reasoning, and find $\text{Var}(X)$. [5]
- 9 The continuous random variable X has probability density function $f(x) = \frac{x}{2}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. Find $E(\frac{1}{X})$ and $E(X^3)$. [4]
- 10 The continuous random variable X has probability density function $f(x) = \frac{x}{8}$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. A company's profit, in £ thousands, is $P = 5X - 2$. Find $E(P)$ and $\text{Var}(P)$. [6]
- 11 X and Y are independent random variables with $E(X) = 4$, $\text{Var}(X) = 3$, $E(Y) = 2$ and $\text{Var}(Y) = 5$. Find $E(3X - 2Y)$ and $\text{Var}(3X - 2Y)$. [4]
- 12 The continuous random variable X has probability density function $f(x) = e^{-x}$ for $x \geq 0$, and $f(x) = 0$ otherwise. The mean, median and mode of X are compared.
(a) Write down the mode and show that the median is $\ln 2$. [3]
(b) Given that $E(X) = 1$, state with a reason the type of skew of this distribution. [2]
- 13 The continuous random variable X has probability density function $f(x) = \frac{x}{6}$ for $0 \leq x \leq 2$, $f(x) = \frac{6-x}{12}$ for $2 < x \leq 6$, and $f(x) = 0$ otherwise. Find $E(X)$. [4]
- 14 The lengths of telephone conversations, T minutes, are modelled by a continuous random variable with probability density function $f(t) = \frac{t}{72}$ for $0 \leq t \leq 12$, and $f(t) = 0$ otherwise. Show by calculation that $E(T) = \text{Var}(T)$. [6]

- 15** The continuous random variable X has probability density function $f(x) = ax + b$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. It is given that $E(X) = \frac{7}{6}$. Find the values of a and b . [6]
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- 16** The continuous random variable X has probability density function $f(x) = \frac{5x^4}{32}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. A measure of skewness is defined as $\frac{\text{mean} - \text{mode}}{\text{standard deviation}}$.
- (a) Find $E(X)$ and $\text{Var}(X)$. [4]
- (b) State the mode of X , evaluate the measure of skewness to 3 significant figures, and comment on its sign. [3]
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- 17** The independent random variables X and Y have $E(X) = 12$, $\text{Var}(X) = 4$, $E(Y) = 8$ and $\text{Var}(Y) = 9$. Find $E(X - Y)$, $\text{Var}(X - Y)$ and the standard deviation of $\frac{X + Y}{2}$. [6]
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- 18** The continuous random variable X has probability density function $f(x) = \frac{1}{x}$ for $1 \leq x \leq e$, and $f(x) = 0$ otherwise. Find $E(X)$, $E(\ln X)$ and $\text{Var}(\ln X)$. [7]
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- 19** The continuous random variable X has probability density function $f(x) = \frac{1}{2\sqrt{x}}$ for $0 < x \leq 1$, and $f(x) = 0$ otherwise.
- (a) Verify that f is a valid probability density function. [2]
- (b) Find $E(X)$ and $\text{Var}(X)$. [4]
- (c) Find $E(\sqrt{X})$. [2]
-
- 20** The continuous random variable X has mean $\frac{8}{3}$ and $E(X^2) = 8$. A student calculates the variance as $\text{Var}(X) = E(X^2) - E(X) = 8 - \frac{8}{3} = \frac{16}{3}$. Identify the error and find the correct variance. Hence find the standard deviation of $Y = 3X + 4$. [5]
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- 21** The continuous random variable X has probability density function $f(x) = xe^{-x}$ for $x \geq 0$, and $f(x) = 0$ otherwise. Use integration by parts to show that $E(X) = 2$ and $E(X^2) = 6$, and hence find $\text{Var}(X)$. [8]
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- 22** The continuous random variable X has probability density function $f(x) = \frac{6}{5}x^2$ for $0 \leq x \leq 1$, $f(x) = \frac{6}{5}(2 - x)$ for $1 < x \leq 2$, and $f(x) = 0$ otherwise. Find $E(X)$ and $\text{Var}(X)$. [8]
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- 23** The continuous random variable X has probability density function $f(x) = 12x^2(1 - x)$ for $0 \leq x \leq 1$, and $f(x) = 0$ otherwise. The mode is $\frac{2}{3}$ and the median is approximately 0.614.
- (a) Find $E(X)$ and $\text{Var}(X)$. [5]
- (b) Use the mean, median and mode to describe the skewness of the distribution. [2]
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- 24** The continuous random variable X has probability density function $f(x) = \frac{3x^2}{8}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. The cost, in pounds, of a repair taking X hours is $C = 50 + 20X^2$.
- (a) Find the expected cost of a repair. [4]
- (b) Find the probability that a repair costs more than £120. [4]

