

Introduction to Complex Numbers

1 (2 marks)

(a) $\sqrt{-49} = 7i$

Answer: 7i

B1
1 mark

(b) $\sqrt{-12} = \sqrt{12}i = 2\sqrt{3}i$

Answer: $2\sqrt{3}i$

B1
1 mark

2 (2 marks)

$i^{10} = (i^4)^2 \times i^2 = -1$

$i^{23} = (i^4)^5 \times i^3 = -i$

Answer: $i^{10} = -1$, $i^{23} = -i$

B1
B1
2 marks

3 (3 marks)

(a) Real part 5, imaginary part -9

Answer: 5 and -9

B1
1 mark

(b) Real part 0, imaginary part 4

Answer: 0 and 4

B1
1 mark

(c) Real part 8, imaginary part 0

Answer: 8 and 0

B1
1 mark

4 (2 marks)

$x^2 = -16$

$x = \pm 4i$

Answer: $x = \pm 4i$

M1
A1
2 marks

5 (2 marks)

Real parts: $2a = 8$, so $a = 4$

Imaginary parts: $3b = -12$, so $b = -4$

Answer: $a = 4$, $b = -4$

B1
B1
2 marks

6 (3 marks)

(a) $i^2 = -1$, so it is real

Answer: real

B1
1 mark

(b) $3i^3 = -3i$, so it is purely imaginary

Answer: purely imaginary

B1
1 mark

(c) $2 + i^4 = 3$, so it is real

Answer: real

B1
1 mark

7 (3 marks)

$x = \frac{6 \pm \sqrt{36 - 52}}{2}$

$= \frac{6 \pm 4i}{2}$

$x = 3 \pm 2i$

Answer: $x = 3 + 2i$ or $x = 3 - 2i$

M1
A1
A1
3 marks

8 (3 marks)

(a) $i^{47} = (i^4)^{11} \times i^3$

$= -i$

Answer: $-i$

M1
A1
2 marks

(b) $\frac{1}{i^3} = \frac{1}{-i} = \frac{i}{-i^2} = i$

Answer: i

B1
1 mark

9 (3 marks)

$$a + b = 7 \text{ and } a - b = 3$$

M1

$$\text{Adding: } 2a = 10, \text{ so } a = 5$$

A1

$$b = 2$$

A1

$$\text{Answer: } a = 5, b = 2$$

3 marks

10 (3 marks)

$$x = \frac{-6 \pm \sqrt{36 - 56}}{4}$$

M1

$$= \frac{-6 \pm 2\sqrt{5}i}{4}$$

A1

$$x = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}i$$

A1

$$\text{Answer: } x = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}i$$

3 marks

11 (3 marks)

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \text{ only holds when at least one of } a \text{ and } b \text{ is positive}$$

E1

$$\sqrt{-2} = \sqrt{2}i \text{ and } \sqrt{-8} = 2\sqrt{2}i$$

M1

$$\text{Product} = 4i^2 = -4$$

A1

$$\text{Answer: } -4$$

3 marks

12 (3 marks)

$$\text{Sum of roots} = 4, \text{ product of roots} = 2^2 + 3^2 = 13$$

M1 A1

$$x^2 - 4x + 13 = 0$$

A1

$$\text{Answer: } x^2 - 4x + 13 = 0$$

3 marks

13 (2 marks)

$$i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n(1 + i + i^2 + i^3)$$

M1

$$= i^n(1 + i - 1 - i) = 0$$

A1

$$\text{Answer: } i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0 \text{ for every integer } n$$

2 marks

14 (4 marks)

$$\text{Complex roots of a real quadratic occur in conjugate pairs, so } 1 - 2i \text{ is also a root}$$

B1

$$\text{Sum of roots} = 2 = -p, \text{ so } p = -2$$

M1 A1

$$\text{Product of roots} = 1 + 4 = 5 = q$$

A1

$$\text{Answer: } p = -2, q = 5$$

4 marks

15 (5 marks)

$$1 - 3 + 4 - 2 = 0, \text{ so } x = 1 \text{ is a root}$$

B1

$$x^3 - 3x^2 + 4x - 2 = (x - 1)(x^2 - 2x + 2)$$

M1 A1

$$x = \frac{2 \pm \sqrt{4 - 8}}{2}$$

M1

$$x = 1 \pm i$$

A1

$$\text{Answer: } x = 1, 1 + i, 1 - i$$

5 marks

16 (4 marks)

$$\text{Every four consecutive terms sum to zero } (i - 1 - i + 1 = 0)$$

M1

$$\text{The first 48 terms form 12 such groups and sum to 0}$$

A1

$$\text{Remaining terms } i^{49} + i^{50} = i + (-1)$$

M1

$$\text{Sum} = -1 + i$$

A1

$$\text{Answer: } -1 + i$$

4 marks

17 (4 marks)

No real roots, so $b^2 - 4ac < 0$: $4k^2 - 4(3k + 4) < 0$

M1

$$k^2 - 3k - 4 < 0$$

A1

$$(k - 4)(k + 1) < 0$$

M1

$$-1 < k < 4$$

A1

Answer: $-1 < k < 4$

4 marks

18 (4 marks)

$$(a + 3b) + (2a - b)i = 5 + 3i$$

M1

$$a + 3b = 5 \text{ and } 2a - b = 3$$

A1

From the second, $b = 2a - 3$, so $a + 6a - 9 = 5$

M1

$$a = 2, b = 1$$

A1

Answer: $a = 2, b = 1$

4 marks

19 (4 marks)

$$(x^2 - 4)(x^2 + 4) = 0$$

M1

$$x^2 = 4 \text{ gives } x = \pm 2$$

A1

$$x^2 = -4$$

M1

$$x = \pm 2i$$

A1

Answer: $x = 2, -2, 2i, -2i$

4 marks

20 (4 marks)

The roots are conjugates $-3 + bi$ and $-3 - bi$ (sum of roots is -6)

M1

$$\text{Difference} = 2bi = 10i, \text{ so } b = 5$$

A1

$$\text{Roots } -3 \pm 5i$$

A1

$$k = (-3)^2 + 5^2 = 34$$

A1

Answer: $z = -3 \pm 5i, k = 34$

4 marks

21 (6 marks)

Another root is $2 - i$, so a quadratic factor is $z^2 - 4z + 5$

B1 M1

$$z^3 + az^2 + bz + 10 = (z^2 - 4z + 5)(z + c); \text{ constant term } 5c = 10, \text{ so } c = 2$$

M1

$$\text{Expanding: } z^3 - 2z^2 - 3z + 10$$

M1

$$a = -2, b = -3$$

A1

The other roots are $2 - i$ and -2

A1

Answer: $a = -2, b = -3$; other roots $2 - i$ and -2

6 marks

22 (5 marks)

(a) $\sqrt{-36} = 6i$ is correct, but the whole numerator must be divided by 2

E1

The student divided only the -4 and left $6i$ undivided

E1

Answer: $6i$ should have been divided by 2

2 marks

$$(b) x = \frac{-4 \pm 6i}{2}$$

M1

$$= -2 \pm 3i$$

A1

Check: sum of roots -4 and product $4 + 9 = 13$

B1

Answer: $x = -2 \pm 3i$

3 marks

23 (6 marks)

$$(1 + i)^2 = 1 + 2i + i^2 = 2i$$

M1

$$(1 + i)^4 = (2i)^2 = -4$$

A1

$$(1 + i)^{4n} = ((1 + i)^4)^n = (-4)^n$$

B1

$$(1 + i)^{20} = (-4)^5 = -1024$$

B1

$$(1 - i)^2 = -2i, \text{ so } (1 - i)^4 = -4 \text{ and } (1 - i)^{20} = -1024$$

M1

$$\text{Sum} = -2048$$

A1

Answer: -2048

6 marks

24 (6 marks)

$$(a) x = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

M1

$$\alpha, \beta = -1 \pm 2i$$

A1

Answer: $-1 + 2i$ and $-1 - 2i$

2 marks

$$(b) (-1 + 2i)^2 = 1 - 4i - 4 = -3 - 4i, \text{ and } \beta^2 = -3 + 4i$$

M1 A1

$$\text{Sum} = -6, \text{ product} = 9 + 16 = 25$$

M1

$$x^2 + 6x + 25 = 0$$

A1

$$\text{Answer: } x^2 + 6x + 25 = 0$$

4 marks

25 (7 marks)

$$\text{Let the real root be } t: t^2 + (2 + i)t + k + 3i = 0$$

M1

$$\text{Real part: } t^2 + 2t + k = 0, \text{ imaginary part: } t + 3 = 0$$

M1 A1

$$\text{So } t = -3 \text{ and } 9 - 6 + k = 0, \text{ giving } k = -3$$

A1 A1

$$\text{Sum of the roots} = -(2 + i), \text{ so the other root} = -2 - i + 3$$

M1

$$\text{Other root} = 1 - i$$

A1

$$\text{Answer: } k = -3, \text{ other root } 1 - i$$

7 marks

26 (6 marks)

$$i^{-n} = (i^{-1})^n = (-i)^n, \text{ so the expression is } i^n + (-i)^n$$

M1 A1

$$n = 4m: 1 + 1 = 2$$

B1

$$n = 4m + 1 \text{ or } n = 4m + 3: i + (-i) = 0 \text{ or } -i + i = 0$$

B1

$$n = 4m + 2: -1 + (-1) = -2$$

B1

The powers of i repeat with period 4, so no other values occur

E1

$$\text{Answer: } 2 \text{ when } n \text{ is a multiple of } 4; -2 \text{ when } n = 4m + 2; 0 \text{ when } n \text{ is odd}$$

6 marks

27 (8 marks)

$$(a) (x^2 + 2)^2 - (2x)^2 = x^4 + 4x^2 + 4 - 4x^2$$

M1

$$= x^4 + 4, \text{ and the left side is a difference of two squares}$$

A1

$$\text{Answer: } x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$$

2 marks

$$(b) x^2 + 2x + 2 = 0: x = \frac{-2 \pm \sqrt{4 - 8}}{2}$$

M1

$$x = -1 \pm i$$

A1

$$x^2 - 2x + 2 = 0: x = \frac{2 \pm \sqrt{4 - 8}}{2}$$

M1

$$x = 1 \pm i$$

A1

$$\text{Answer: } x = 1 + i, 1 - i, -1 + i, -1 - i$$

4 marks

$$(c) \text{ Each root gives a factor } (x - \text{root})$$

M1

$$(x - 1 - i)(x - 1 + i)(x + 1 - i)(x + 1 + i)$$

A1

$$\text{Answer: } (x - 1 - i)(x - 1 + i)(x + 1 - i)(x + 1 + i)$$

2 marks

28 (8 marks)

$$\text{Another root is } 1 + 2i, \text{ so a quadratic factor is } z^2 - 2z + 5$$

B1 M1

$$\text{The other factor is } z^2 + pz + c \text{ with } 5c = 5, \text{ so } c = 1$$

M1

$$(z^2 - 2z + 5)(z^2 + pz + 1): z^2 \text{ coefficient } 1 - 2p + 5 = 6$$

M1

$$\text{So } p = 0$$

A1

$$\text{Expanding gives } z^4 - 2z^3 + 6z^2 - 2z + 5, \text{ so } a = -2, b = -2$$

A1 A1

$$z^2 + 1 = 0 \text{ gives } z = \pm i; \text{ other roots } 1 + 2i, i, -i$$

A1

$$\text{Answer: } a = -2, b = -2; \text{ other roots } 1 + 2i, i, -i$$

8 marks

29 (7 marks)

(a) $w^2 + pw + q = 0$

B1Take the conjugate of both sides: $\overline{w^2 + pw + q} = 0$ **M1**Conjugation preserves sums and products and fixes real numbers: $\overline{w^2} + p\overline{w} + q = 0$ **M1**Hence $\overline{w}$ is also a root**A1****Answer: If the complex number w is a root of $z^2 + pz + q = 0$ then its conjugate $\overline{w}$ is also a root****4 marks****(b)** The argument of (a) extends to any real polynomial: non-real roots occur in conjugate pairs**M1**

A cubic has exactly 3 roots (counting multiplicity), so the non-real roots use up an even number of them

M1

At most 2 roots are non-real, so at least 1 is real

A1**Answer: A cubic equation with real coefficients always has at least one real root****3 marks****30 (6 marks)**Group the terms in fours: the group starting at the term $(4m + 1)i^{4m}$ is**M1**

$$(4m + 1) + (4m + 2)i - (4m + 3) - (4m + 4)i$$

A1

$$= -2 - 2i, \text{ which is independent of } m$$

A1There are $100 \div 4 = 25$ groups**B1**

$$\text{Sum} = 25(-2 - 2i)$$

M1

$$= -50 - 50i$$

A1**Answer: $-50 - 50i$** **6 marks****Complex Arithmetic****1 (2 marks)**

(a) $z + w = (3 + 1) + (4 - 2)i = 4 + 2i$

B1**Answer: $4 + 2i$** **1 mark**

(b) $z - w = (3 - 1) + (4 + 2)i = 2 + 6i$

B1**Answer: $2 + 6i$** **1 mark****2 (2 marks)**

$$(2 + i)(3 - i) = 6 - 2i + 3i - i^2$$

M1

$$= 7 + i$$

A1**Answer: $7 + i$** **2 marks****3 (2 marks)**

$$(1 + 3i)^2 = 1 + 6i + 9i^2$$

M1

$$= -8 + 6i$$

A1**Answer: $-8 + 6i$** **2 marks****4 (2 marks)**

$$\frac{10}{2 - i} = \frac{10(2 + i)}{(2 - i)(2 + i)} = \frac{10(2 + i)}{5}$$

M1

$$= 4 + 2i$$

A1**Answer: $4 + 2i$** **2 marks****5 (2 marks)**

$$3(2 - i) = 6 - 3i \text{ and } i(4 + 2i) = 4i - 2$$

M1

$$6 - 3i - 4i + 2 = 8 - 7i$$

A1**Answer: $8 - 7i$** **2 marks****6 (2 marks)**

$$(3 - 2i)(3 + 2i) = 9 - 4i^2$$

M1

$$= 13, \text{ which is real}$$

A1**Answer: 13****2 marks**

7 (3 marks)

$$(1 + i)(2 - i) = 2 - i + 2i - i^2 = 3 + i$$

M1 A1

$$(3 + i)(3 + i) = 9 + 6i - 1 = 8 + 6i$$

A1

Answer: 8 + 6i

3 marks

8 (3 marks)

$$\frac{5 + i}{2 + 3i} = \frac{(5 + i)(2 - 3i)}{(2 + 3i)(2 - 3i)}$$

M1

$$= \frac{10 - 15i + 2i + 3}{13} = \frac{13 - 13i}{13}$$

A1

$$= 1 - i$$

A1

Answer: 1 - i

3 marks

9 (3 marks)

$$z^2 = 1 - 4i + 4i^2 = -3 - 4i$$

M1 A1

$$z^3 = (-3 - 4i)(1 - 2i) = -3 + 6i - 4i + 8i^2 = -11 + 2i$$

A1

Answer: $z^2 = -3 - 4i$, $z^3 = -11 + 2i$

3 marks

10 (4 marks)

$$(x + yi)(2 - i) = (2x + y) + (2y - x)i$$

M1 A1

$$\text{Equating: } 2x + y = 5 \text{ and } 2y - x = -5$$

M1

$$x = 3, y = -1$$

A1

Answer: $x = 3$, $y = -1$

4 marks

11 (4 marks)

$$(a) Z = (1 + 2i) + (2 - i) = 3 + i \Omega$$

B1

Answer: 3 + i Ω

1 mark

$$(b) Z_1 Z_2 = (1 + 2i)(2 - i) = 2 - i + 4i + 2 = 4 + 3i$$

M1

$$Z = \frac{4 + 3i}{3 + i} = \frac{(4 + 3i)(3 - i)}{10}$$

M1

$$= \frac{15 + 5i}{10} = \frac{3}{2} + \frac{1}{2}i \Omega$$

A1

Answer: $\frac{3}{2} + \frac{1}{2}i \Omega$

3 marks

12 (3 marks)

$$(1 + i)^2 = 2i$$

B1

$$\frac{2i}{1 - i} = \frac{2i(1 + i)}{(1 - i)(1 + i)} = \frac{2i - 2}{2}$$

M1

$$= -1 + i$$

A1

Answer: -1 + i

3 marks

13 (3 marks)

$$\frac{i}{1 + i} = \frac{i(1 - i)}{2} = \frac{1 + i}{2}$$

M1

$$\frac{1}{1 - i} = \frac{1 + i}{2}$$

A1

$$\text{Sum} = 1 + i$$

A1

Answer: 1 + i

3 marks

14 (4 marks)

$$\frac{\lambda + 6i}{2 + \lambda i} = \frac{(\lambda + 6i)(2 - \lambda i)}{(2 + \lambda i)(2 - \lambda i)} = \frac{(\lambda + 6i)(2 - \lambda i)}{4 + \lambda^2}$$

M1

$$(\lambda + 6i)(2 - \lambda i) = 2\lambda - \lambda^2 i + 12i + 6\lambda = 8\lambda + (12 - \lambda^2)i$$

A1

$$\text{The number is real when the imaginary part is zero: } 12 - \lambda^2 = 0$$

M1

$$\lambda = \pm 2\sqrt{3}$$

A1

Answer: $\lambda = 2\sqrt{3}$ or $\lambda = -2\sqrt{3}$

4 marks

15 (4 marks)

$$(a) (3 + \lambda i)(2 - i) = 6 + \lambda + (2\lambda - 3)i$$

M1

$$\text{Imaginary part zero: } \lambda = \frac{3}{2}$$

A1

$$\text{Answer: } \lambda = \frac{3}{2}$$

2 marks

$$(b) \text{ Real part zero: } 6 + \lambda = 0$$

M1

$$\lambda = -6$$

A1

$$\text{Answer: } \lambda = -6$$

2 marks

16 (4 marks)

$$(2 - i)^2 = 4 - 4i + i^2$$

M1

$$= 3 - 4i$$

A1

$$(2 - i)^4 = (3 - 4i)^2 = 9 - 24i + 16i^2$$

M1

$$= -7 - 24i$$

A1

$$\text{Answer: } -7 - 24i$$

4 marks

17 (3 marks)

Multiply top and bottom by the conjugate $2 + i$

M1

$$\frac{(1 + 2i)(2 + i)}{(2 - i)(2 + i)} = \frac{2 + i + 4i + 2i^2}{5}$$

A1

$$= \frac{5i}{5} = i$$

A1

$$\text{Answer: } \frac{1 + 2i}{2 - i} = i$$

3 marks

18 (4 marks)

$$\frac{x}{1 + i} = \frac{x(1 - i)}{2}, \frac{y}{2 - i} = \frac{y(2 + i)}{5} \text{ and } \frac{17}{4 + i} = \frac{17(4 - i)}{17} = 4 - i$$

M1 A1

$$\text{Real parts: } \frac{x}{2} + \frac{2y}{5} = 4. \text{ Imaginary parts: } -\frac{x}{2} + \frac{y}{5} = -1$$

M1

$$\text{Adding: } \frac{3y}{5} = 3, \text{ so } y = 5 \text{ and } x = 4$$

A1

$$\text{Answer: } x = 4, y = 5$$

4 marks

19 (4 marks)

$$z^2 = 9 - 6i + i^2 = 8 - 6i$$

M1 A1

$$p + qz = (p + 3q) - qi$$

M1

$$q = 6 \text{ and } p + 18 = 8, \text{ so } p = -10$$

A1

$$\text{Answer: } p = -10, q = 6$$

4 marks

20 (4 marks)

$$z^2 = 1 + 2i + i^2 = 2i$$

M1 A1

$$\frac{1}{z^2} = \frac{1}{2i} = -\frac{i}{2} \text{ and } z^4 = (2i)^2 = -4$$

M1

$$\frac{1}{z^2} + \frac{1}{z^4} = -\frac{1}{4} - \frac{1}{2}i$$

A1

$$\text{Answer: } -\frac{1}{4} - \frac{1}{2}i$$

4 marks

21 (6 marks)

$$(a) \text{ The numerator is correct: } 6 + 5i - 1 = 5 + 5i$$

E1

$$\text{The denominator should be } 4 - i^2 = 4 + 1 = 5, \text{ not } 3$$

E1

$$\text{Answer: } 4 - i^2 = 5, \text{ not } 3$$

2 marks

$$(b) \frac{5 + 5i}{5} = 1 + i$$

M1 A1

$$\text{Check: } (1 + i)(2 - i) = 2 - i + 2i - i^2$$

M1

$$= 3 + i$$

A1

$$\text{Answer: } 1 + i$$

4 marks

22 (6 marks)

$$(a) (a + bi)(a - bi) = a^2 - abi + abi - b^2i^2$$

$$= a^2 + b^2$$

M1

A1

Answer: $(a + bi)(a - bi) = a^2 + b^2$

2 marks

$$(b) \frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)}$$

M1

$$= \frac{a - bi}{a^2 + b^2}, \text{ which gives the stated form}$$

A1

Answer: $\frac{1}{a + bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$

2 marks

$$(c) \frac{1}{3 + 4i} = \frac{3}{25} - \frac{4}{25}i \text{ and } \frac{1}{3 - 4i} = \frac{3}{25} + \frac{4}{25}i$$

M1

$$\text{Difference} = -\frac{8}{25}i$$

A1

Answer: $-\frac{8}{25}i$

2 marks

23 (7 marks)

$$(a) \frac{1}{Z} = \frac{Z_1 + Z_2}{Z_1 Z_2}, \text{ so } Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

M1

$$Z_1 Z_2 = (2 + 3i)(4 - 3i) = 8 - 6i + 12i + 9 = 17 + 6i$$

M1 A1

$$Z_1 + Z_2 = 6, \text{ so } Z = \frac{17 + 6i}{6} = \frac{17}{6} + i$$

A1

Answer: $Z = \frac{17}{6} + i$

4 marks

$$(b) I = \frac{26}{\frac{17 + 6i}{6}} = \frac{156}{17 + 6i}$$

M1

$$= \frac{156(17 - 6i)}{289 + 36} = \frac{156(17 - 6i)}{325}$$

M1

$$I = \frac{204}{25} - \frac{72}{25}i \text{ A}$$

A1

Answer: $I = \frac{204}{25} - \frac{72}{25}i \text{ A}$

3 marks

24 (8 marks)

$$(a) z^2 = x^2 + 2xyi + y^2i^2$$

M1

$$\text{Real part } x^2 - y^2, \text{ imaginary part } 2xy$$

A1

Answer: $\text{Re} = x^2 - y^2, \text{Im} = 2xy$

2 marks

$$(b) z^3 = (x^2 - y^2 + 2xyi)(x + iy)$$

M1

$$\text{Real: } x(x^2 - y^2) - 2xy^2 = x^3 - 3xy^2$$

A1

$$\text{Imaginary: } y(x^2 - y^2) + 2x^2y = 3x^2y - y^3$$

A1

Answer: $\text{Re} = x^3 - 3xy^2, \text{Im} = 3x^2y - y^3$

3 marks

$$(c) \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)}$$

M1

$$= \frac{x - iy}{x^2 + y^2}$$

A1

$$\text{Re} = \frac{x}{x^2 + y^2}, \text{Im} = -\frac{y}{x^2 + y^2}$$

A1

Answer: $\text{Re} = \frac{x}{x^2 + y^2}, \text{Im} = -\frac{y}{x^2 + y^2}$

3 marks

25 (6 marks)

$$(a) (1 + ai)^3 = 1 + 3ai + 3a^2i^2 + a^3i^3$$

M1

$$= (1 - 3a^2) + (3a - a^3)i, \text{ so } q = 3a - a^3$$

A1

Answer: $q = 3a - a^3$

2 marks

$$(b) 3a - a^3 = a(3 - a^2) = 0$$

M1

$$a = 0 \text{ or } a = \pm\sqrt{3}$$

A1

$$\text{Answer: } a = 0, \sqrt{3}, -\sqrt{3}$$

2 marks

$$(c) 1 - 3a^2 = 0$$

M1

$$a = \pm \frac{1}{\sqrt{3}}$$

A1

$$\text{Answer: } a = \pm \frac{1}{\sqrt{3}}$$

2 marks

26 (6 marks)

$$\text{From the first equation } w = 3 + i - z$$

B1

$$iz + 2(3 + i - z) = 4 + 3i$$

M1

$$(i - 2)z = 4 + 3i - 6 - 2i = -2 + i$$

M1 A1

$$z = \frac{-2 + i}{-2 + i} = 1$$

A1

$$w = 3 + i - 1 = 2 + i$$

A1

$$\text{Answer: } z = 1, w = 2 + i$$

6 marks

27 (10 marks)

$$(a) w = \frac{x + (y + 2)i}{(x - 1) + yi} = \frac{[x + (y + 2)i][(x - 1) - yi]}{(x - 1)^2 + y^2}$$

M1 A1

$$\text{Real part of numerator: } x(x - 1) + y(y + 2) = x^2 - x + y^2 + 2y$$

M1 A1

$$\text{Imaginary part of numerator: } (y + 2)(x - 1) - xy = 2x - y - 2, \text{ so } \text{Im } w = \frac{2x - y - 2}{(x - 1)^2 + y^2}$$

A1

$$\text{Answer: } \text{Im } w = \frac{2x - y - 2}{(x - 1)^2 + y^2}$$

5 marks

$$(b) \text{Im } w = 0: 2x - y - 2 = 0$$

M1

$$y = 2x - 2 \text{ (with } z \neq 1)$$

A1

$$\text{Answer: } y = 2x - 2$$

2 marks

$$(c) z + 2i = i(z - 1) = iz - i$$

M1

$$z(1 - i) = -3i$$

A1

$$z = \frac{-3i(1 + i)}{2} = \frac{3}{2} - \frac{3}{2}i$$

A1

$$\text{Answer: } z = \frac{3}{2} - \frac{3}{2}i$$

3 marks

28 (8 marks)

$$(a) u^2 = 1 + 2\sqrt{3}i + 3i^2 = -2 + 2\sqrt{3}i$$

M1 A1

$$u^3 = (-2 + 2\sqrt{3}i)(1 + \sqrt{3}i) = -2 - 2\sqrt{3}i + 2\sqrt{3}i + 6i^2 = -8$$

A1

$$\text{Answer: } u^3 = -8$$

3 marks

$$(b) u^{12} = (u^3)^4 = (-8)^4$$

M1

$$= 4096$$

A1

$$\text{Answer: } 4096$$

2 marks

$$(c) u^n \text{ is real when } n = 3 \text{ (value } -8) \text{ and } u^1, u^2 \text{ are not real}$$

M1

$$u^4 = -8u \text{ and } u^5 = -8u^2 \text{ are not real, } u^6 = 64$$

M1

$$\text{So } n = 6$$

A1

$$\text{Answer: } n = 6$$

3 marks

29 (8 marks)

$$(a) (1 + i)^2 = 1 + 2i + i^2 = 2i$$

M1

$$(1 + i)^4 = (2i)^2 = -4$$

A1

$$(1 + i)^8 = (-4)^2 = 16$$

A1

$$\text{Answer: } (1 + i)^8 = 16$$

3 marks

$$(b) \text{Geometric series with first term 1, common ratio } 1 + i \text{ and 8 terms: } S = \frac{(1 + i)^8 - 1}{(1 + i) - 1}$$

M1

$$= \frac{15}{i}$$

A1

$$S = -15i$$

A1

Answer: $S = -15i$

3 marks

(c) Sum of the real parts = $\text{Re}(S) = 0$

B1

Sum of the imaginary parts = $\text{Im}(S) = -15$

B1

Answer: 0 and -15

2 marks

30 (8 marks)

(a) $(2 + i)(2 - i) = 4 - i^2 = 5$

B1

$(3 + 2i)(3 - 2i) = 9 - 4i^2 = 13$

B1

Answer: $(2 + i)(2 - i) = 5$ and $(3 + 2i)(3 - 2i) = 13$

2 marks

(b) $(2 + i)(3 + 2i) = 6 + 4i + 3i + 2i^2 = 4 + 7i$

M1 A1

$(4 + 7i)(4 - 7i) = (2 + i)(3 + 2i)(2 - i)(3 - 2i) = 5 \times 13 = 65$, and also $4^2 + 7^2 = 65$

A1

Answer: $65 = 4^2 + 7^2$

3 marks

(c) $(2 + i)(3 - 2i) = 6 - 4i + 3i - 2i^2 = 8 - i$

M1 A1

$65 = 8^2 + 1^2$

A1

Answer: $65 = 8^2 + 1^2$

3 marks

Complex Conjugates

1 (3 marks)

(a) Change the sign of the imaginary part

B1

Answer: $3 - 5i$

1 mark

(b) Change the sign of the imaginary part

B1

Answer: $-2 + i$

1 mark

(c) Change the sign of the imaginary part

B1

Answer: $-7i$

1 mark

2 (2 marks)

(a) $(4 - 3i) + (4 + 3i) = 8$

B1

Answer: 8

1 mark

(b) $(4 - 3i)(4 + 3i) = 16 + 9 = 25$

B1

Answer: 25

1 mark

3 (2 marks)

$(1 + i)(2 - i) = 2 - i + 2i + 1 = 3 + i$

M1

Conjugate = $3 - i$

A1

Answer: $3 - i$

2 marks

4 (2 marks)

$$\frac{1}{3 + i} = \frac{3 - i}{(3 + i)(3 - i)} = \frac{3 - i}{10}$$

M1

$$= \frac{3}{10} - \frac{1}{10}i$$

A1

Answer: $\frac{3}{10} - \frac{1}{10}i$

2 marks

5 (2 marks)

(a) True: the imaginary part is 0

B1

Answer: True

1 mark

(b) True: $z = bi$ gives $\bar{z} = -bi = -z$

B1

Answer: True

1 mark

6 (2 marks)

(a) Complex roots of a real quadratic form a conjugate pair: $3 - 4i$

B1

Answer: $3 - 4i$

1 mark

(b) Sum = 6, product = $3^2 + 4^2 = 25$

B1

Answer: Sum 6, product 25**1 mark****7 (4 marks)**

$$(a) \ z + \bar{z} = (a + bi) + (a - bi) \\ = 2a, \text{ which is real}$$

M1**A1****Answer: $z + \bar{z}$ is real****2 marks**

$$(b) \ z\bar{z} = (a + bi)(a - bi) = a^2 - b^2i^2 \\ = a^2 + b^2 \geq 0$$

M1**A1****Answer: $z\bar{z}$ is real and non-negative****2 marks****8 (3 marks)**

$$z = \frac{(2 + i)(1 + i)}{(1 - i)(1 + i)} \\ = \frac{2 + 2i + i - 1}{2} = \frac{1 + 3i}{2}$$

M1**A1**

$$\bar{z} = \frac{1}{2} - \frac{3}{2}i$$

A1

$$\text{Answer: } \frac{1}{2} - \frac{3}{2}i$$

3 marks**9 (3 marks)**

$$z + \bar{z} = 6 \text{ and } z\bar{z} = 9 + 4 = 13$$

B1

$$z^2 + \bar{z}^2 = 6^2 - 2 \times 13 \\ = 10$$

M1**A1****Answer: 10****3 marks****10 (4 marks)**The third root is $1 - i$ **B1**

$$(x - 1 - i)(x - 1 + i) = x^2 - 2x + 2$$

M1

$$(x - 2)(x^2 - 2x + 2)$$

M1

$$x^3 - 4x^2 + 6x - 4 = 0$$

A1

$$\text{Answer: } x^3 - 4x^2 + 6x - 4 = 0$$

4 marks**11 (3 marks)**

$$\text{Let } z = a + bi: a + bi + 2a - 2bi = 3a - bi$$

M1

$$3a = 6 \text{ and } -b = 3$$

A1

$$z = 2 - 3i$$

A1

$$\text{Answer: } z = 2 - 3i$$

3 marks**12 (3 marks)**

$$zw = (2 + i)(1 - 3i) = 2 - 6i + i + 3 = 5 - 5i, \text{ so } \overline{zw} = 5 + 5i$$

M1 A1

$$\overline{zw} = (2 - i)(1 + 3i) = 2 + 6i - i + 3 = 5 + 5i, \text{ the two are equal}$$

A1**Answer: Both equal $5 + 5i$** **3 marks****13 (3 marks)**

$$\frac{\bar{z}}{z} = \frac{1 - 2i}{1 + 2i} = \frac{(1 - 2i)^2}{(1 + 2i)(1 - 2i)}$$

M1

$$= \frac{1 - 4i - 4}{5}$$

A1

$$= -\frac{3}{5} - \frac{4}{5}i$$

A1

$$\text{Answer: } -\frac{3}{5} - \frac{4}{5}i$$

3 marks

14 (4 marks)

$2 - 3i$ is also a root (real coefficients)

B1

Sum of the roots = 5, so the third root is $5 - (2 + 3i) - (2 - 3i) = 1$

M1 A1

Check with the product of the roots: $13 \times 1 = 13$

A1

Answer: $z = 2 - 3i$ and $z = 1$

4 marks

15 (5 marks)

Let $z = a + bi$: $a^2 + b^2 + 3a + 3bi = 14 + 6i$

M1 A1

Imaginary parts: $3b = 6$, so $b = 2$

A1

$a^2 + 3a + 4 = 14$, so $(a - 2)(a + 5) = 0$

M1

$z = 2 + 2i$ or $z = -5 + 2i$

A1

Answer: $z = 2 + 2i$ or $z = -5 + 2i$

5 marks

16 (4 marks)

(a) $(3 - i)^n$ is the complex conjugate of $(3 + i)^n$

M1

The sum of a number and its conjugate is $2\text{Re}((3 + i)^n)$, which is real

A1

Answer: $(3 + i)^n + (3 - i)^n$ is a real number

2 marks

(b) $(3 + i)^2 = 8 + 6i$ and $(3 + i)^4 = (8 + 6i)^2 = 28 + 96i$

M1

Sum = $2 \times 28 = 56$

A1

Answer: 56

2 marks

17 (4 marks)

Let $u = \frac{w}{\bar{w}}$; then the conjugate of u is $\frac{\bar{w}}{w}$

M1 A1

So the expression is $u + \bar{u}$

M1

$u + \bar{u} = 2\text{Re}(u)$, which is real

A1

Answer: $\frac{w}{\bar{w}} + \frac{\bar{w}}{w}$ is real

4 marks

18 (4 marks)

$$\frac{3-i}{1+i} = \frac{(3-i)(1-i)}{2} = \frac{3-3i-i-1}{2} = 1-2i$$

M1 A1

The other root is $1 + 2i$: sum 2, product 5

M1

$$x^2 - 2x + 5 = 0$$

A1

Answer: $x^2 - 2x + 5 = 0$

4 marks

19 (3 marks)

$$w = \frac{25}{z} = \frac{25}{3-4i}$$

M1

$$= \frac{25(3+4i)}{(3-4i)(3+4i)} = \frac{25(3+4i)}{25}$$

M1

$$w = 3 + 4i$$

A1

Answer: $w = 3 + 4i$

3 marks

20 (4 marks)

Let $z = a + bi$: $2a + 2bi + i(a - bi) = 2a + b + (a + 2b)i$

M1 A1

$2a + b = 4$ and $a + 2b = 5$

M1

$a = 1$, $b = 2$, so $z = 1 + 2i$

A1

Answer: $z = 1 + 2i$

4 marks

21 (6 marks)

(a) The conjugate root theorem only applies when all the coefficients are real; here they are not

E1

Answer: Coefficients not all real

1 mark

(b) $(1 - i)^2 - (3 + i)(1 - i) + 2 + 2i = -2i - (4 - 2i) + 2 + 2i$

M1

$$= -2 + 2i \neq 0$$

A1

Answer: Value is $-2 + 2i$, not zero

2 marks

(c) Sum of the roots = $3 + i$

M1

$$\begin{aligned}\text{Other root} &= 3 + i - (1 + i) \\ &= 2\end{aligned}$$

Answer: 2

M1

A1

3 marks

22 (6 marks)

$$\text{Multiply } \frac{z}{z} \text{ by } \frac{z}{z}: \frac{z}{z} = \frac{z^2}{zz}$$

M1

$$\text{So } z^2 = z\bar{z} \times \left(\frac{3}{5} + \frac{4}{5}i\right) = 20\left(\frac{3}{5} + \frac{4}{5}i\right)$$

M1 A1

$$z^2 = 12 + 16i$$

A1

$$\text{Let } z = a + bi: a^2 - b^2 = 12 \text{ and } 2ab = 16, \text{ so } b = \frac{8}{a} \text{ and } a^4 - 12a^2 - 64 = 0$$

M1

$$(a^2 - 16)(a^2 + 4) = 0, \text{ so } a = \pm 4, b = \pm 2: z = 4 + 2i \text{ or } z = -4 - 2i$$

A1

Answer: $z = 4 + 2i$ or $z = -4 - 2i$

6 marks

23 (6 marks)

$$\text{Let } z = a + bi: a^2 - b^2 + 2abi = a - bi$$

M1 A1

$$\text{Real: } a^2 - b^2 = a. \text{ Imaginary: } 2ab = -b, \text{ so } b(2a + 1) = 0$$

A1

$$\text{If } b = 0: a^2 = a, \text{ so } z = 0 \text{ or } z = 1$$

M1 A1

$$\text{If } a = -\frac{1}{2}: b^2 = a^2 - a = \frac{3}{4}, \text{ so } z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

A1

$$\text{Answer: } z = 0, 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

6 marks

24 (6 marks)

$$1 - 2i \text{ is also a root, giving the factor } z^2 - 2z + 5$$

B1 M1

$$z^4 - 6z^3 + 18z^2 - 30z + 25 = (z^2 - 2z + 5)(z^2 + pz + 5)$$

M1

$$\text{Coefficient of } z^3: p - 2 = -6, \text{ so } p = -4$$

A1

$$z^2 - 4z + 5 = 0: z = \frac{4 \pm \sqrt{16 - 20}}{2}$$

M1

$$z = 1 \pm 2i \text{ or } z = 2 \pm i$$

A1

Answer: $z = 1 + 2i, 1 - 2i, 2 + i, 2 - i$

6 marks

25 (8 marks)

$$(a) \ z\bar{w} = (a + bi)(c - di) = (ac + bd) + (bc - ad)i$$

M1 A1

$$\bar{z}w \text{ is the conjugate of } z\bar{w}, \text{ so the sum is } 2(ac + bd)$$

A1

Answer: $z\bar{w} + \bar{z}w = 2(ac + bd)$

3 marks

$$(b) \ 2(2p + 12) = 0$$

M1

$$p = -6$$

A1

Answer: $p = -6$

2 marks

$$(c) \ \frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} \text{ and } w\bar{w} = c^2 + d^2 \text{ is real and positive}$$

M1

$$\text{Real part of } z\bar{w} \text{ is } ac + bd$$

A1

$$ac + bd = 0, \text{ so } \frac{z}{w} \text{ has zero real part}$$

A1

Answer: If $z\bar{w} + \bar{z}w = 0$ and $w \neq 0$, then $\frac{z}{w}$ is purely imaginary