- 25** The continuous random variable X has cumulative distribution function $F(x) = 1 - \frac{8}{x^3}$ for $x \geq 2$, and $F(x) = 0$ for $x < 2$.
- (a) Find the probability density function of X . [2]
 (b) Find $E(X)$ and $\text{Var}(X)$. [4]
 (c) Find the median of X and use it to describe the skewness. [2]
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- 26** X is a continuous random variable with $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$, and a and b are constants. Using the definitions $E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x) dx$ and $\text{Var}(X) = E((X - \mu)^2)$, prove that $E(aX + b) = a\mu + b$ and $\text{Var}(aX + b) = a^2\sigma^2$. [6]
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- 27** The continuous random variable X has probability density function $f(x) = \frac{6}{125}x(5 - x)$ for $0 \leq x \leq 5$, and $f(x) = 0$ otherwise. Find $E(X)$ and $\text{Var}(X)$, and the probability that X lies within one standard deviation of its mean. [8]
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- 28** The continuous random variable X has probability density function $f(x) = \frac{3}{x^4}$ for $x \geq 1$, and $f(x) = 0$ otherwise. The continuous random variable Y has probability density function $g(y) = \frac{2}{y^3}$ for $y \geq 1$, and $g(y) = 0$ otherwise.
- (a) Find $E(X)$ and $\text{Var}(X)$. [5]
 (b) Show that $E(Y)$ exists and find its value, but that the variance of Y does not exist. [5]
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- 29** The continuous random variable X has probability density function $f(x) = ax^2 + bx$ for $0 \leq x \leq 1$, and $f(x) = 0$ otherwise. It is given that $E(X) = 0.6$.
- (a) Find the values of a and b . [5]
 (b) Show that f is non-negative on the range, and find the mode. [4]
 (c) Find $\text{Var}(X)$. [3]
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- 30** The continuous random variable X takes non-negative values, has probability density function f and cumulative distribution function F , and $x(1 - F(x)) \rightarrow 0$ as $x \rightarrow \infty$.
- (a) Show that $E(X) = \int_0^{\infty} [1 - F(x)] dx$. [4]
 (b) X_1 and X_2 are independent, each uniformly distributed on $[0, 1]$. The random variable M is the larger of X_1 and X_2 . Find the cumulative distribution function of M and use part (a) to find $E(M)$. [4]
 (c) Find $E(m)$ where m is the smaller of X_1 and X_2 , and verify that $E(M) + E(m) = E(X_1) + E(X_2)$. [3]

Transformations of Random Variables

- 1** The continuous random variable X has probability density function $f(x) = \frac{1}{2}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. $Y = 3X + 1$. State the range of Y and find its probability density function. [3]
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- 2** $X \sim U[0, 1]$ and $Y = 2X$. Find the cumulative distribution function of Y . [2]
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- 3** X has cumulative distribution function $F(x) = \frac{x^2}{4}$ for $0 \leq x \leq 2$. $Y = X + 1$. Find the cumulative distribution function $G(y)$ of Y . [2]

- 4** The continuous random variable X has probability density function $f(x) = \frac{2x}{9}$ for $0 \leq x \leq 3$, and $f(x) = 0$ otherwise. $Y = X^2$. State the range of Y and find $P(Y \leq 4)$. [3]
- 5** $X \sim U[0, 4]$ and $Y = -X$. Find $P(Y \leq -1)$. [2]
- 6** $X \sim U[1, 4]$ and $Y = \frac{1}{X}$. State the range of Y and find $P(Y > \frac{1}{2})$. [3]
- 7** The continuous random variable X has probability density function $f(x) = \frac{3x^2}{8}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. $Y = 2X$. Find the cumulative distribution function and the probability density function of Y . [5]
- 8** The continuous random variable X has probability density function $f(x) = \frac{x}{2}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. $Y = X + 3$. Find $E(Y)$ and the probability density function of Y . [5]
- 9** $X \sim U[0, 2]$ and $Y = X^2$. Show that the probability density function of Y is $g(y) = \frac{1}{4\sqrt{y}}$ for $0 < y \leq 4$. [4]
- 10** The continuous random variable X has probability density function $f(x) = 2x$ for $0 < x \leq 1$, and $f(x) = 0$ otherwise. $Y = \frac{1}{X}$. Find the probability density function of Y . [5]
- 11** $X \sim U[0, 1]$ and $Y = e^X$. Find the cumulative distribution function and probability density function of Y . [4]
- 12** $X \sim U[0, 1]$ and $Y = -\ln X$. Show that Y has an exponential distribution and state its parameter. [4]
- 13** $X \sim U[0, 9]$ and $Y = \sqrt{X}$. Find the probability density function of Y and find $P(Y > 2)$. [4]
- 14** $X \sim U[0, 2]$ and $Y = 4 - X^2$. Find the cumulative distribution function of Y , and hence its probability density function. [6]
- 15** $X \sim U[1, 2]$ and $Y = \frac{1}{X}$. Find the probability density function of Y and use it to find $E(Y)$, giving your answer in terms of $\ln 2$. [6]
- 16** The continuous random variable X has probability density function $f(x) = \frac{3x^2}{8}$ for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. $Y = X^2$. Find the probability density function of Y , and verify that it integrates to 1. [6]
- 17** The random variable X has probability density function $f(x) = \frac{1}{x}$ for $1 \leq x \leq e$, and $f(x) = 0$ otherwise. Show that $Y = \ln X$ is uniformly distributed, stating the interval. [4]
- 18** The cost, £C, of a job is $C = 2X + 5$, where the time X hours has probability density function $f(x) = \frac{x+1}{12}$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. Find the probability density function of C and the probability that the cost exceeds £11. [6]