3 marks

26 (6 marks)

$$(a) \ \alpha + \beta = 2 \text{ and } \alpha\beta = 5$$

B1

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

M1

$$= 8 - 30 = -22$$

A1

Answer: -22

3 marks

$$(b) \ \alpha = 1 + 2i, \beta = 1 - 2i$$

B1

$$\alpha^2 = -3 + 4i \text{ and } \alpha^3 = (-3 + 4i)(1 + 2i) = -11 - 2i$$

M1

$$\beta^3 = -11 + 2i, \text{ so } \alpha^3 + \beta^3 = -22$$

A1

Answer: -22

3 marks

27 (8 marks)

(a) $\overline{w} = \frac{\overline{z} - 1}{\overline{z} + 1}$

B1

Using $\overline{z} = \frac{1}{z}$: $\overline{w} = \frac{\frac{1}{z} - 1}{\frac{1}{z} + 1}$

M1

$= \frac{(1 - z)}{(1 + z)}$ (multiplying top and bottom by z)

M1

$= -\frac{z - 1}{z + 1} = -w$

A1

Answer: $\overline{w} = -w$

4 marks

(b) $w + \overline{w} = 0$, and $w + \overline{w} = 2\text{Re}(w)$

M1

So $\text{Re}(w) = 0$

A1

Answer: $\text{Re}(w) = 0$

2 marks

(c) $w = \frac{-\frac{2}{5} + \frac{4}{5}i}{\frac{8}{5} + \frac{4}{5}i} = \frac{-1 + 2i}{4 + 2i}$

M1

$= \frac{(-1 + 2i)(4 - 2i)}{20} = \frac{10i}{20} = \frac{1}{2}i$

A1

Answer: $w = \frac{1}{2}i$

2 marks

28 (7 marks)

Take the conjugate of the second equation: $2z + \overline{w} = 5 + 5i$

M1 A1

Now solve $z + 2\overline{w} = 4 + 7i$ and $2z + \overline{w} = 5 + 5i$ for z and $\overline{w}$

M1

Eliminate $\overline{w}$: $2(2z + \overline{w}) - (z + 2\overline{w}) = 3z = 10 + 10i - 4 - 7i = 6 + 3i$

M1 A1

$z = 2 + i$

A1

$\overline{w} = 5 + 5i - 2(2 + i) = 1 + 3i$, so $w = 1 - 3i$

A1

Answer: $z = 2 + i$, $w = 1 - 3i$

7 marks

29 (8 marks)

(a) $\frac{z}{z} = \frac{z^2}{z\overline{z}} = \frac{(a + bi)^2}{a^2 + b^2}$

M1 A1

$(a + bi)^2 = a^2 - b^2 + 2abi$, which gives the result

A1

Answer: $\frac{z}{z} = \frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$

3 marks

(b) Imaginary part zero: $2ab = 0$

M1

So $a = 0$ or $b = 0$

A1

Answer: $\frac{z}{z}$ is real if and only if z is real or purely imaginary

2 marks

(c) Real part zero: $a^2 = b^2$

M1

$b = a$ or $b = -a$, so $z = a \pm ai$ ($a \neq 0$)

A1

They lie on the lines $y = x$ and $y = -x$, excluding the origin

A1

Answer: $z = a \pm ai$, on $y = \pm x$, excluding O

3 marks

30 (8 marks)

(a) $n = 1$ is true: $\overline{z} = \overline{z}$

B1

Assume $\overline{z^k} = (\overline{z})^k$

M1

$\overline{z^{k+1}} = \overline{z^k z} = \overline{z^k} \overline{z} = (\overline{z})^k \overline{z} = (\overline{z})^{k+1}$

M1 A1

Answer: $\overline{z^n} = (\overline{z})^n$ for every positive integer n

4 marks

(b) $a_n w^n + \dots + a_0 = 0$

B1

Take conjugates of both sides, using that the conjugate of a sum is the sum of conjugates

M1

$a_n (\overline{w})^n + \dots + a_0 = 0$, since each a_k is real so $\overline{a_k} = a_k$

M1 A1

Answer: If the polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, with real coefficients, has a root w , then $\bar{w}$ is also a root

Solving Complex Equations

1 (2 marks)

$$z = \frac{-4 \pm \sqrt{16 - 32}}{2} = \frac{-4 \pm 4i}{2}$$

M1

$$z = -2 \pm 2i$$

A1

Answer: $z = -2 + 2i$ or $z = -2 - 2i$

2 marks

2 (2 marks)

$$2z - z = 4 + 3i$$

M1

$$z = 4 + 3i$$

A1

Answer: $z = 4 + 3i$

2 marks

3 (2 marks)

$$z = \frac{4}{1+i} = \frac{4(1-i)}{2}$$

M1

$$z = 2 - 2i$$

A1

Answer: $z = 2 - 2i$

2 marks

4 (2 marks)

$$x + 2y = 5 \text{ and } x - y = 2$$

M1

$$x = 3, y = 1$$

A1

Answer: $x = 3, y = 1$

2 marks

5 (2 marks)

$$(1 + i)^2 = 1 + 2i - 1 = 2i$$

M1

$$2i - 2(1 + i) + 2 = 2i - 2 - 2i + 2 = 0$$

A1

Answer: $z = 1 + i$ is a solution of the equation $z^2 - 2z + 2 = 0$

2 marks

6 (3 marks)

$$z - 2 = 0 \text{ gives } z = 2$$

B1

$$z^2 = -9$$

M1

$$z = \pm 3i$$

A1

Answer: $z = 2, 3i, -3i$

3 marks

7 (4 marks)

$$\text{Let } (a + bi)^2 = 5 + 12i: a^2 - b^2 = 5 \text{ and } 2ab = 12$$

M1 A1

$$b = \frac{6}{a}: a^4 - 5a^2 - 36 = 0, \text{ so } (a^2 - 9)(a^2 + 4) = 0$$

M1

$$a = \pm 3, b = \pm 2: \text{ the roots are } \pm(3 + 2i)$$

A1

Answer: $3 + 2i$ and $-3 - 2i$

4 marks

8 (4 marks)

$$\text{Discriminant: } (2 + i)^2 - 4(1 + i) = 3 + 4i - 4 - 4i = -1$$

M1 A1

$$z = \frac{2 + i \pm i}{2}$$

M1

$$z = 1 + i \text{ or } z = 1$$

A1

Answer: $z = 1$ or $z = 1 + i$

4 marks

9 (3 marks)

$$z + 2i = \frac{4 + 2i}{1 - i} = \frac{(4 + 2i)(1 + i)}{2}$$

M1

$$= \frac{4 + 4i + 2i - 2}{2} = 1 + 3i$$

A1

$$z = 1 + i$$

A1

Answer: $z = 1 + i$

3 marks

10 (3 marks)

$$z = \frac{-2 \pm \sqrt{4 - 40}}{4}$$

M1

$$= \frac{-2 \pm 6i}{4}$$

A1

$$z = -\frac{1}{2} \pm \frac{3}{2}i$$

A1

Answer: $z = -\frac{1}{2} \pm \frac{3}{2}i$

3 marks

11 (3 marks)z and w are the roots of $t^2 - 4t + 5 = 0$

M1

$$t = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

A1

z = 2 + i, w = 2 - i or z = 2 - i, w = 2 + i

A1

Answer: (z, w) = (2 + i, 2 - i) or (2 - i, 2 + i)

3 marks

12 (4 marks)

$$(2i)^3 - 2(2i)^2 + 4(2i) - 8 = -8i + 8 + 8i - 8 = 0$$

M1 A1

$$z^3 - 2z^2 + 4z - 8 = (z - 2)(z^2 + 4)$$

M1

Other roots: z = 2 and z = -2i

A1

Answer: z = 2 and z = -2i

4 marks

13 (3 marks)

$$iz - 2z = -i - 3, \text{ so } z(2 - i) = 3 + i$$

M1

$$z = \frac{3 + i}{2 - i} = \frac{(3 + i)(2 + i)}{5}$$

M1

$$z = \frac{5 + 5i}{5} = 1 + i$$

A1

Answer: $z = 1 + i$

3 marks

14 (4 marks)

$$(a) z = \frac{4 \pm \sqrt{16 - 52}}{2}$$

M1

$$z = 2 \pm 3i$$

A1

Answer: $z = 2 \pm 3i$

2 marks

(b) Let z = w + 1 so w = z - 1

M1

$$w = 1 \pm 3i$$

A1

Answer: $w = 1 \pm 3i$

2 marks

15 (5 marks)Let the real root be t: $t^2 - (5 + i)t + m + 2i = 0$

M1

Imaginary part: $-t + 2 = 0$, so t = 2

A1

Real part: $t^2 - 5t + m = 0$, so $4 - 10 + m = 0$ and m = 6

M1 A1

The sum of the roots is 5 + i, so the other root is $5 + i - 2 = 3 + i$

A1

Answer: m = 6; the roots are 2 and 3 + i

5 marks

16 (4 marks)

$$2z + i = (3 + i)(z - 1)$$

M1

$$2z + i = (3 + i)z - 3 - i, \text{ so } z(-1 - i) = -3 - 2i$$

A1

$$z = \frac{3 + 2i}{1 + i} = \frac{(3 + 2i)(1 - i)}{2}$$

M1

$$z = \frac{5 - i}{2}$$

A1

$$\text{Answer: } z = \frac{5}{2} - \frac{1}{2}i$$

4 marks

17 (4 marks)

$$(z^2 + 4)(z^2 - 1) = 0$$

M1 A1

$$z^2 = 1 \text{ gives } z = \pm 1$$

A1

$$z^2 = -4 \text{ gives } z = \pm 2i$$

A1

$$\text{Answer: } z = 1, -1, 2i, -2i$$

4 marks

18 (4 marks)

$$\text{Discriminant: } (-2i)^2 - 4 \times 3 = -4 - 12 = -16$$

M1 A1

$$z = \frac{2i \pm 4i}{2}$$

M1

$$z = 3i \text{ or } z = -i$$

A1

$$\text{Answer: } z = 3i \text{ or } z = -i$$

4 marks

19 (5 marks)

$$\text{Try } z = -2: -8 + 8 - 18 + 18 = 0, \text{ so } z = -2 \text{ is a root}$$

M1 A1

$$z^3 + 2z^2 + 9z + 18 = (z + 2)(z^2 + 9)$$

M1 A1

$$z^2 = -9, \text{ so } z = \pm 3i; \text{ roots } -2, 3i, -3i$$

A1

$$\text{Answer: } z = -2, 3i, -3i$$

5 marks

20 (5 marks)

$$\text{Let } z = a + bi: (2 + i)(a + bi) = (2a - b) + (a + 2b)i$$

M1 A1

$$(1 - 3i)(a - bi) = (a - 3b) - (3a + b)i$$

M1

$$\text{Sum: } (3a - 4b) + (-2a + b)i = 10 - 5i, \text{ so } 3a - 4b = 10 \text{ and } -2a + b = -5$$

A1

$$a = 2, b = -1: z = 2 - i$$

A1

$$\text{Answer: } z = 2 - i$$

5 marks

21 (7 marks)

$$\text{Discriminant: } (5 + i)^2 - 4(6 + 3i) = 24 + 10i - 24 - 12i = -2i$$

M1 A1

$$\text{Let } (a + bi)^2 = -2i: a^2 = b^2, 2ab = -2$$

M1

$$\text{So } ab = -1 \text{ with } a = \pm b: a = 1, b = -1 \text{ or } a = -1, b = 1$$

A1

$$\sqrt{-2i} = \pm(1 - i)$$

A1

$$z = \frac{5 + i \pm (1 - i)}{2}$$

M1

$$z = 3 \text{ or } z = 2 + i$$

A1

$$\text{Answer: } z = 3 \text{ or } z = 2 + i$$

7 marks

22 (7 marks)

$$\text{Complete the square: } (z - 1)^2 = 7 - 24i$$

M1 A1

$$\text{Let } (a + bi)^2 = 7 - 24i: a^2 - b^2 = 7 \text{ and } ab = -12$$

M1

$$a^4 - 7a^2 - 144 = 0, \text{ so } (a^2 - 16)(a^2 + 9) = 0$$

M1

$$a = \pm 4, b = \mp 3: z - 1 = \pm(4 - 3i)$$

A1

$$z = 5 - 3i$$

A1

$$z = -3 + 3i$$

A1

$$\text{Answer: } z = 5 - 3i \text{ or } z = -3 + 3i$$

7 marks

23 (6 marks)

Let the real root be t . Real part: $t^3 - 2t^2 - 3t + 6 = 0$. Imaginary part: $-3t^2 + 7t - 2 = 0$

M1 A1

Imaginary part: $(3t - 1)(t - 2) = 0$, so $t = \frac{1}{3}$ or $t = 2$

M1

Real part: $(t - 2)(t^2 - 3) = 0$, so only $t = 2$ satisfies both parts

A1

Divide by $(z - 2)$: the quadratic factor is $z^2 - 3iz - 3 + i$

M1

Discriminant $(-3i)^2 - 4(-3 + i) = 3 - 4i = (2 - i)^2$, so $z = \frac{3i \pm (2 - i)}{2}$: $z = 1 + i$ or $z = -1 + 2i$

A1

Answer: $z = 2, 1 + i, -1 + 2i$

6 marks

24 (7 marks)

Treat as a quadratic in z^2 : $z^2 = \frac{-6 \pm \sqrt{36 - 100}}{2}$

M1

$z^2 = -3 \pm 4i$

A1

Let $(a + bi)^2 = -3 + 4i$: $a^2 - b^2 = -3$, $ab = 2$

M1

$a^4 + 3a^2 - 4 = 0$, so $a = \pm 1$, $b = \pm 2$

M1 A1

$z = \pm(1 + 2i)$

A1

$z^2 = -3 - 4i$ gives (conjugates) $z = \pm(1 - 2i)$

A1

Answer: $z = 1 + 2i, -1 - 2i, 1 - 2i, -1 + 2i$

7 marks

25 (6 marks)

Let $z = a + bi$: $z\bar{z} = a^2 + b^2$ and $z - \bar{z} = 2bi$

M1 A1

$a^2 + b^2 + 4bi = 13 + 8i$

A1

Imaginary parts: $4b = 8$, so $b = 2$

M1

$a^2 + 4 = 13$, so $a = \pm 3$

A1

$z = 3 + 2i$ or $z = -3 + 2i$

A1

Answer: $z = 3 + 2i$ or $z = -3 + 2i$

6 marks

26 (6 marks)

(a) $(a + bi)^2 = a^2 - b^2 + 2abi$, which has a middle term

E1

The real parts give $a^2 - b^2 = 5$ and the imaginary parts $2ab = 12$, not $a^2 = 5$ and $b^2 = 12$

E1

Answer: Cross term $2abi$ ignored

2 marks

(b) $a^2 - b^2 = 5$ and $ab = 6$

M1

$a^4 - 5a^2 - 36 = 0$, so $a^2 = 9$

M1 A1

$z = \pm(3 + 2i)$

A1

Answer: $z = 3 + 2i$ or $z = -3 - 2i$

4 marks

27 (8 marks)

(a) $(3 + 4i)^2 = 9 + 24i + 16i^2$

M1

$= -7 + 24i$

A1

Answer: $(3 + 4i)^2 = -7 + 24i$

2 marks

(b) $z^2 = 3 + 4i$ or $z^2 = -3 - 4i$

B1

Answer: $3 + 4i$ or $-3 - 4i$

1 mark

(c) $z^2 = 3 + 4i$: $a^2 - b^2 = 3$, $ab = 2$

M1

$a^4 - 3a^2 - 4 = 0$, so $a^2 = 4$

M1 A1

$z = \pm(2 + i)$

A1

$(1 - 2i)^2 = -3 - 4i$, so $z^2 = -3 - 4i$ gives $z = \pm(1 - 2i)$

A1

Answer: $z = 2 + i, -2 - i, 1 - 2i, -1 + 2i$

5 marks

28 (7 marks)

Let $z = a + bi$: $a^2 - b^2 + 2abi + 2a - 2bi + 1 = 0$

M1 A1

Real part: $a^2 - b^2 + 2a + 1 = 0$. Imaginary part: $2b(a - 1) = 0$

A1

If $b = 0$: $a^2 + 2a + 1 = 0$, so $a = -1$ and $z = -1$

M1 A1

If $a = 1$: $1 - b^2 + 2 + 1 = 0$, so $b^2 = 4$ and $b = \pm 2$

M1

$z = 1 + 2i$ or $z = 1 - 2i$

A1

Answer: $z = -1, 1 + 2i, 1 - 2i$

7 marks

29 (8 marks)

(a) $w^2 = (z^2 - z)^2 = z^4 - 2z^3 + z^2$

M1

$z^4 - 2z^3 - 3z^2 + 4z - 12 = (z^4 - 2z^3 + z^2) - 4(z^2 - z) - 12$

M1

$= w^2 - 4w - 12$

A1

Answer: $w^2 - 4w - 12 = 0$

3 marks

(b) $(w - 6)(w + 2) = 0$, so $w = 6$ or $w = -2$

M1 A1

$w = 6$: $z^2 - z - 6 = 0$, so $(z - 3)(z + 2) = 0$ and $z = 3$ or $z = -2$

M1 A1

$w = -2$: $z^2 - z + 2 = 0$, so $z = \frac{1 \pm i\sqrt{7}}{2}$

A1

Answer: $z = 3, -2, \frac{1}{2} + \frac{\sqrt{7}}{2}i, \frac{1}{2} - \frac{\sqrt{7}}{2}i$

5 marks

30 (8 marks)

(a) $(3i)^4 = 81$, $(3i)^3 = -27i$, $(3i)^2 = -9$

M1

$f(3i) = 81 + 108i - 126 - 108i + 45$

A1

$= 0$

A1

Answer: $z = 3i$ is a root of $f(z) = 0$

3 marks

(b) $-3i$ is also a root, giving the factor $z^2 + 9$

B1 M1

$f(z) = (z^2 + 9)(z^2 + pz + 5)$; comparing z^3 terms: $p = -4$

M1 A1

$z^2 - 4z + 5 = 0$ gives $z = 2 \pm i$

A1

Answer: $z = 3i, -3i, 2 + i, 2 - i$

5 marks

Argand Diagrams**1 (3 marks)**

(a) A is at $(3, 2)$, so $a = 3 + 2i$

B1

Answer: $3 + 2i$

1 mark

(b) B is at $(-2, 1)$, so $b = -2 + i$

B1

Answer: $-2 + i$

1 mark

(c) C is at $(-1, -3)$, so $c = -1 - 3i$

B1

Answer: $-1 - 3i$

1 mark

2 (3 marks)

P is plotted at $(2, 3)$

B1

Q is plotted at $(-1, 1)$

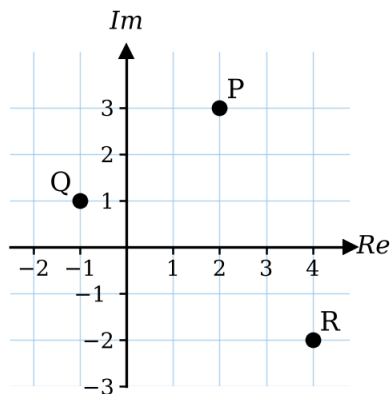
B1

R is plotted at $(4, -2)$

B1

Answer: P(2, 3), Q(-1, 1), R(4, -2)

3 marks

**3 (3 marks)**

P is $(5, -2)$

B1

$\bar{z} = 5 + 2i$ is the point $(5, 2)$

B1

$-z = -5 + 2i$ is the point $(-5, 2)$

B1

Answer: $(5, -2), (5, 2), (-5, 2)$

3 marks

4 (2 marks)

$z + w = (3 + 1) + (2 + 4)i$

M1

$= 4 + 6i$, so the point is $(4, 6)$

A1

Answer: $(4, 6)$

2 marks

5 (2 marks)

$AB^2 = (4 - 1)^2 + (6 - 2)^2 = 9 + 16$

M1

$AB = 5$

A1

Answer: 5

2 marks

6 (2 marks)

(a) P is at $(0, -3)$, on the imaginary axis

B1

Answer: The imaginary axis

1 mark

(b) Reflection in the real axis changes $(0, -3)$ to $(0, 3)$, which is $3i$

B1

Answer: $3i$

1 mark

7 (6 marks)

(a) $z + w = 4 + 4i$, so C is $(4, 4)$

B1

$z - w = 2 - 2i$, so D is $(2, -2)$

B1

A $(3, 1)$ and B $(1, 3)$ plotted correctly

B1

Answer: C $(4, 4)$, D $(2, -2)$

3 marks

(b) $OA = (3, 1) = BC$ and $OB = (1, 3) = AC$, so OACB is a parallelogram

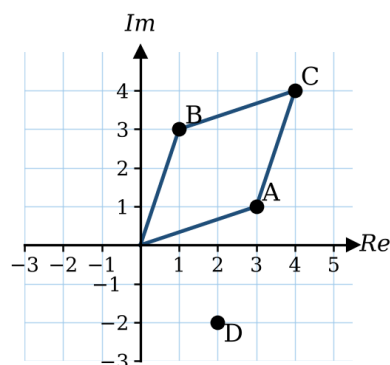
M1 A1

$OA^2 = 10 = OB^2$, so all four sides are equal and it is a rhombus

A1

Answer: A rhombus

3 marks



8 (4 marks)

$$(a) M = \frac{1}{2}[(2 - 3i) + (6 + 5i)]$$

M1

$$= 4 + i$$

A1

Answer: 4 + i

2 marks

$$(b) P = A + \frac{1}{4}(B - A) = (2 - 3i) + \frac{1}{4}(4 + 8i)$$

M1

$$= 3 - i$$

A1

Answer: 3 - i

2 marks

9 (3 marks)

In a parallelogram ABCD, the vector AB equals the vector DC

M1

$$d = a + c - b$$

M1

$$d = (1 + i) + (5 + 5i) - (4 + 2i) = 2 + 4i$$

A1

Answer: 2 + 4i

3 marks

10 (3 marks)

$$(a) iz = 3i + 2i^2 = -2 + 3i$$

B1

Answer: -2 + 3i

1 mark

(b) OP is the vector (3, 2) and OQ is (-2, 3), and their scalar product is $-6 + 6 = 0$

M1

They are equal in length and perpendicular: Q is the image of P under a 90° anticlockwise rotation about O

A1

Answer: OQ is OP rotated 90° anticlockwise

2 marks

11 (3 marks)

$$AB = (4 + 5i) - (1 + 2i) = 3 + 3i$$

M1 A1

BC = $(7 + 8i) - (4 + 5i) = 3 + 3i$, so AB and BC are parallel and share the point B, so A, B, C lie on one line

A1

Answer: A, B and C are collinear

3 marks

12 (4 marks)

$$z = \frac{4 \pm \sqrt{16 - 52}}{2}$$

M1

$$z = 2 \pm 3i$$

A1

Points (2, 3) and (2, -3) plotted

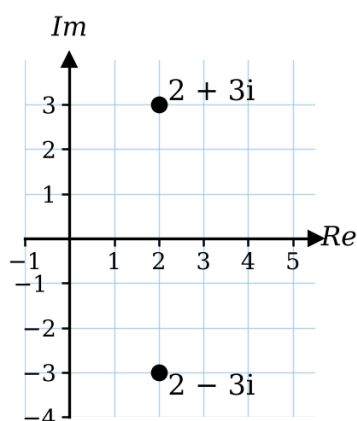
B1

They are reflections of each other in the real axis

B1

Answer: $z = 2 \pm 3i$; reflections in the real axis

4 marks

**13 (4 marks)**

(a) AB = (4, 0) is horizontal and BC = (0, 3) is vertical

M1

So AB is perpendicular to BC

A1

Answer: Angle ABC is a right angle

2 marks

(b) AB = 4 and BC = 3

B1

$$\text{Area} = \frac{1}{2} \times 4 \times 3 = 6$$

B1

Answer: 6**2 marks****14 (4 marks)**OC is OA rotated through 90° anticlockwise, so $c = i(4 + 2i)$ **M1 A1**

$$c = -2 + 4i$$

A1

$$b = a + c = (4 + 2i) + (-2 + 4i) = 2 + 6i$$

A1**Answer: C: $-2 + 4i$, B: $2 + 6i$** **4 marks****15 (5 marks)**

$$AB = (6 + 3i) - (2 + i) = 4 + 2i$$

M1 A1AC is AB rotated 90° anticlockwise: $AC = i(4 + 2i)$ **M1**

$$= -2 + 4i$$

A1

$$c = (2 + i) + (-2 + 4i) = 5i$$

A1**Answer: $5i$** **5 marks****16 (4 marks)**

A (3, 2), B (3, -2), C (-3, -2), D (-3, 2)

B1

AB is vertical with length 4 and BC is horizontal with length 6

M1 A1So the sides are perpendicular, opposite sides are equal: a rectangle of area $4 \times 6 = 24$ **A1****Answer: Area 24****4 marks****17 (4 marks)**(a) Reflection in $y = x$ swaps the coordinates: $(3, 4) \rightarrow (4, 3)$ **M1**

$$q = 4 + 3i$$

A1**Answer: $4 + 3i$** **2 marks**(b) $\bar{z} = 3 - 4i$ **B1**

$$i(3 - 4i) = 3i - 4i^2 = 4 + 3i = q$$

M1**Answer: $q = i\bar{z}$** **2 marks****18 (5 marks)**

$$AB = 2 + i \text{ and } AC = 1 + 2i$$

B1

$$AB^2 = 2^2 + 1^2 = 5 \text{ and } AC^2 = 1^2 + 2^2 = 5, \text{ so } AB = AC \text{ and the triangle is isosceles}$$

M1 A1

$$\text{Area} = \frac{1}{2}|2 \times 2 - 1 \times 1|$$

M1

$$= \frac{3}{2}$$

A1**Answer: Area $\frac{3}{2}$** **5 marks****19 (5 marks)**

$$iz = -1 + 3i, -z = -3 - i, -iz = 1 - 3i$$

B1Each of A, B, C, D is at distance $\sqrt{10}$ from O, and each is obtained from the previous by a 90° rotation about O**M1**So the diagonals AC and BD have equal length $2\sqrt{10}$, bisect each other at O and are perpendicular**A1**ABCD is a square with diagonal $2\sqrt{10}$ **A1**

$$\text{Area} = \frac{1}{2} \times (2\sqrt{10})^2 = 20$$

A1**Answer: Area 20****5 marks****20 (4 marks)**

$$(a) g = \frac{1}{3}[(2 + i) + (5 + 7i) + (-1 + 4i)]$$

M1

$$= \frac{6 + 12i}{3} = 2 + 4i$$

A1**Answer: $2 + 4i$** **2 marks**(b) $d = 2g - a$ **M1**

$$= (4 + 8i) - (2 + i) = 2 + 7i$$

A1**Answer: $2 + 7i$** **2 marks**

21 (6 marks)

(a) $OB = OA + OC$

M1

$b = (4 + i) + (1 + 3i) = 5 + 4i$

A1

Answer: $5 + 4i$

2 marks

(b) Midpoint of $OB = \frac{1}{2}(5 + 4i)$

M1

Midpoint of $AC = \frac{1}{2}[(4 + i) + (1 + 3i)] = \frac{1}{2}(5 + 4i)$, the same

A1

Answer: The diagonals OB and AC bisect each other

2 marks

(c) Area = $|4 \times 3 - 1 \times 1|$

M1

$= 11$

A1

Answer: 11

2 marks

22 (5 marks)

(a) $iz = i(3 + 2i) = 3i + 2i^2$

M1

$= -2 + 3i = w$

A1

Answer: $w = iz$

2 marks

(b) P is (3, 2) in the first quadrant and Q is (-2, 3) in the second quadrant

B1

Moving from the first quadrant to the second is an anticlockwise turn, and $OP \cdot OQ = -6 + 6 = 0$

M1

Multiplying by i is a 90° rotation anticlockwise about O, not clockwise

E1

Answer: 90° anticlockwise

3 marks

23 (7 marks)

$AB = (5 + 2i) - (2 + i) = 3 + i$

B1

$DC = (6 + 5i) - (3 + 4i) = 3 + i$, so AB and DC are equal and parallel: ABCD is a parallelogram

M1 A1

$AD = (3 + 4i) - (2 + i) = 1 + 3i$

B1

$AB^2 = 9 + 1 = 10$ and $AD^2 = 1 + 9 = 10$, so adjacent sides are equal: a rhombus

M1 A1

$AB \cdot AD = 3 \times 1 + 1 \times 3 = 6 \neq 0$, so the angle at A is not 90° : not a rectangle, so not a square

A1

Answer: A rhombus, but not a rectangle or a square

7 marks

24 (5 marks)

Relative to P, A is at $a - p = (4 + 3i) - (1 + i) = 3 + 2i$

M1 A1

Rotating 90° anticlockwise multiplies by i : $i(3 + 2i) = -2 + 3i$

M1 A1

$b = p + (-2 + 3i) = -1 + 4i$

A1

Answer: B represents $-1 + 4i$

5 marks

25 (7 marks)

(a) $z = \frac{2 \pm \sqrt{4 - 40}}{2}$

M1

$z = 1 \pm 3i$

A1

Answer: $z = 1 \pm 3i$

2 marks

(b) $OA^2 = 1^2 + 3^2 = 10$ and $OB^2 = 1^2 + (-3)^2 = 10$

M1

$OA = OB$

A1

Answer: Triangle OAB is isosceles

2 marks

(c) AB is vertical with length 6

B1

The perpendicular distance from O to line AB ($x = 1$) is 1

M1

Area = $\frac{1}{2} \times 6 \times 1 = 3$

A1

Answer: 3

3 marks

26 (6 marks)

(a) Gradient of AB = $\frac{2}{3}$

B1

Gradient of AC = $\frac{p+2-1}{p-1} = \frac{2}{3}$, so $3(p+1) = 2(p-1)$

M1

$p = -5$

A1

Answer: $p = -5$

3 marks

(b) $AB = 3 + 2i$, $OC = p + (p + 2)i$

B1

Perpendicular: $3p + 2(p + 2) = 0$

M1

$$p = -\frac{4}{5}$$

A1

Answer: $p = -\frac{4}{5}$

3 marks

27 (8 marks)

(a) B is A rotated through $\frac{2\pi}{3}$ about O: $b = (4 + 2i)(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3})$

M1 A1

$$= (4 + 2i)(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$$

M1

$$= -2 + 2\sqrt{3}i - i - \sqrt{3}$$

A1

$$b = (-2 - \sqrt{3}) + (2\sqrt{3} - 1)i \text{ and } c = (-2 + \sqrt{3}) - (2\sqrt{3} + 1)i$$

A1

Answer: B: $(-2 - \sqrt{3}) + (2\sqrt{3} - 1)i$; C: $(-2 + \sqrt{3}) - (2\sqrt{3} + 1)i$

5 marks

(b) $OA^2 = 20$; for an equilateral triangle with centre O, $AB^2 = 3 \times OA^2 = 60$

M1

$$\text{Area } ABC = \frac{\sqrt{3}}{4} \times 60 = 15\sqrt{3}$$

A1

Triangle DEF is similar to ABC with scale factor $\frac{1}{2}$, so its area is $\frac{15\sqrt{3}}{4}$

A1

Answer: $\frac{15\sqrt{3}}{4}$

3 marks

28 (9 marks)

(a) ABCD is a parallelogram exactly when the diagonals AC and BD bisect each other

B1

Midpoint of AC is $\frac{1}{2}(a + c)$ and midpoint of BD is $\frac{1}{2}(b + d)$

M1

These coincide exactly when $a + c = b + d$

A1

Answer: ABCD (in that order) is a parallelogram if and only if $a + c = b + d$

3 marks

(b) $d = a + c - b$

M1

$$d = (1 + 2i) + (7 + 4i) - (5 + i) = 3 + 5i$$

A1

Answer: $3 + 5i$

2 marks

(c) Midpoints: $P = \frac{1}{2}(a + b)$, $Q = \frac{1}{2}(b + c)$, $R = \frac{1}{2}(c + d)$, $S = \frac{1}{2}(d + a)$

B1

$$p + r = \frac{1}{2}(a + b + c + d)$$

M1

$$q + s = \frac{1}{2}(a + b + c + d)$$

M1

$p + r = q + s$, so PQRS is a parallelogram by part (a)

A1

Answer: The midpoints of the four sides of any quadrilateral ABCD are the vertices of a parallelogram

4 marks

29 (8 marks)

The centre M is the midpoint of AC: $m = \frac{1}{2}[(1 + i) + (7 + 5i)] = 4 + 3i$

M1 A1

Relative to M, A is $a - m = -3 - 2i$

B1

B is A rotated 90° anticlockwise about M: $b - m = i(-3 - 2i) = 2 - 3i$

M1 A1

$b = 6$; D is opposite B: $d - m = -(2 - 3i)$, so $d = 2 + 6i$

A1

$AB = 5 - i$ and $BC = (7 + 5i) - 6 = 1 + 5i$

B1

$AB \cdot BC = 5 \times 1 + (-1) \times 5 = 0$ and $AB^2 = 26 = BC^2$

A1

Answer: B represents 6 and D represents $2 + 6i$

8 marks

30 (8 marks)

(a) $\bar{z}w = (a - bi)(c + di) = (ac + bd) + (ad - bc)i$

M1 A1

Area of triangle OAB = $\frac{1}{2}|ad - bc| = \frac{1}{2}|\text{Im}(\bar{z}w)|$

A1

Answer: The imaginary part of $\bar{z}w$ is $ad - bc$; the area of triangle OAB is $\frac{1}{2}|\text{Im}(\bar{z}w)|$

3 marks

(b) $\bar{z}w = (3 - 2i)(-1 + 4i) = 5 + 14i$

M1

$$\text{Area} = \frac{1}{2} \times 14 = 7$$

A1

Answer: 7

2 marks

(c) Collinear means the area of the triangle is zero, so $\text{Im}(\bar{z}w) = 0$

M1

$$(2 - i)(k + 3i) = (2k + 3) + (6 - k)i$$

A1

$$6 - k = 0, \text{ so } k = 6$$

A1

Answer: k = 6

3 marks

Modulus and Argument

1 (2 marks)

$$(a) |3 + 4i| = \sqrt{3^2 + 4^2} = 5$$

B1

Answer: 5

1 mark

$$(b) |5 - 12i| = \sqrt{25 + 144} = 13$$

B1

Answer: 13

1 mark

2 (3 marks)

$$(a) |-7| = 7$$

B1

Answer: 7

1 mark

$$(b) |4i| = 4$$

B1

Answer: 4

1 mark

$$(c) |1 + i| = \sqrt{1 + 1} = \sqrt{2}$$

B1

Answer: $\sqrt{2}$

1 mark

3 (3 marks)

(a) The point (1, 1) is in the first quadrant with $\tan \theta = 1$

B1

Answer: $\frac{\pi}{4}$

1 mark

(b) The point (-1, 1) is in the second quadrant: $\pi - \frac{\pi}{4}$

B1

Answer: $\frac{3\pi}{4}$

1 mark

(c) The point (-1, -1) is in the third quadrant: $-\pi + \frac{\pi}{4}$

B1

Answer: $-\frac{3\pi}{4}$

1 mark

4 (3 marks)

(a) -5 lies on the negative real axis

B1

Answer: π

1 mark

(b) 5i lies on the positive imaginary axis

B1

Answer: $\frac{\pi}{2}$

1 mark

(c) -2i lies on the negative imaginary axis

B1

Answer: $-\frac{\pi}{2}$

1 mark

5 (2 marks)

 $\tan \theta = \frac{1}{\sqrt{3}}$, and the point is in the first quadrant

M1

$$\theta = \frac{\pi}{6}$$

A1

Answer: $\frac{\pi}{6}$

2 marks

6 (2 marks)

$$z = 3 - i + 6i - 2i^2 = 5 + 5i$$

M1

$$|z| = \sqrt{25 + 25} = 5\sqrt{2}$$

A1

$$\text{Answer: } 5\sqrt{2}$$

2 marks

7 (3 marks)

The point $(-2, 3)$ is in the second quadrant

B1

$$\theta = \pi - \tan^{-1}\left(\frac{3}{2}\right)$$

M1

$$= 2.16$$

A1

$$\text{Answer: } 2.16$$

3 marks

8 (4 marks)

$$|z| = \sqrt{9 + 27} = 6$$

M1 A1

The point is in the third quadrant with reference angle $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

M1

$$\arg z = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$$

A1

$$\text{Answer: } |z| = 6, \arg z = -\frac{2\pi}{3}$$

4 marks

9 (3 marks)

$$a = 4\cos\frac{2\pi}{3} = -2$$

M1 A1

$$b = 4\sin\frac{2\pi}{3} = 2\sqrt{3}, \text{ so } z = -2 + 2\sqrt{3}i$$

A1

$$\text{Answer: } z = -2 + 2\sqrt{3}i$$

3 marks

10 (4 marks)

$$|z| = \sqrt{1 + 3} = 2 \text{ and } |w| = \sqrt{4 + 4} = 2\sqrt{2}$$

B1 B1

$$|zw| = |z||w| = 4\sqrt{2}$$

M1

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$$

A1

$$\text{Answer: } |zw| = 4\sqrt{2}, \left|\frac{z}{w}\right| = \frac{\sqrt{2}}{2}$$

4 marks

11 (4 marks)

$$w = \frac{-5 + 5i}{2 + i} = \frac{(-5 + 5i)(2 - i)}{5}$$

M1

$$= \frac{-10 + 5i + 10i + 5}{5} = -1 + 3i$$

A1

$$|w| = \sqrt{1 + 9} = \sqrt{10}$$

B1

$$\arg w = \pi - \tan^{-1}3 = 1.89$$

A1

$$\text{Answer: } w = -1 + 3i, |w| = \sqrt{10}, \arg w = 1.89$$

4 marks

12 (3 marks)

$$|(2 + i)^3| = |2 + i|^3$$

M1

$$|2 + i| = \sqrt{5}$$

A1

$$(\sqrt{5})^3 = 5\sqrt{5}$$

A1

$$\text{Answer: } |(2 + i)^3| = 5\sqrt{5}$$

3 marks

13 (4 marks)

$$\text{If } z = 5 + yi \text{ then } 25 + y^2 = 169, \text{ so } y^2 = 144$$

M1

$$y = -12, \text{ so } z = 5 - 12i$$

A1

$$\arg z = -\tan^{-1}\left(\frac{12}{5}\right)$$

M1

$$= -1.18$$

A1

$$\text{Answer: } z = 5 - 12i, \arg z = -1.18$$

4 marks

14 (4 marks)

$$(-1 + i)^2 = 1 - 2i + i^2 = -2i$$

M1

$$z = -2i(-1 + i) = 2i - 2i^2 = 2 + 2i$$

A1

$$|z| = \sqrt{4 + 4} = 2\sqrt{2}$$

A1

$$\arg z = \frac{\pi}{4}$$

A1

$$\text{Answer: } |z| = 2\sqrt{2}, \arg z = \frac{\pi}{4}$$

4 marks

15 (4 marks)

$$|zw|^2 = (zw)(\overline{zw})$$

M1

$$= z\overline{w}z\overline{w}$$

A1

$$= (z\overline{z})(w\overline{w}) = |z|^2|w|^2$$

M1

$$\text{Taking positive square roots, } |zw| = |z||w|$$

A1

$$\text{Answer: } |zw| = |z||w| \text{ for all complex numbers } z \text{ and } w$$