- 19** The continuous random variable X has probability density function $f(x) = \frac{2}{x^3}$ for $x \geq 1$, and $f(x) = 0$ otherwise. $Y = X^2$. Find the cumulative distribution function and the probability density function of Y , and find $P(Y > 9)$. [6]
- 20** The continuous random variable $X \sim U[0, 4]$ and $Y = X^2$. A student writes: "The probability density function of X is $\frac{1}{4}$, so the probability density function of Y is $\frac{1}{4}$ for $0 \leq y \leq 16$." Explain why this cannot be correct, and find the correct probability density function of Y . [5]
- 21** $X \sim U[0, 2]$ and $Y = X^3$. Find the cumulative distribution function and probability density function of Y , and find $E(Y)$ in two ways. [8]
- 22** The continuous random variable X has probability density function $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \leq x \leq 1$, and $f(x) = 0$ otherwise. $Y = |X|$. Show that the probability density function of Y is $g(y) = \frac{3}{2}(1 - y^2)$ for $0 \leq y \leq 1$. [6]
- 23** $X \sim U[0, 3]$ and $Y = (X - 2)^2$.
 (a) State the range of Y . [2]
 (b) Show that the cumulative distribution function of Y is $G(y) = \frac{2\sqrt{y}}{3}$ for $0 \leq y \leq 1$ and $G(y) = \frac{1 + \sqrt{y}}{3}$ for $1 < y \leq 4$, and find the probability density function of Y . [7]
- 24** A computer produces random numbers U from the continuous uniform distribution on $[0, 1]$. Show that the transformation $X = -\frac{1}{\lambda} \ln(1 - U)$ produces a variable with an exponential distribution with parameter λ . Hence find the values of X , to 3 significant figures, obtained from $U = 0.2, 0.5$ and 0.9 when $\lambda = 0.5$. [7]
- 25** X_1, X_2 and X_3 are independent random variables, each with an exponential distribution with parameter 1. M is the largest and m the smallest of the three. Find the cumulative distribution functions of M and m , and find $P(m > 0.5)$. [7]
- 26** $X \sim U[0, 3]$ and $Y = \frac{1}{1 + X}$. Find the probability density function of Y and find $P(Y > 0.4)$. [7]
- 27** X_1 and X_2 are independent random variables, each uniformly distributed on $[0, 1]$. The random variable $S = X_1 + X_2$.
 (a) By considering the area of a suitable region of the unit square, show that the cumulative distribution function of S is $G(s) = \frac{s^2}{2}$ for $0 \leq s \leq 1$ and $G(s) = 1 - \frac{(2 - s)^2}{2}$ for $1 < s \leq 2$. [5]
 (b) Find the probability density function of S and use it to find $P(S > 1.5)$. [3]
- 28** X has probability density function $f(x)$ and $Y = aX + b$, where $a \neq 0$.
 (a) Show that the probability density function of Y is $g(y) = \frac{1}{|a|} f\left(\frac{y - b}{a}\right)$, considering separately $a > 0$ and $a < 0$. [5]
 (b) X has probability density function $f(x) = \frac{3x^2}{8}$ for $0 \leq x \leq 2$. Find the probability density function of $Y = 5 - 2X$ and state its range. [4]

29 The continuous random variable X has probability density function $f(x) = \frac{1+x}{2}$ for $-1 \leq x \leq 1$, and $f(x) = 0$ otherwise. $Y = X^2$.

- (a) Show that the cumulative distribution function of Y is $G(y) = \sqrt{y}$ for $0 \leq y \leq 1$, and find the probability density function of Y . [5]
- (b) Find $E(Y)$ in two ways: using $g(y)$, and using $f(x)$ directly. [5]

30 $Z \sim N(0, 1)$ with probability density function $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, and $Y = Z^2$.

- (a) Show that the cumulative distribution function of Y is $G(y) = 2\Phi(\sqrt{y}) - 1$ for $y \geq 0$, where Φ is the cumulative distribution function of Z , and find the probability density function of Y . [6]
- (b) Find $P(Y < 1)$, to 4 decimal places, and state $E(Y)$ given that $\text{Var}(Z) = 1$. [3]