4 marks

16 (5 marks)

$$\arg z = \pi - \tan^{-1}\left(\frac{1}{3}\right) = 2.820$$

M1 A1

$$\arg w = \tan^{-1}\left(\frac{1}{2}\right) = 0.4636$$

A1

$$\arg z - \arg w = 2.36 \text{ (3 s.f.)}$$

A1

$$\frac{z}{w} = \frac{(-3 + i)(4 - 2i)}{20} = -\frac{1}{2} + \frac{1}{2}i, \text{ whose argument is } \frac{3\pi}{4} (\approx 2.36)$$

A1

$$\text{Answer: } \arg z - \arg w = 2.36 \text{ and } \arg\left(\frac{z}{w}\right) = \frac{3\pi}{4}$$

5 marks

17 (6 marks)

$$\text{(a) } z^2 = 4 + 8i - 4 = 8i$$

M1

$$\text{Modulus } 8, \text{ argument } \frac{\pi}{2}$$

A1

$$\text{Answer: } |z^2| = 8, \arg = \frac{\pi}{2}$$

2 marks

$$\text{(b) } \frac{1}{z} = \frac{2 - 2i}{8} = \frac{1 - i}{4}$$

M1

$$\text{Modulus } \frac{\sqrt{2}}{4}, \text{ argument } -\frac{\pi}{4}$$

A1

$$\text{Answer: } \left|\frac{1}{z}\right| = \frac{\sqrt{2}}{4}, \arg = -\frac{\pi}{4}$$

2 marks

$$\text{(c) } \bar{z} = 2 - 2i$$

M1

$$\text{Modulus } 2\sqrt{2}, \text{ argument } -\frac{\pi}{4}$$

A1

$$\text{Answer: } |\bar{z}| = 2\sqrt{2}, \arg = -\frac{\pi}{4}$$

2 marks

18 (5 marks)

$$V = 4 + 8i$$

B1

$$\text{Amplitude } |V| = \sqrt{16 + 64}$$

M1

$$= 4\sqrt{5} \approx 8.94 \text{ volts}$$

A1

$$\text{Phase angle} = \tan^{-1}\left(\frac{8}{4}\right) = \tan^{-1}2$$

M1

$$= 1.11 \text{ radians}$$

A1

Answer: Amplitude $4\sqrt{5}$ V, phase angle 1.11 rad

5 marks

19 (5 marks)

$$|z| = \sqrt{9 + 16} = 5$$

B1

$$\arg z = -\tan^{-1}\left(\frac{4}{3}\right) = -0.927$$

M1 A1

$$-z = -3 + 4i \text{ is in the second quadrant: } \arg(-z) = -0.927 + \pi = 2.21$$

A1

$$iz = 4 + 3i: \arg(iz) = \tan^{-1}\left(\frac{3}{4}\right) = 0.644 (= \arg z + \frac{\pi}{2})$$

A1

Answer: $|z| = 5$, $\arg z = -0.927$, $\arg(-z) = 2.21$, $\arg(iz) = 0.644$

5 marks

20 (6 marks)

$$|zw| = |z||w| = 6$$

B1

$$\arg(zw) = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

B1

$$zw = 6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 6i$$

A1

$$\left|\frac{z}{w}\right| = \frac{3}{2} \text{ and } \arg\left(\frac{z}{w}\right) = \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6}$$

B1

$$\frac{z}{w} = \frac{3}{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

M1

$$= \frac{3\sqrt{3}}{4} - \frac{3}{4}i$$

A1

Answer: $zw = 6i$, $\frac{z}{w} = \frac{3\sqrt{3}}{4} - \frac{3}{4}i$

6 marks

21 (6 marks)

(a) The point $(-1, -1)$ is in the third quadrant

E1

$\tan^{-1}1$ gives an angle in the first quadrant; the sign information is lost when dividing -1 by -1

E1

Answer: Wrong quadrant

2 marks

(b) Reference angle $\frac{\pi}{4}$, third quadrant

M1

$$\arg = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

A1

Answer: $-\frac{3\pi}{4}$

2 marks

(c) Second quadrant, reference angle $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

M1

$$\arg = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

A1

Answer: $\frac{5\pi}{6}$

2 marks

22 (6 marks)

$$(a) |z + w|^2 = (a + c)^2 + (b + d)^2$$

M1

$$|z - w|^2 = (a - c)^2 + (b - d)^2$$

A1

$$\text{Sum} = 2a^2 + 2c^2 + 2b^2 + 2d^2$$

M1

$$= 2(a^2 + b^2) + 2(c^2 + d^2) = 2|z|^2 + 2|w|^2$$

A1

Answer: $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$

4 marks

(b) $|z + w|^2 = |4 - i|^2 = 17$ and $|z - w|^2 = |2 + 3i|^2 = 13$, sum 30

M1

$$2|z|^2 + 2|w|^2 = 2 \times 10 + 2 \times 5 = 30$$

A1

Answer: Both sides equal 30**2 marks****23 (6 marks)**

Let $z = x + iy$: $x^2 + y^2 = 25$

B1

$z^2 = x^2 - y^2 + 2xyi$ is purely imaginary, so $x^2 = y^2$

M1 A1

$2x^2 = 25$, so $x = \pm \frac{5}{\sqrt{2}}$, $y = \pm \frac{5}{\sqrt{2}}$

M1Imaginary part $2xy < 0$, so x and y have opposite signs**M1**

$z = \pm \frac{5\sqrt{2}}{2}(1 - i)$

A1

Answer: $z = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$ or $z = -\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$

6 marks**24 (9 marks)**

(a) $|a + i| = \sqrt{a^2 + 1}$ and $|1 + ai| = \sqrt{1 + a^2}$

M1

$|z| = \frac{|a + i|}{|1 + ai|} = 1$

A1**Answer: $|z| = 1$ for every real value of a** **2 marks**

(b) $z = \frac{(a + i)(1 - ai)}{(1 + ai)(1 - ai)}$

M1

$= \frac{2a + (1 - a^2)i}{1 + a^2}$

A1

Answer: $z = \frac{2a + (1 - a^2)i}{1 + a^2}$

2 marks

(c) $1 - a^2 = 0$

M1

$a = 1$ or $a = -1$

A1**Answer: $a = \pm 1$** **2 marks**

(d) Real and imaginary parts equal and positive: $2a = 1 - a^2$

M1

$a^2 + 2a - 1 = 0$, so $a = -1 \pm \sqrt{2}$

A1

Real part must be positive, so $a = \sqrt{2} - 1$

A1

Answer: $a = \sqrt{2} - 1$

3 marks**25 (6 marks)**

$\frac{z + w}{z - w} = \frac{(z + w)(\bar{z} - \bar{w})}{(z - w)(\bar{z} - \bar{w})}$

M1Denominator $= |z - w|^2$, which is real and positive**A1**

Numerator $= z\bar{z} - z\bar{w} + w\bar{z} - w\bar{w}$

M1

$= |z|^2 - |w|^2 + (w\bar{z} - z\bar{w})$

A1

$|z| = |w|$, so the numerator is $w\bar{z} - z\bar{w}$

B1

$w\bar{z} - z\bar{w} = u - \bar{u}$ where $u = w\bar{z}$, which is $2i \operatorname{Im}(u)$: purely imaginary or zero

A1

Answer: $\frac{z + w}{z - w}$ is purely imaginary or zero

6 marks**26 (7 marks)**

(a) $OA = |a| = 1$ and $OB = |b| = 1$, so adjacent sides are equal

M1

$OC = a + b$, so $OACB$ is a parallelogram ($AC = b = OB$), with equal adjacent sides: a rhombus

A1**Answer: $OACB$ is a rhombus****2 marks**

(b) The diagonal OC of a rhombus bisects the angle AOB

M1

Angle $AOB = \arg b = \frac{\pi}{4}$, so $\arg(a + b) = \frac{\pi}{8}$

A1

Answer: $\arg(a + b) = \frac{\pi}{8}$

2 marks

(c) $a + b = 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

B1

$$\tan \frac{\pi}{8} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2 + \sqrt{2}} \quad \text{M1}$$

$$= \frac{\sqrt{2}(2 - \sqrt{2})}{(2 + \sqrt{2})(2 - \sqrt{2})} = \frac{2\sqrt{2} - 2}{2} = \sqrt{2} - 1 \quad \text{A1}$$

Answer: $\tan \frac{\pi}{8} = \sqrt{2} - 1$ 3 marks

27 (8 marks)

(a) $|z + w|^2 = (z + w)(\bar{z} + \bar{w})$ M1

$$= |z|^2 + |w|^2 + z\bar{w} + \bar{z}w \quad \text{A1}$$

$$z\bar{w} + \bar{z}w = 2\operatorname{Re}(z\bar{w}) \leq 2|z\bar{w}| = 2|z||w| \quad \text{M1 A1}$$

So $|z + w|^2 \leq (|z| + |w|)^2$, and taking square roots gives the result A1

Answer: $|z + w| \leq |z| + |w|$ 5 marks

(b) $|z - 3 + 4i| \leq |z| + |-3 + 4i| = 5 + 5 = 10$ M1 A1

Equality when z points in the same direction as $-3 + 4i$: $z = -3 + 4i$ A1

Answer: 10, attained at $z = -3 + 4i$ 3 marks

28 (8 marks)

(a) $|z| = 1$ gives $\bar{z} = \frac{1}{z}$, and similarly $\bar{w} = \frac{1}{w}$ B1 M1

The conjugate of the fraction is $\frac{\bar{z} + \bar{w}}{1 + \bar{z}\bar{w}}$ M1

$$= \frac{\frac{1}{z} + \frac{1}{w}}{1 + \frac{1}{zw}} \quad \text{A1}$$

Multiply numerator and denominator by zw : $\frac{w + z}{zw + 1}$ M1

The fraction equals its conjugate, so it is real A1

Answer: $\frac{z + w}{1 + zw}$ is real 6 marks

(b) $z + w = \frac{64 - 8i}{65}$ and $1 + zw = 1 + \frac{63 - 16i}{65} = \frac{128 - 16i}{65}$ M1

The quotient is $\frac{64 - 8i}{128 - 16i} = \frac{1}{2}$, which is real A1

Answer: $\frac{1}{2}$ 2 marks

29 (8 marks)

(a) $(1 + i)(2 + i) = 2 + i + 2i - 1 = 1 + 3i$ M1 A1

$$(1 + 3i)(3 + i) = 3 + i + 9i - 3 = 10i \quad \text{A1}$$

Answer: $(1 + i)(2 + i)(3 + i) = 10i$ 3 marks

(b) $\arg$ of the product $= \tan^{-1}1 + \tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3}) \pmod{2\pi}$, as the arguments of the factors add M1

Each term is acute, so the sum lies strictly between 0 and $\frac{3\pi}{2}$ M1

$\arg(10i) = \frac{\pi}{2}$ is the only value in that range, so the sum is $\frac{\pi}{2}$ A1

Answer: $\tan^{-1}1 + \tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3}) = \frac{\pi}{2}$ 3 marks

(c) $(1 + 2i)(1 + 3i) = -5 + 5i$ M1

Both arguments are acute and the sum of arguments is less than π , so the argument is $\arg(-5 +$ A1

$$5i) = \frac{3\pi}{4}$$

Answer: $\tan^{-1}2 + \tan^{-1}3 = \frac{3\pi}{4}$ 2 marks

30 (7 marks)

Let $z = x + iy$ so that $w = (2 - x) + (2 - y)i$

B1

$$x^2 + y^2 = 4$$

B1

$$|w|^2 = (2 - x)^2 + (2 - y)^2 = 4$$

M1

Expanding: $8 - 4(x + y) + x^2 + y^2 = 4$, so $x + y = 2$

M1 A1

Then $x^2 + (2 - x)^2 = 4$ gives $x(x - 2) = 0$

M1

$(z, w) = (2, 2i)$ or $(2i, 2)$

A1

Answer: $(z, w) = (2, 2i)$ or $(2i, 2)$

7 marks

Complex Number Loci

1 (2 marks)

(a) Points at a constant distance 4 from the origin

B1

Answer: Circle, centre O, radius 4

1 mark

(b) Points at distance 3 from $2i$

B1

Answer: Circle, centre $2i$, radius 3

1 mark

2 (2 marks)

Circle with centre $(3, 0)$ and radius 2

M1

$$(x - 3)^2 + y^2 = 4$$

A1

Answer: $(x - 3)^2 + y^2 = 4$

2 marks

3 (2 marks)

The points make a constant angle $\frac{\pi}{3}$ with the positive real axis

B1

A half-line starting at the origin (excluding O), at angle $\frac{\pi}{3}$ to the real axis

B1

Answer: Half-line from O at angle $\frac{\pi}{3}$

2 marks

4 (2 marks)

z is equidistant from 2 and -2

M1

The perpendicular bisector of the points $(2, 0)$ and $(-2, 0)$: the imaginary axis ($x = 0$)

A1

Answer: The line $x = 0$ (imaginary axis)

2 marks

5 (2 marks)

(a) The vertical line $x = 3$

B1

Answer: Line $x = 3$

1 mark

(b) The horizontal line $y = -2$

B1

Answer: Line $y = -2$

1 mark

6 (2 marks)

(a) Centre $1 + 2i$, radius 5

B1

Answer: Centre $(1, 2)$, radius 5

1 mark

(b) $z + 3i = z - (-3i)$: centre $-3i$, radius 2

B1

Answer: Centre $(0, -3)$, radius 2

1 mark

7 (3 marks)

The locus is a circle

B1

Centre $(2, 1)$

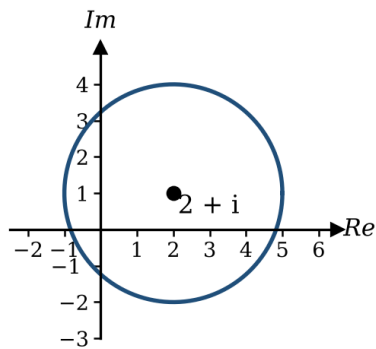
B1

Radius 3

B1

Answer: Circle, centre $2 + i$, radius 3

3 marks

**8 (3 marks)**

$\arg(z - (-1 + 2i)) = \frac{3\pi}{4}$: a half-line starting at $-1 + 2i$

B1

The point $-1 + 2i$ is excluded (open end)

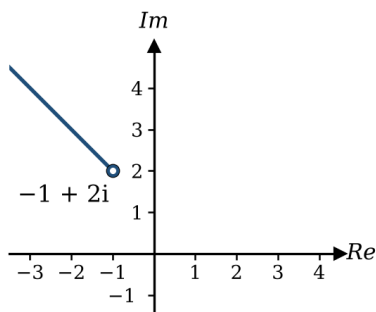
B1

Direction at $\frac{3\pi}{4}$ to the positive real axis, i.e. up and to the left

B1

Answer: Half-line from $-1 + 2i$ at angle $\frac{3\pi}{4}$

3 marks

**9 (4 marks)**

Let $z = x + iy$: $(x - 3)^2 + y^2 = x^2 + (y - 1)^2$

M1 A1

$x^2 - 6x + 9 + y^2 = x^2 + y^2 - 2y + 1$

M1

$y = 3x - 4$

A1

Answer: $y = 3x - 4$

4 marks

10 (3 marks)

The boundary $|z + 2| = |z - 2|$ is the perpendicular bisector of -2 and 2 , the line $x = 0$

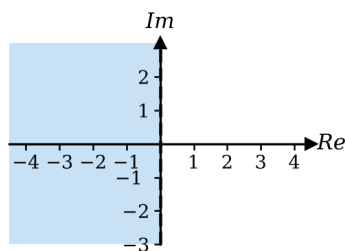
M1 A1

Closer to -2 : the region $x < 0$ is shaded, with a dashed boundary since the inequality is strict

A1

Answer: The half-plane $x < 0$ (boundary dashed)

3 marks

**11 (3 marks)**

$z - (1 - i)$ has argument $\frac{\pi}{4}$: a half-line from $(1, -1)$

B1

Gradient $\tan \frac{\pi}{4} = 1$: $y + 1 = x - 1$, so $y = x - 2$

M1

$x > 1$ (the point $(1, -1)$ is excluded)

A1

Answer: $y = x - 2, x > 1$

3 marks

12 (3 marks)

The centre C represents $a = -2 + i$

B1

The radius is $r = 2$

B1

$$|z + 2 - i| \leq 2$$

B1

Answer: $|z + 2 - i| \leq 2$

3 marks

13 (3 marks)

Let $z = -3 + iy$: $9 + y^2 = 25$

M1

$$y^2 = 16, \text{ so } y = \pm 4$$

A1

$$z = -3 + 4i \text{ or } z = -3 - 4i$$

A1

Answer: $z = -3 \pm 4i$

3 marks

14 (4 marks)

$$(x - 4)^2 + y^2 = 4[(x - 1)^2 + y^2]$$

M1

$$x^2 - 8x + 16 + y^2 = 4x^2 - 8x + 4 + 4y^2$$

A1

$$3x^2 + 3y^2 = 12, \text{ so } x^2 + y^2 = 4$$

A1

Circle, centre the origin, radius 2

A1

Answer: $x^2 + y^2 = 4$: centre O, radius 2

4 marks

15 (4 marks)

Let $z = x + iy$: $z\bar{z} = x^2 + y^2$ and $z + \bar{z} = 2x$

M1

$$x^2 + y^2 + 4x - 5 = 0$$

A1

$$(x + 2)^2 + y^2 = 9$$

M1

A circle with centre $(-2, 0)$, that is -2 , and radius 3

A1

Answer: Circle, centre -2 , radius 3

4 marks

16 (5 marks)

Circle centre $(2, 0)$, radius 2, passing through the origin

B1

Half-line from the origin at $\frac{\pi}{4}$: $y = x$ with $x > 0$

B1

$$(x - 2)^2 + x^2 = 4$$

M1

$$2x^2 - 4x = 0, \text{ so } x = 0 \text{ or } x = 2; x = 0 \text{ is the excluded origin}$$

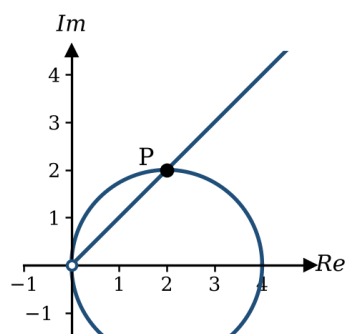
A1

$$z = 2 + 2i$$

A1

Answer: $z = 2 + 2i$

5 marks

**17 (4 marks)**

Two concentric circles with centre $(0, 2)$ and radii 1 and 3; R is the region between them, including both circles

B1 B1

$$\text{Area} = \pi \times 3^2 - \pi \times 1^2$$

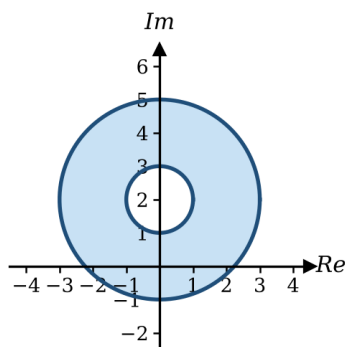
M1

$$= 8\pi$$

A1

Answer: 8π

4 marks



18 (5 marks)

Circle centre O, radius 4

B1

Half-lines at angles $\pm \frac{\pi}{4}$ from the origin

B1

R is the sector inside the circle between the half-lines

B1

$$\text{Area} = \frac{1}{2}r^2\theta = \frac{1}{2} \times 16 \times \frac{\pi}{2}$$

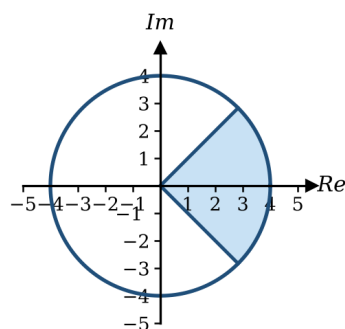
M1

$$= 4\pi$$

A1

Answer: 4π

5 marks



19 (5 marks)

(a) $|z| = |z - 8|$

B1

$$x^2 + y^2 = (x - 8)^2 + y^2, \text{ so } x = 4$$

A1

Answer: $|z| = |z - 8|$, $x = 4$

2 marks

(b) $|z| = 5$ and $x = 4$

B1

$$16 + y^2 = 25, \text{ so } y = \pm 3$$

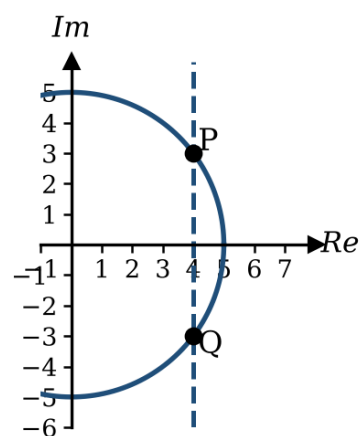
M1

$$z = 4 + 3i \text{ or } z = 4 - 3i$$

A1

Answer: $4 + 3i$ or $4 - 3i$

3 marks



20 (4 marks)

$$\arg z = \frac{\pi}{4} \text{ gives } z = a + ai \text{ with } a > 0$$

B1

$$(a - 1)^2 + a^2 = (a - 3)^2 + (a - 4)^2$$

M1

$$2a^2 - 2a + 1 = 2a^2 - 14a + 25, \text{ so } 12a = 24$$

A1

$$z = 2 + 2i$$

A1

$$\text{Answer: } z = 2 + 2i$$

4 marks

21 (6 marks)

(a) The circle has centre 4 and radius 2; the distance from O to the centre is 4

B1

The radius to the point of contact is perpendicular to the tangent, so $\sin \alpha = \frac{2}{4}$

M1

$$\alpha = \frac{\pi}{6}$$

A1

$$\text{Answer: } \alpha = \frac{\pi}{6}$$

3 marks

(b) The tangent length is $\sqrt{4^2 - 2^2} = 2\sqrt{3}$

M1

$$z = 2\sqrt{3}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$$

M1

$$= 2\sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 3 + \sqrt{3}i$$

A1

$$\text{Answer: } z = 3 + \sqrt{3}i$$

3 marks

22 (8 marks)

(a) $|z - 3| \leq 3$: the disc with centre (3, 0), radius 3

B1 B1

$|z| \leq |z - 3|$: closer to 0 than to 3, so $x \leq \frac{3}{2}$

M1

R is the part of the disc to the left of $x = \frac{3}{2}$

A1

$$\text{Answer: Segment of the disc left of } x = \frac{3}{2}$$

4 marks

(b) The line $x = \frac{3}{2}$ is at distance $d = \frac{3}{2}$ from the centre; half-angle α with $\cos \alpha = \frac{1}{2}$, so $\alpha = \frac{\pi}{3}$

M1 A1

$$\text{Sector area} = \frac{1}{2} \times 9 \times \frac{2\pi}{3} = 3\pi$$

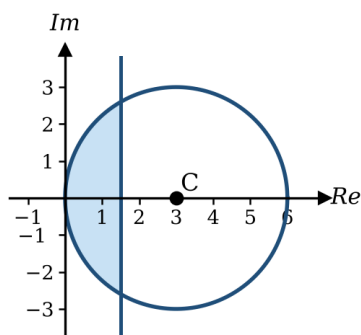
M1

$$\text{Subtract the triangle } \frac{1}{2} \times 3\sqrt{3} \times \frac{3}{2} = \frac{9\sqrt{3}}{4}; \text{ area} = 3\pi - \frac{9\sqrt{3}}{4}$$

A1

$$\text{Answer: } 3\pi - \frac{9\sqrt{3}}{4}$$

4 marks



23 (6 marks)

The locus $\arg(z - 3i) = \frac{\pi}{6}$ is the half-line from $3i$ at angle $\frac{\pi}{6}$

B1

$$z = 3i + t(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) = \frac{\sqrt{3}}{2}t + (3 + \frac{t}{2})i, t > 0$$

M1 A1

$$|z|^2 = \frac{3}{4}t^2 + 9 + 3t + \frac{1}{4}t^2 = t^2 + 3t + 9 = 13$$

M1

$$t^2 + 3t - 4 = 0, \text{ so } t = 1 \text{ (} t > 0 \text{)}$$

A1

$$z = \frac{\sqrt{3}}{2} + \frac{7}{2}i$$

A1

$$\text{Answer: } z = \frac{\sqrt{3}}{2} + \frac{7}{2}i$$

6 marks**24 (6 marks)**

(a) $|z - 1 + 2i| = |z - (1 - 2i)|$

M1

So the centre is $(1, -2)$, not $(1, 2)$

E1

Answer: Centre should be $(1, -2)$

2 marks

(b) Centre $(1, -2)$, radius 2

M1

$$(x - 1)^2 + (y + 2)^2 = 4$$

A1

$$\text{Answer: } (x - 1)^2 + (y + 2)^2 = 4$$

2 marks

(c) Greatest y = centre y -coordinate + radius

M1

$$= -2 + 2 = 0$$

A1

Answer: 0

2 marks**25 (6 marks)**

(a) $(x - 1)^2 + y^2 = 4 = (x + 1)^2 + y^2$, so $x = 0$

M1 A1

$$1 + y^2 = 4, \text{ so } y = \pm\sqrt{3}: z = \pm\sqrt{3}i$$

A1

$$\text{Answer: } z = \sqrt{3}i \text{ and } z = -\sqrt{3}i$$

3 marks

(b) $AB = 2$

B1

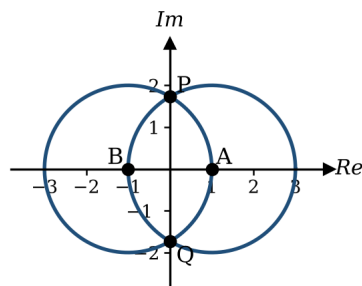
$$AP = |\sqrt{3}i - 1| = \sqrt{1 + 3} = 2$$

M1

$$BP = |\sqrt{3}i + 1| = 2, \text{ so all sides are equal}$$

A1

Answer: Triangle ABP is equilateral

3 marks**26 (7 marks)**

(a) Centre $C(5, 12)$, radius 4, and $OC = \sqrt{25 + 144} = 13$

B1

$$\text{Greatest } |z| = 13 + 4 = 17$$

M1

$$\text{Least } |z| = 13 - 4 = 9$$

A1

Answer: 17 and 9

3 marks

(b) The extreme arguments occur on tangents from O to the circle

M1

$$\text{Angle between } OC \text{ and a tangent: } \sin \alpha = \frac{4}{13}, \text{ so } \alpha = 0.3129$$

A1

$$\arg C = \tan^{-1}\left(\frac{12}{5}\right) = 1.1760$$

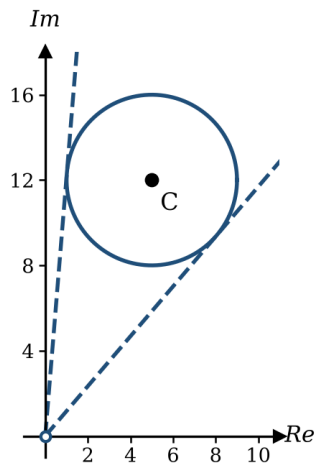
B1

$$\text{Greatest arg} = 1.176 + 0.313 = 1.49; \text{ least arg} = 1.176 - 0.313 = 0.863$$

A1

Answer: Greatest 1.49, least 0.863

4 marks



27 (8 marks)

(a) $(x - 3)^2 + y^2 = 4[x^2 + (y + 3)^2]$

M1 A1

$x^2 - 6x + 9 + y^2 = 4x^2 + 4y^2 + 24y + 36$

A1

$3x^2 + 6x + 3y^2 + 24y + 27 = 0$, so $x^2 + 2x + y^2 + 8y + 9 = 0$

M1

$(x + 1)^2 + (y + 4)^2 = 8$

A1

Centre $(-1, -4)$, radius $2\sqrt{2}$

A1

Answer: Circle, centre $-1 - 4i$, radius $2\sqrt{2}$

6 marks

(b) Distance from O to the centre is $\sqrt{1 + 16} = \sqrt{17}$

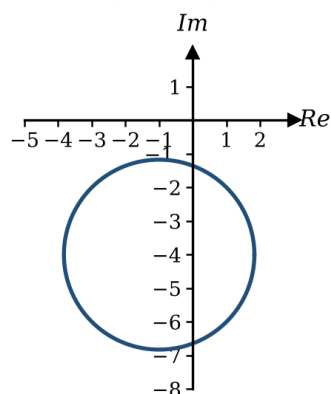
M1

Greatest $|z| = \sqrt{17} + 2\sqrt{2}$

A1

Answer: $\sqrt{17} + 2\sqrt{2}$

2 marks



28 (8 marks)

(a) Disc, centre $(1, 0)$, radius 1, and disc, centre $(0, 1)$, radius 1

B1 B1

R is the lens-shaped overlap of the two discs

B1

Answer: Lens-shaped region

3 marks

(b) $(x - 1)^2 + y^2 = 1 = x^2 + (y - 1)^2$ gives $x = y$, and then $x^2 - x = 0$

M1

The points are 0 and $1 + i$

A1

Answer: 0 and $1 + i$

2 marks

(c) The chord joining 0 and $1 + i$ has length $\sqrt{2}$; each circle subtends a right angle at its centre

M1

Each circular segment = quarter disc - triangle = $\frac{\pi}{4} - \frac{1}{2}$

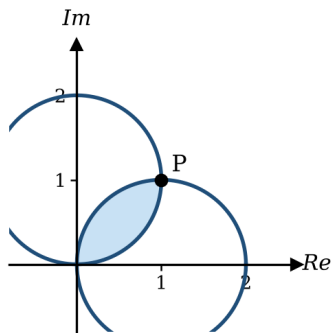
M1

R consists of two such segments: area = $\frac{\pi}{2} - 1$

A1

Answer: $\frac{\pi}{2} - 1$

3 marks



29 (10 marks)

$$(a) \frac{z-1}{z+1} = \frac{(z-1)(\bar{z}+1)}{(z+1)(\bar{z}+1)}$$

M1

$$\text{Numerator: } (x-1+iy)(x+1-iy) = x^2 - 1 + y^2 + 2iy$$

A1

$$\text{Denominator: } |z+1|^2 = (x+1)^2 + y^2$$

A1

$$\text{Answer: } \frac{z-1}{z+1} = \frac{x^2 + y^2 - 1 + 2iy}{(x+1)^2 + y^2}$$

3 marks

(b) Argument $\frac{\pi}{2}$: real part zero and imaginary part positive

M1

$$x^2 + y^2 = 1 \text{ and } y > 0$$

A1

The upper semicircle of the unit circle, excluding the points 1 and -1

A1

Answer: Upper unit semicircle, ends excluded

3 marks

(c) Real part = imaginary part > 0 : $x^2 + y^2 - 1 = 2y$ and $y > 0$

M1 A1

$$x^2 + (y-1)^2 = 2$$

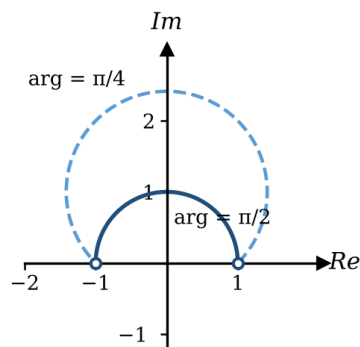
A1

The arc above the real axis of the circle with centre i and radius $\sqrt{2}$ (the major arc)

A1

Answer: $x^2 + (y-1)^2 = 2, y > 0$

4 marks



30 (8 marks)

(a) Circle

B1

Centre $(0, 5)$, radius 3

B1

Answer: Circle, centre $5i$, radius 3

2 marks

(b) Greatest $\arg z$ occurs when Oz is a tangent to the circle, with the tangent point T on the left

M1

$$\sin \angle COT = \frac{3}{5}, \text{ so } \angle COT = 0.6435$$

A1

$$\text{Greatest } \arg z = \frac{\pi}{2} + 0.6435 = 2.21$$

A1

$$OT^2 = 5^2 - 3^2, \text{ so } OT = 4$$

M1

$$z = 4(\cos \theta + i \sin \theta) \text{ with } \cos \theta = -\sin 0.6435 = -\frac{3}{5} \text{ and } \sin \theta = \cos 0.6435 = \frac{4}{5}$$

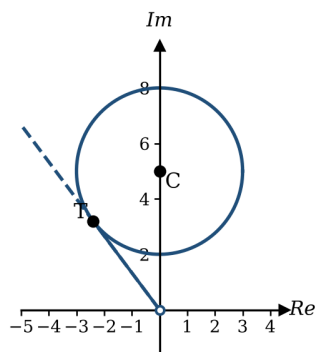
M1

$$z = -\frac{12}{5} + \frac{16}{5}i$$

A1

Answer: $\arg z = 2.21, z = -\frac{12}{5} + \frac{16}{5}i$

6 marks



Modulus-Argument Form

1 (2 marks)

$$\cos \frac{\pi}{3} = \frac{1}{2} \text{ and } \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

B1

$$z = 1 + \sqrt{3}i$$

B1

Answer: $1 + \sqrt{3}i$

2 marks

2 (2 marks)

Modulus $\sqrt{1+1} = \sqrt{2}$, argument $\frac{\pi}{4}$

B1 B1

Answer: $\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$

2 marks

3 (2 marks)

-3 lies on the negative real axis: modulus 3, argument π

B1 B1

Answer: $3(\cos \pi + i \sin \pi)$

2 marks

4 (2 marks)

Modulus 5, argument $\frac{\pi}{2}$

B1 B1

Answer: $5(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$

2 marks

5 (3 marks)

Modulus $\sqrt{3+1} = 2$

B1

The point is in the fourth quadrant with $\tan \theta = -\frac{1}{\sqrt{3}}$

M1

$$2(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}))$$

A1

Answer: $2(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}))$

3 marks

6 (2 marks)

$$|zw| = 4 \times 3 = 12$$

B1

$$\arg(zw) = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}, \text{ so } zw = 12(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$$

B1

Answer: $12(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$

2 marks

7 (3 marks)

$$|z| = \sqrt{1+3} = 2$$

B1

Fourth quadrant, reference angle $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$

M1

$$2(\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3}))$$

A1

Answer: $2(\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3}))$

3 marks

8 (3 marks)

$$|\frac{z}{w}| = \frac{8}{2} = 4$$

B1

$$\arg(\frac{z}{w}) = \frac{5\pi}{6} - \frac{\pi}{3} = \frac{\pi}{2}$$

M1

$$\frac{z}{w} = 4i$$

A1

Answer: $4i$

3 marks

9 (4 marks)

$$|zw| = \sqrt{2} \times 2\sqrt{2} = 4$$

B1

$$\arg(zw) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

M1

$$zw = 4(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}) = 4i$$

A1

Check: $z = 1 + i$, $w = 2 + 2i$, and $(1 + i)(2 + 2i) = 2 + 4i - 2 = 4i$

B1

Answer: $4(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}) = 4i$

4 marks

10 (3 marks)

$$2(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}) = \sqrt{2} + \sqrt{2}i \text{ and } 2(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}) = -\sqrt{2} + \sqrt{2}i$$

M1

$$z = 2\sqrt{2}i$$

A1

$$z = 2\sqrt{2}(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2})$$

A1

Answer: $2\sqrt{2}(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2})$

3 marks

11 (3 marks)

$$(a) \bar{z} = 3(\cos\frac{\pi}{5} - i \sin\frac{\pi}{5}) = 3(\cos(-\frac{\pi}{5}) + i \sin(-\frac{\pi}{5}))$$

B1

Answer: $3(\cos(-\frac{\pi}{5}) + i \sin(-\frac{\pi}{5}))$

1 mark

$$(b) |\frac{1}{z}| = \frac{1}{3}$$

B1

$$\arg(\frac{1}{z}) = -\frac{\pi}{5}, \text{ so } \frac{1}{z} = \frac{1}{3}(\cos(-\frac{\pi}{5}) + i \sin(-\frac{\pi}{5}))$$

B1

Answer: $\frac{1}{3}(\cos(-\frac{\pi}{5}) + i \sin(-\frac{\pi}{5}))$

2 marks

12 (3 marks)

The form $r(\cos \theta + i \sin \theta)$ needs a plus sign between the terms

E1

$$\cos\frac{\pi}{3} - i \sin\frac{\pi}{3} = \cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3})$$

M1

The argument is $-\frac{\pi}{3}$: $2(\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3}))$

A1

Answer: $2(\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3}))$

3 marks

13 (3 marks)

$$iz = i \cos \theta - \sin \theta$$

M1

$$= -\sin \theta + i \cos \theta = \cos(\theta + \frac{\pi}{2}) + i \sin(\theta + \frac{\pi}{2})$$

A1

Multiplication by i rotates the point through $\frac{\pi}{2}$ anticlockwise about O

B1

Answer: Rotation through $\frac{\pi}{2}$ anticlockwise

3 marks

14 (4 marks)

$$1 + i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$$

B1

$$\sqrt{3} - i = 2(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}))$$

B1

$$\text{Modulus} = \frac{\sqrt{2}}{2}, \text{ argument} = \frac{\pi}{4} - (-\frac{\pi}{6}) = \frac{5\pi}{12}$$

M1

$$\frac{\sqrt{2}}{2}(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12})$$

A1

$$\text{Answer: } \frac{\sqrt{2}}{2}(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12})$$

4 marks

15 (3 marks)

$-1 = \cos \pi + i \sin \pi$, so the argument changes by π and the modulus is 3

M1

$$\frac{\pi}{4} + \pi = \frac{5\pi}{4}, \text{ which is outside } (-\pi, \pi]$$

A1

$$\text{Principal argument } \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}: 3(\cos(-\frac{3\pi}{4}) + i \sin(-\frac{3\pi}{4}))$$

A1

$$\text{Answer: } 3(\cos(-\frac{3\pi}{4}) + i \sin(-\frac{3\pi}{4}))$$

3 marks

16 (4 marks)

$$|zw| = 6$$

B1

$$\arg(zw) = \frac{3\pi}{4} + \frac{2\pi}{3} = \frac{17\pi}{12}$$

M1

$$\frac{17\pi}{12} \text{ is greater than } \pi, \text{ so subtract } 2\pi: -\frac{7\pi}{12}$$

A1

$$zw = 6(\cos(-\frac{7\pi}{12}) + i \sin(-\frac{7\pi}{12}))$$

A1

$$\text{Answer: } 6(\cos(-\frac{7\pi}{12}) + i \sin(-\frac{7\pi}{12}))$$

4 marks

17 (3 marks)

$$\frac{1}{\cos \theta + i \sin \theta} = \frac{\cos \theta - i \sin \theta}{(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)}$$

M1

$$\text{Denominator} = \cos^2 \theta + \sin^2 \theta = 1$$

A1

$$\text{So } \frac{1}{\cos \theta + i \sin \theta} = \cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta)$$

A1

$$\text{Answer: For any real } \theta, \frac{1}{\cos \theta + i \sin \theta} = \cos \theta - i \sin \theta$$

3 marks

18 (6 marks)

$$(a) 1 + i\sqrt{3} = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) \text{ and } 1 + i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$$

B1

$$\text{Product: modulus } 2\sqrt{2}, \text{ argument } \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$$

M1

$$2\sqrt{2}(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12})$$

A1

$$\text{Answer: } 2\sqrt{2}(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12})$$

3 marks

$$(b) (1 + i\sqrt{3})(1 + i) = (1 - \sqrt{3}) + (1 + \sqrt{3})i$$

M1

$$\text{Equate real and imaginary parts: } 2\sqrt{2}\cos\frac{7\pi}{12} = 1 - \sqrt{3}$$

A1

$$\cos\frac{7\pi}{12} = \frac{1 - \sqrt{3}}{2\sqrt{2}}, \sin\frac{7\pi}{12} = \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

A1

$$\text{Answer: } \cos\frac{7\pi}{12} = \frac{1 - \sqrt{3}}{2\sqrt{2}}, \sin\frac{7\pi}{12} = \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

3 marks

19 (4 marks)

$$zw = 6(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$$

M1

$$zw = 6i$$

A1

$$\frac{z}{w} = \frac{2}{3}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$$

M1

$$= \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3}i$$

A1

$$\text{Answer: } zw = 6i, \frac{z}{w} = \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3}i$$

4 marks

20 (4 marks)

$$\sec\alpha(\cos\alpha + i\sin\alpha) = \frac{1}{\cos\alpha}(\cos\alpha + i\sin\alpha)$$

M1

$$= 1 + i\frac{\sin\alpha}{\cos\alpha} = 1 + i\tan\alpha$$

A1

Since $\sec\alpha > 0$, the modulus is $\sec\alpha$

B1

and the argument is α

B1

Answer: Modulus $\sec\alpha$, argument α

4 marks

21 (6 marks)

$$1 - \cos\alpha = 2\sin^2\frac{\alpha}{2}$$

B1

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

B1

$$\text{So the number is } 2\sin\frac{\alpha}{2}(\sin\frac{\alpha}{2} + i\cos\frac{\alpha}{2})$$

M1 A1

$$\sin\frac{\alpha}{2} = \cos(\frac{\pi}{2} - \frac{\alpha}{2}) \text{ and } \cos\frac{\alpha}{2} = \sin(\frac{\pi}{2} - \frac{\alpha}{2})$$

M1

$$\text{Since } 0 < \alpha < \pi, 2\sin\frac{\alpha}{2} > 0: \text{ modulus } 2\sin\frac{\alpha}{2}, \text{ argument } \frac{\pi}{2} - \frac{\alpha}{2}$$

A1

$$\text{Answer: Modulus } 2\sin\frac{\alpha}{2}, \text{ argument } \frac{\pi}{2} - \frac{\alpha}{2}$$

6 marks

22 (7 marks)

(a) The arguments should be added, not multiplied

E1

Also $\cos\theta \times \varphi$ is not $\cos(\theta + \varphi)$; the angle must appear inside cos and sin

E1

Answer: Arguments must be added

2 marks

$$(b) |zw| = 6, \arg(zw) = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

M1 A1

$$zw = 6(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 6i$$

A1

Answer: 6i

3 marks

$$(c) \frac{z}{w} = \frac{2}{3}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$$

M1

$$= \frac{\sqrt{3}}{3} + \frac{1}{3}i$$

A1

$$\text{Answer: } \frac{\sqrt{3}}{3} + \frac{1}{3}i$$

2 marks

23 (6 marks)

$$(a) \bar{z} = r(\cos \theta - i \sin \theta), \text{ so } \frac{z}{\bar{z}} = \frac{\cos \theta + i \sin \theta}{\cos \theta - i \sin \theta} \quad \text{M1}$$

$$= (\cos \theta + i \sin \theta)^2, \text{ using } \frac{1}{\cos \theta - i \sin \theta} = \cos \theta + i \sin \theta \quad \text{M1}$$

$$\arg \frac{z}{\bar{z}} = \theta - (-\theta) = 2\theta \text{ with modulus } 1 \quad \text{A1}$$

$$\text{Answer: } \frac{z}{\bar{z}} = \cos 2\theta + i \sin 2\theta \quad \text{3 marks}$$

$$(b) \frac{z}{\bar{z}} = \frac{3+4i}{3-4i} = \frac{(3+4i)^2}{25} \quad \text{M1}$$

$$= \frac{-7+24i}{25} \quad \text{A1}$$

$$\cos 2\theta = -\frac{7}{25} \quad \text{A1}$$

$$\text{Answer: } -\frac{7}{25} \quad \text{3 marks}$$

24 (5 marks)

$$OA = 4 \text{ and } OB = 4 \quad \text{B1}$$

$$\text{Angle } AOB = \arg B - \arg A = \frac{\pi}{3} \quad \text{B1}$$

$$\text{A triangle with two equal sides enclosing an angle } \frac{\pi}{3} \text{ is equilateral} \quad \text{M1}$$

$$B = 4\cos\frac{\pi}{3} + 4i\sin\frac{\pi}{3} \quad \text{M1}$$

$$B = 2 + 2\sqrt{3}i \quad \text{A1}$$

$$\text{Answer: } B = 2 + 2\sqrt{3}i \quad \text{5 marks}$$

25 (6 marks)

$$(a) |z| = |w| = \sqrt{5} \quad \text{B1}$$

$$\arg z = \tan^{-1}2 = 1.11 \quad \text{B1}$$

$$\arg w = \tan^{-1}\left(\frac{1}{2}\right) = 0.464; z = \sqrt{5}(\cos 1.11 + i \sin 1.11), w = \sqrt{5}(\cos 0.464 + i \sin 0.464) \quad \text{B1}$$

$$\text{Answer: } z = \sqrt{5}(\cos 1.11 + i \sin 1.11), w = \sqrt{5}(\cos 0.464 + i \sin 0.464) \quad \text{3 marks}$$

$$(b) zw = (1+2i)(2+i) = 2+i+4i-2 = 5i \quad \text{M1}$$

$$\arg(zw) = \frac{\pi}{2} \quad \text{A1}$$

$$\arg(zw) = \arg z + \arg w = \tan^{-1}2 + \tan^{-1}\left(\frac{1}{2}\right) \quad \text{A1}$$

$$\text{Answer: } \tan^{-1}2 + \tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} \quad \text{3 marks}$$

26 (6 marks)

$$\text{The argument of } V_2 \text{ is } -\frac{\pi}{3} = \frac{\pi}{6} - \frac{\pi}{2} \quad \text{M1}$$

$$-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right), \text{ so } V_2 = -12i(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) \quad \text{A1}$$

$$V_1 + V_2 = (5 - 12i)(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) \quad \text{M1}$$

$$\text{Amplitude} = |5 - 12i| = 13 \text{ volts} \quad \text{A1}$$

$$\text{Phase angle} = \arg(5 - 12i) + \frac{\pi}{6} = -1.176 + 0.524 \quad \text{M1}$$

$$= -0.652 \text{ radians} \quad \text{A1}$$

$$\text{Answer: Amplitude } 13 \text{ V, phase angle } -0.652 \text{ rad} \quad \text{6 marks}$$

27 (8 marks)

$$(a) 1 + \cos \theta = 2\cos^2 \frac{\theta}{2} \quad \text{B1}$$

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \quad \text{B1}$$

$$z = 2\cos \frac{\theta}{2} (\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}) \quad \text{M1}$$

$$-\frac{\pi}{2} < \frac{\theta}{2} < \frac{\pi}{2}, \text{ so } \cos \frac{\theta}{2} > 0: \text{ modulus } 2\cos \frac{\theta}{2}, \text{ argument } \frac{\theta}{2} \quad \text{A1}$$

$$\text{Answer: Modulus } 2\cos \frac{\theta}{2}, \text{ argument } \frac{\theta}{2} \quad \text{4 marks}$$

$$(b) \text{ The denominator is } \bar{z}, \text{ so the quotient is } \frac{z}{\bar{z}} \text{ with modulus 1 and argument } \frac{\theta}{2} - (-\frac{\theta}{2}) = \theta \quad \text{M1}$$

$$\frac{z}{\bar{z}} = \cos \theta + i \sin \theta \quad \text{A1}$$

$$\text{Answer: } \frac{1 + \cos \theta + i \sin \theta}{1 + \cos \theta - i \sin \theta} = \cos \theta + i \sin \theta \quad \text{2 marks}$$

$$(c) z = 2\cos \frac{\pi}{6} (\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) = \sqrt{3} (\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) \quad \text{M1}$$

$$= \frac{3}{2} + \frac{\sqrt{3}}{2}i \quad \text{A1}$$

$$\text{Answer: } \frac{3}{2} + \frac{\sqrt{3}}{2}i \quad \text{2 marks}$$

28 (7 marks)

$$(a) \cos \alpha + \cos \beta = 2\cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \quad \text{M1 A1}$$

$$\sin \alpha + \sin \beta = 2\sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \quad \text{M1 A1}$$

$$\text{Factor out } 2\cos \frac{\alpha - \beta}{2} \quad \text{A1}$$

$$\text{Answer: } (\cos \alpha + i \sin \alpha) + (\cos \beta + i \sin \beta) = 2\cos \frac{\alpha - \beta}{2} (\cos \frac{\alpha + \beta}{2} + i \sin \frac{\alpha + \beta}{2}) \quad \text{5 marks}$$

$$(b) 2\cos \frac{\pi}{6} (\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) = \sqrt{3} (\frac{1}{2} + \frac{\sqrt{3}}{2}i) \quad \text{M1}$$

$$= \frac{\sqrt{3}}{2} + \frac{3}{2}i \quad \text{A1}$$

$$\text{Answer: } i + \frac{\sqrt{3}}{2} + \frac{1}{2}i = \frac{\sqrt{3}}{2} + \frac{3}{2}i \quad \text{2 marks}$$

29 (7 marks)

$$(a) z = \frac{2\cos \alpha \pm \sqrt{4\cos^2 \alpha - 4}}{2} \quad \text{M1}$$

$$= \cos \alpha \pm \sqrt{-\sin^2 \alpha} \quad \text{A1}$$

$$= \cos \alpha \pm i \sin \alpha, \text{ as } \sin \alpha > 0 \quad \text{A1}$$

$$\text{Answer: The roots are } \cos \alpha \pm i \sin \alpha \quad \text{3 marks}$$

$$(b) \text{ Both roots have modulus } \sqrt{\cos^2 \alpha + \sin^2 \alpha} = 1 \quad \text{B1}$$

$$\cos \alpha + i \sin \alpha \text{ has argument } \alpha; \cos \alpha - i \sin \alpha \text{ has argument } -\alpha \quad \text{B1}$$

$$\text{Answer: } \cos \alpha + i \sin \alpha \text{ and } \cos(-\alpha) + i \sin(-\alpha) \quad \text{2 marks}$$

$$(c) \text{ Product of the roots} = 1 \quad \text{B1}$$

$$(\cos \alpha + i \sin \alpha)(\cos \alpha - i \sin \alpha) = \cos^2 \alpha + \sin^2 \alpha = 1, \text{ so each root is the reciprocal of the other} \quad \text{B1}$$

$$\text{Answer: Product 1: each root is the reciprocal of the other} \quad \text{2 marks}$$

30 (8 marks)

$$(a) 1 + i \tan \alpha = \sec \alpha (\cos \alpha + i \sin \alpha) \text{ and } 1 - i \tan \alpha = \sec \alpha (\cos \alpha - i \sin \alpha) \quad \text{M1 A1}$$

$$w = \frac{\cos \alpha + i \sin \alpha}{\cos \alpha - i \sin \alpha} = (\cos \alpha + i \sin \alpha)^2 \quad \text{M1}$$

$$= \cos 2\alpha + i \sin 2\alpha \text{ (arguments add)} \quad \text{A1}$$

$$\text{Answer: } w = \cos 2\alpha + i \sin 2\alpha \quad \text{4 marks}$$

$$(b) \cos 2\alpha + i \sin 2\alpha = i, \text{ so } 2\alpha = \frac{\pi}{2}$$

M1

$$\alpha = \frac{\pi}{4}, \tan \alpha = 1$$

A1

Answer: $\tan \alpha = 1$

2 marks

$$(c) w = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$$

M1

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

A1

$$\text{Answer: } \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

2 marks

Exponential Form and Euler's Formula

1 (3 marks)

$$(a) \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$$

B1

Answer: i

1 mark

$$(b) \cos \pi + i \sin \pi = -1$$

B1

Answer: -1

1 mark

$$(c) \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) = -i$$

B1

Answer: $-i$

1 mark

2 (2 marks)

$$2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$$

M1

$$= 1 + \sqrt{3}i$$

A1

$$\text{Answer: } 1 + \sqrt{3}i$$

2 marks

3 (2 marks)

$$r = \sqrt{1 + 1} = \sqrt{2}$$

B1

$$\theta = \frac{\pi}{4}, \text{ so } 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

B1

$$\text{Answer: } \sqrt{2}e^{i\frac{\pi}{4}}$$

2 marks

4 (2 marks)

$$-4 \text{ lies on the negative real axis: } r = 4$$

B1

$$\theta = \pi, \text{ so } -4 = 4e^{i\pi}$$

B1

$$\text{Answer: } 4e^{i\pi}$$

2 marks

5 (2 marks)

$$zw = 3 \times 2 e^{i(\frac{\pi}{6} + \frac{\pi}{3})} = 6e^{i\frac{\pi}{2}}$$

B1

$$= 6i$$

B1

$$\text{Answer: } 6e^{i\frac{\pi}{2}} = 6i$$

2 marks

6 (2 marks)

$$\frac{z}{w} = \frac{8}{2}e^{i(\frac{5\pi}{6} - \frac{\pi}{3})} = 4e^{i\frac{\pi}{2}}$$

B1

$$= 4i$$

B1

$$\text{Answer: } 4e^{i\frac{\pi}{2}} = 4i$$

2 marks

7 (3 marks)

$$r = \sqrt{3+1} = 2$$

B1

$$\text{Fourth quadrant, reference angle } \frac{\pi}{6}: \theta = -\frac{\pi}{6}$$

M1

$$\sqrt{3} - i = 2e^{-i\frac{\pi}{6}}$$

A1

$$\text{Answer: } 2e^{-i\frac{\pi}{6}}$$

3 marks

8 (3 marks)

$$e^{i\frac{2\pi}{3}} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$$

M1

$$\cos \frac{2\pi}{3} = -\frac{1}{2}$$

A1

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \text{ so the number is } -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

A1

$$\text{Answer: } -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

3 marks

9 (3 marks)

$$1 - i = \sqrt{2}e^{-i\frac{\pi}{4}} \text{ and } \sqrt{3} + i = 2e^{i\frac{\pi}{6}}$$

B1

$$(1 - i)^5 = 4\sqrt{2}e^{-i\frac{5\pi}{4}} = 4\sqrt{2}e^{i\frac{3\pi}{4}} \text{ and } (\sqrt{3} + i)^2 = 4e^{i\frac{\pi}{3}}$$

M1

$$\text{Quotient} = \sqrt{2}e^{i(\frac{3\pi}{4} - \frac{\pi}{3})} = \sqrt{2}e^{i\frac{5\pi}{12}}$$

A1

$$\text{Answer: } \sqrt{2}e^{i\frac{5\pi}{12}}$$

3 marks

10 (3 marks)

$$e^{2+i\frac{\pi}{3}} = e^2 \times e^{i\frac{\pi}{3}}$$

M1

$$\text{Modulus } e^2, \text{ argument } \frac{\pi}{3}: e^2 e^{i\frac{\pi}{3}}$$

A1

$$= e^2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = \frac{e^2}{2} + \frac{\sqrt{3}e^2}{2}i$$

A1

$$\text{Answer: } e^2 e^{i\frac{\pi}{3}} = \frac{e^2}{2} + \frac{\sqrt{3}e^2}{2}i$$

3 marks

11 (3 marks)

$$e^{i\theta} e^{i\varphi} = (\cos \theta + i \sin \theta)(\cos \varphi + i \sin \varphi)$$

M1

$$= (\cos \theta \cos \varphi - \sin \theta \sin \varphi) + i(\sin \theta \cos \varphi + \cos \theta \sin \varphi)$$

A1

$$= \cos(\theta + \varphi) + i \sin(\theta + \varphi) = e^{i(\theta + \varphi)}$$

A1

$$\text{Answer: } e^{i\theta} \times e^{i\varphi} = e^{i(\theta + \varphi)}$$

3 marks

12 (3 marks)

$$(2e^{i\frac{\pi}{4}})^3 = 8e^{i\frac{3\pi}{4}}$$

M1

$$= 8\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

A1

$$= -4\sqrt{2} + 4\sqrt{2}i$$

A1

$$\text{Answer: } -4\sqrt{2} + 4\sqrt{2}i$$

3 marks

13 (3 marks)

$$\cos \theta = -\frac{\sqrt{3}}{2} \text{ and } \sin \theta = -\frac{1}{2}$$

M1

$$\text{Third quadrant with reference angle } \frac{\pi}{6}$$

M1

$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

A1

$$\text{Answer: } \theta = \frac{7\pi}{6}$$

3 marks

14 (4 marks)

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta$$

B1

$$\text{Adding: } e^{i\theta} + e^{-i\theta} = 2\cos \theta$$

M1 A1

$$\text{Subtracting: } e^{i\theta} - e^{-i\theta} = 2i \sin \theta$$

A1

$$\text{Answer: } \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

4 marks

15 (4 marks)

$$\sqrt{3} + i = 2e^{i\frac{\pi}{6}}$$

B1

$$1 - i = \sqrt{2}e^{-i\frac{\pi}{4}}$$

B1

$$\frac{\sqrt{3} + i}{1 - i} = \frac{2}{\sqrt{2}}e^{i(\frac{\pi}{6} + \frac{\pi}{4})}$$

M1

$$= \sqrt{2}e^{i\frac{5\pi}{12}}$$

A1

$$\text{Answer: } \sqrt{2}e^{i\frac{5\pi}{12}}$$

4 marks

16 (4 marks)

$$-2 + 2i = 2\sqrt{2}e^{i\frac{3\pi}{4}}$$

M1 A1

$$\ln(-2 + 2i) = \ln(2\sqrt{2}) + i\frac{3\pi}{4}$$

M1

$$= 1.04 + 2.36i$$

A1

$$\text{Answer: } 1.04 + 2.36i$$

4 marks

17 (4 marks)

$$\text{Let } z = x + iy: e^x e^{iy} = -e$$

M1

$$\text{Modulus: } e^x = e, \text{ so } x = 1$$

A1

$$\text{Argument: } y = \pi + 2k\pi$$

M1

$$z = 1 + (2k + 1)\pi i, \text{ for any integer } k$$

A1

$$\text{Answer: } z = 1 + (2k + 1)\pi i, k \text{ an integer}$$

4 marks

18 (4 marks)

$$\text{Write } 1 = e^{i\frac{\theta}{2}} e^{-i\frac{\theta}{2}}$$

M1

$$1 + e^{i\theta} = e^{i\frac{\theta}{2}}(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}})$$

A1

$$e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} = 2\cos \frac{\theta}{2}$$

M1

$$\text{So } 1 + e^{i\theta} = 2\cos \frac{\theta}{2} e^{i\frac{\theta}{2}}$$

A1

$$\text{Answer: } 1 + e^{i\theta} = 2\cos \frac{\theta}{2} e^{i\frac{\theta}{2}}$$

4 marks

19 (4 marks)

$$(a) e^{i\theta} + e^{-i\theta} = 2\cos \theta$$

M1

$$= 2\cos \frac{\pi}{3} = 1$$

A1

$$\text{Answer: } 1$$

2 marks

$$(b) e^{i\theta} - e^{-i\theta} = 2i \sin \theta$$

M1

$$= 2i \sin \frac{\pi}{4} = \sqrt{2}i$$

A1

$$\text{Answer: } \sqrt{2}i$$

2 marks

20 (5 marks)

$$z^2 = 16e^{i\frac{2\pi}{3}}$$

M1 A1

$$\frac{z^2}{w} = \frac{16}{2}e^{i(\frac{2\pi}{3} + \frac{\pi}{6})}$$

M1

$$= 8e^{i\frac{5\pi}{6}}$$

A1

$$= 8(-\frac{\sqrt{3}}{2} + \frac{1}{2}i) = -4\sqrt{3} + 4i$$

A1

$$\text{Answer: } 8e^{i\frac{5\pi}{6}} = -4\sqrt{3} + 4i$$

5 marks

21 (6 marks)

$$(a) C + iS = 1 + \frac{1}{3}e^{i\theta} + \frac{1}{9}e^{2i\theta} + \frac{1}{27}e^{3i\theta} + \dots$$

M1

This is a geometric series with first term 1 and common ratio $\frac{1}{3}e^{i\theta}$, whose modulus is $\frac{1}{3} < 1$

A1

$$\text{Sum to infinity} = \frac{1}{1 - \frac{1}{3}e^{i\theta}} = \frac{3}{3 - e^{i\theta}}$$

A1

$$\text{Answer: } C + iS = \frac{3}{3 - e^{i\theta}}$$

3 marks

$$(b) \frac{3}{3 - e^{i\theta}} \times \frac{3 - e^{-i\theta}}{3 - e^{-i\theta}} = \frac{3(3 - \cos \theta + i \sin \theta)}{(3 - \cos \theta)^2 + \sin^2 \theta}$$

M1

$$(3 - \cos \theta)^2 + \sin^2 \theta = 9 - 6\cos \theta + 1 = 10 - 6\cos \theta, \text{ so } C = \frac{3(3 - \cos \theta)}{10 - 6\cos \theta}$$

A1

$$S = \frac{3 \sin \theta}{10 - 6\cos \theta}$$

A1

$$\text{Answer: } C = \frac{3(3 - \cos \theta)}{10 - 6\cos \theta}, S = \frac{3 \sin \theta}{10 - 6\cos \theta}$$

3 marks

22 (7 marks)

$$(a) e^z \text{ is not one-to-one on the complex numbers: } e^{z+2\pi i} = e^z$$

E1

So $e^a = e^b$ does not imply $a = b$

E1

Answer: e^z is periodic with period $2\pi i$

2 marks

$$(b) \text{ Let } z = x + iy: e^x = 1, \text{ so } x = 0$$

M1

$$y = 2k\pi: z = 2k\pi i, k \text{ an integer}$$

A1

Answer: $z = 2k\pi i$

2 marks

$$(c) e^x = |2i| = 2, \text{ so } x = \ln 2$$

B1

$$\arg(2i) = \frac{\pi}{2}, \text{ so } y = \frac{\pi}{2} + 2k\pi$$

M1

$$z = \ln 2 + (\frac{\pi}{2} + 2k\pi)i$$

A1

$$\text{Answer: } z = \ln 2 + (\frac{\pi}{2} + 2k\pi)i$$

3 marks

23 (6 marks)

$$(a) \text{ Distance} = |z| = 3$$

B1

$$z(0) = 3e^{i\frac{\pi}{6}} = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

B1

$$\text{Answer: } 3; \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

2 marks

$$(b) \arg z = 2t + \frac{\pi}{6} = \frac{\pi}{2}$$

M1

$$t = \frac{\pi}{6} \text{ seconds}$$

A1

$$\text{Answer: } t = \frac{\pi}{6}$$

2 marks

$$(c) z = 3e^{i(\frac{\pi}{2} + \frac{\pi}{6})} = 3e^{i\frac{2\pi}{3}}$$

M1

$$= -\frac{3}{2} + \frac{3\sqrt{3}}{2}i \quad \text{A1}$$

Answer: $-\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ 2 marks

24 (6 marks)

Numerator: $e^{i\theta} - 1 = e^{i\frac{\theta}{2}}(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}) = e^{i\frac{\theta}{2}} \times 2i \sin \frac{\theta}{2}$ M1 A1

Denominator: $e^{i\theta} + 1 = e^{i\frac{\theta}{2}} \times 2\cos \frac{\theta}{2}$ M1 A1

Quotient = $\frac{2i \sin \frac{\theta}{2}}{2\cos \frac{\theta}{2}} = i \tan \frac{\theta}{2}$ A1

$\theta = \frac{\pi}{2}: \frac{i-1}{i+1} = \frac{(i-1)(1-i)}{2} = \frac{2i}{2} = i = i \tan \frac{\pi}{4}$ B1

Answer: $\frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i \tan \frac{\theta}{2}$ 6 marks

25 (6 marks)

$e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ B1

$e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ B1

Product: $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i)$ M1

Real part: $\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$ A1

Imaginary part: $-\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$ A1

$\cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}, \sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$ A1

Answer: $\cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}, \sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$ 6 marks

26 (6 marks)

(a) $(e^{i\frac{\pi}{4}})^8 = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$ B1

Answer: $(e^{i\frac{\pi}{4}})^8 = 1$ 1 mark

(b) $2026 = 8 \times 253 + 2$ M1

$(e^{i\frac{\pi}{4}})^{2026} = (e^{i\frac{\pi}{4}})^2$ (as the power 8 gives 1) A1

$= e^{i\frac{\pi}{2}} = i$ A1

Answer: i 3 marks

(c) $(1+i)^{2026} = (\sqrt{2})^{2026} \times i = 2^{1013}i$ M1 A1

Answer: $2^{1013}i$ 2 marks

27 (7 marks)

(a) $i^i = e^{i \times i(\frac{\pi}{2} + 2k\pi)}$ M1

$= e^{i^2(\frac{\pi}{2} + 2k\pi)}$ A1

$= e^{-(\frac{\pi}{2} + 2k\pi)}$ A1

Answer: i^i takes the values $e^{-(\frac{\pi}{2} + 2k\pi)}$ for integer k 3 marks

(b) The exponent $-(\frac{\pi}{2} + 2k\pi)$ is real B1

e raised to a real power is positive E1

Answer: Real exponent of e 2 marks

(c) $e^{-\frac{\pi}{2}}$ M1

$$= 0.208$$

Answer: 0.208

A1

2 marks

28 (9 marks)

(a) Geometric series with first term 1, ratio $e^{i\theta}$ and n terms

M1

$$S = \frac{e^{in\theta} - 1}{e^{i\theta} - 1}$$

A1

Answer: $S = \frac{e^{in\theta} - 1}{e^{i\theta} - 1}$

2 marks

(b) $e^{in\theta} - 1 = e^{i\frac{n\theta}{2}} \times 2i \sin \frac{n\theta}{2}$

M1 A1

$$e^{i\theta} - 1 = e^{i\frac{\theta}{2}} \times 2i \sin \frac{\theta}{2}$$

A1

$$S = \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} e^{i\frac{(n-1)\theta}{2}}$$

A1

Answer: $S = \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} e^{i\frac{(n-1)\theta}{2}}$

4 marks

(c) Take real parts of the result in (b)

M1 A1

$$n = 4, \theta = \frac{\pi}{2}: \sin \pi \cos \frac{3\pi}{4} / \sin \frac{\pi}{4} = 0, \text{ and } 1 + 0 - 1 + 0 = 0$$

B1

Answer: $1 + \cos \theta + \cos 2\theta + \dots + \cos(n-1)\theta = \frac{\sin \frac{n\theta}{2} \cos \frac{(n-1)\theta}{2}}{\sin \frac{\theta}{2}}$

3 marks

29 (8 marks)

$$e^{2x} \cos x = \operatorname{Re}(e^{2x} e^{ix}) = \operatorname{Re}(e^{(2+i)x})$$

M1 A1

$$\int e^{(2+i)x} dx = \frac{e^{(2+i)x}}{2+i} + c$$

M1 A1

$$\frac{1}{2+i} = \frac{2-i}{5}$$

M1

$$e^{(2+i)x} = e^{2x}(\cos x + i \sin x)$$

B1

$$\text{Real part of } \frac{e^{2x}(2-i)(\cos x + i \sin x)}{5} = \frac{e^{2x}(2\cos x + \sin x)}{5}$$

M1

$$\int e^{2x} \cos x dx = \frac{e^{2x}}{5}(2\cos x + \sin x) + C$$

A1

Answer: $\frac{e^{2x}}{5}(2\cos x + \sin x) + C$

8 marks

30 (7 marks)

$$\text{Let } w = e^z: w^2 - 2w + 2 = 0$$

M1

$$w = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

M1 A1

$$w = 1 + i = \sqrt{2} e^{i\frac{\pi}{4}}: e^x = \sqrt{2}, \text{ so } x = \frac{1}{2} \ln 2, \text{ and } y = \frac{\pi}{4} + 2k\pi$$

M1 A1

$$w = 1 - i: x = \frac{1}{2} \ln 2 \text{ and } y = -\frac{\pi}{4} + 2k\pi$$

M1

$$z = \frac{1}{2} \ln 2 + i(2k\pi \pm \frac{\pi}{4}), k \text{ an integer}$$

A1

Answer: $z = \frac{1}{2} \ln 2 + i(2k\pi \pm \frac{\pi}{4})$

7 marks

De Moivre's Theorem

1 (2 marks)

$$\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)^3 = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

$$= i$$

M1

A1

Answer: i

2 marks

2 (2 marks)

$$\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)^5 = \cos \pi + i \sin \pi$$

$$= -1 + 0i = -1$$

M1

A1

Answer: $\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)^5 = -1$

2 marks

3 (2 marks)

$$2^6 \left(\cos \frac{6\pi}{12} + i \sin \frac{6\pi}{12}\right)$$

$$= 64i$$

M1

A1

Answer: 64i

2 marks

4 (2 marks)

$$(\cos \theta + i \sin \theta)^{5-2}$$

$$= \cos 3\theta + i \sin 3\theta$$

M1

A1

Answer: $\cos 3\theta + i \sin 3\theta$

2 marks

5 (2 marks)

$$\cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta)$$

$$(\cos \theta - i \sin \theta)^4 = \cos(-4\theta) + i \sin(-4\theta)$$

M1

A1

Answer: $\cos(-4\theta) + i \sin(-4\theta)$

2 marks

6 (3 marks)

$$1 + i = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$$

$$(1 + i)^4 = (\sqrt{2})^4 (\cos \pi + i \sin \pi)$$

$$= 4 \times (-1) = -4$$

B1

M1

A1

Answer: $(1 + i)^4 = -4$

3 marks

7 (4 marks)

$$1 + i\sqrt{3} = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$$

$$(1 + i\sqrt{3})^5 = 2^5 \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$$

$$= 32 \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 16 - 16\sqrt{3}i$$

B1

M1 A1

A1

Answer: $16 - 16\sqrt{3}i$

4 marks

8 (4 marks)

$$\sqrt{3} - i = 2 \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right)$$

$$(\sqrt{3} - i)^6 = 2^6 (\cos(-\pi) + i \sin(-\pi))$$

$$= 64 \times (-1) = -64$$

B1 B1

M1

A1

Answer: $(\sqrt{3} - i)^6 = -64$

4 marks

9 (3 marks)

$$1 - i = \sqrt{2}(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}))$$

B1

$$(1 - i)^8 = (\sqrt{2})^8(\cos(-2\pi) + i \sin(-2\pi))$$

M1

$$= 16$$

A1**Answer: 16****3 marks****10 (3 marks)**

$$\text{De Moivre holds for negative integers: } \cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2})$$

M1 A1

$$= -i$$

A1**Answer: -i****3 marks****11 (4 marks)**

$$1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4})$$

B1

$$(1 + i)^6 = 8(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}) = -8i$$

M1 A1

$$\frac{1}{-8i} = \frac{i}{8}$$

A1**Answer: $\frac{i}{8}$** **4 marks****12 (3 marks)**

$$\text{Numerator: } (\cos 6\theta + i \sin 6\theta)(\cos(-6\theta) + i \sin(-6\theta)) = 1$$

M1 A1

$$\text{Denominator: } \cos 4\theta + i \sin 4\theta, \text{ so the result is } \cos(-4\theta) + i \sin(-4\theta) = \cos 4\theta - i \sin 4\theta$$

A1**Answer: $\cos 4\theta - i \sin 4\theta$** **3 marks****13 (3 marks)**

$$\sqrt{3} + i = 2(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}), \text{ so } (\sqrt{3} + i)^n = 2^n(\cos\frac{n\pi}{6} + i \sin\frac{n\pi}{6})$$

M1

$$\text{Purely imaginary when } \frac{n\pi}{6} = \frac{\pi}{2} + k\pi, \text{ so } n = 3 + 6k; \text{ the smallest is } n = 3$$

A1

$$(\sqrt{3} + i)^3 = 8i$$

A1**Answer: $n = 3$, value $8i$** **3 marks****14 (5 marks)**

$$(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$$

B1

$$\text{Binomial expansion: } \cos^3\theta + 3i \cos^2\theta \sin \theta - 3\cos \theta \sin^2\theta - i \sin^3\theta$$

M1

$$\text{Real part: } \cos 3\theta = \cos^3\theta - 3\cos \theta(1 - \cos^2\theta) = 4\cos^3\theta - 3\cos \theta$$

A1

$$\text{Imaginary part: } \sin 3\theta = 3(1 - \sin^2\theta)\sin \theta - \sin^3\theta$$

M1

$$= 3 \sin \theta - 4 \sin^3\theta$$

A1**Answer: $\cos 3\theta = 4\cos^3\theta - 3\cos \theta$ and $\sin 3\theta = 3 \sin \theta - 4 \sin^3\theta$** **5 marks****15 (4 marks)**

$$\sqrt{3} - i = 2(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}))$$

B1

$$(\sqrt{3} - i)^{-5} = 2^{-5}(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6})$$

M1 A1

$$= \frac{1}{32}(-\frac{\sqrt{3}}{2} + \frac{1}{2}i) = -\frac{\sqrt{3}}{64} + \frac{1}{64}i$$

A1**Answer: $-\frac{\sqrt{3}}{64} + \frac{1}{64}i$** **4 marks**

16 (4 marks)

$$(\cos \theta - i \sin \theta)^n = \cos(-n\theta) + i \sin(-n\theta) = \cos n\theta - i \sin n\theta \quad \text{M1}$$

$$\text{Sum} = 2\cos n\theta \quad \text{A1}$$

$$(1 \pm i)^8 = (\sqrt{2})^8 (\cos 2\pi \pm i \sin 2\pi), \text{ so the sum is } 16 \times 2\cos 2\pi \quad \text{M1}$$

$$= 32 \quad \text{A1}$$

Answer: 32

4 marks

17 (4 marks)

$$1 + z^2 = 1 + \cos 2\theta + i \sin 2\theta = 2\cos^2\theta + 2i \sin \theta \cos \theta \quad \text{M1 A1}$$

$$= 2\cos \theta (\cos \theta + i \sin \theta) = 2\cos \theta \times z \quad \text{A1}$$

$$\frac{(1 + z^2)^n}{z^n} = \frac{(2\cos \theta)^n z^n}{z^n} = 2^n \cos^n \theta \quad \text{A1}$$

Answer: $1 + z^2 = 2\cos \theta \times z$; $\frac{(1 + z^2)^n}{z^n} = 2^n \cos^n \theta$

4 marks

18 (5 marks)

$$\cos 4\theta + i \sin 4\theta = (\cos \theta + i \sin \theta)^4 \quad \text{B1}$$

$$\text{Real part: } \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta \quad \text{M1 A1}$$

$$\text{Replace } \sin^2\theta \text{ by } 1 - \cos^2\theta: \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2 \quad \text{M1}$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1 \quad \text{A1}$$

Answer: $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

5 marks

19 (5 marks)

$$\frac{1}{z^n} = z^{-n} = \cos n\theta - i \sin n\theta \quad \text{M1}$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{A1}$$

$$z + \frac{1}{z} = 2\cos \theta = 1, \text{ so } \cos \theta = \frac{1}{2} \quad \text{M1}$$

$$z^2 + \frac{1}{z^2} = 2\cos 2\theta = 2(2\cos^2\theta - 1) \quad \text{M1}$$

$$= 2\left(\frac{1}{2} - 1\right) = -1 \quad \text{A1}$$

Answer: -1

5 marks

20 (5 marks)

$$1 + i \tan \theta = \sec \theta (\cos \theta + i \sin \theta) \quad \text{M1 A1}$$

$$(1 + i \tan \theta)^n = \sec^n \theta (\cos n\theta + i \sin n\theta) \quad \text{A1}$$

$$\sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}, \text{ so } (1 + i \tan \frac{\pi}{6})^3 = \frac{8}{3\sqrt{3}} (\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \quad \text{M1}$$

$$= \frac{8\sqrt{3}}{9} i \quad \text{A1}$$

Answer: $\frac{8\sqrt{3}}{9} i$

5 marks

21 (7 marks)

$$z + z^{-1} = 2\cos \theta \quad \text{B1}$$

$$(2\cos \theta)^5 = (z + z^{-1})^5 \quad \text{M1}$$

$$= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5} \quad \text{M1 A1}$$

$$= (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1}) \quad \text{A1}$$

$$= 2\cos 5\theta + 10\cos 3\theta + 20\cos \theta \quad \text{A1}$$

$$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta) \quad \text{A1}$$

Answer: $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$

7 marks

22 (8 marks)

(a) Let $z = \cos \theta + i \sin \theta$: $(z - z^{-1})^4 = (2i \sin \theta)^4 = 16 \sin^4 \theta$ M1 A1

$$(z - z^{-1})^4 = z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$$
M1 A1

$$= 2\cos 4\theta - 8\cos 2\theta + 6, \text{ so } \sin^4 \theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$$
A1

Answer: $\sin^4 \theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$ 5 marks

(b) $\int_0^{\frac{\pi}{2}} \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta) d\theta = \frac{1}{8}[3\theta - 2 \sin 2\theta + \frac{1}{4}\sin 4\theta]$ M1

At $\frac{\pi}{2}$ the sine terms are zero, and at 0 everything is zero A1

$$= \frac{1}{8} \times \frac{3\pi}{2} = \frac{3\pi}{16}$$
A1

Answer: $\frac{3\pi}{16}$ 3 marks

23 (6 marks)

(a) The modulus of $1 + i$ is $\sqrt{2}$, not 1 E1

$$\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}(1 + i) \neq 1 + i$$
E1

Answer: Modulus ignored 2 marks

(b) $1 + i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ B1

$$(1 + i)^8 = (\sqrt{2})^8 (\cos 2\pi + i \sin 2\pi)$$

$$= 16$$
M1
A1

Answer: 16 3 marks

(c) $(1 + i)^2 = 2i$, $(2i)^2 = -4$ and $(-4)^2 = 16$ B1

Answer: 16 1 mark

24 (7 marks)

$$(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$$
B1

Binomial expansion with $c = \cos \theta$, $s = \sin \theta$: $c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$ M1 A1

Imaginary part: $\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$ A1

Replace c^2 by $1 - s^2$: $5(1 - s^2)^2s - 10(1 - s^2)s^3 + s^5$ M1

$$= 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5$$
A1

$$= 16s^5 - 20s^3 + 5s$$
A1

Answer: $\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$ 7 marks

25 (7 marks)

(a) $z + z^{-1} = 2\cos \theta$, so $(2\cos \theta)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$ M1 A1

$$z^n + z^{-n} = 2\cos n\theta, \text{ so } 16\cos^4 \theta = (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$$
M1

$$= 2\cos 4\theta + 8\cos 2\theta + 6, \text{ so } 8\cos^4 \theta = \cos 4\theta + 4\cos 2\theta + 3$$
A1

Answer: $8\cos^4 \theta = \cos 4\theta + 4\cos 2\theta + 3$ 4 marks

(b) By part (a), $\cos 4\theta + 4\cos 2\theta + 1 = 8\cos^4 \theta - 2$ M1

$$8\cos^4 \theta = 2, \text{ so } \cos^2 \theta = \frac{1}{2} \text{ and } \cos \theta = \pm \frac{1}{\sqrt{2}}$$
A1

$$\theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4}$$
A1

Answer: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ 3 marks

26 (7 marks)

(a) $(\cos \theta + i \sin \theta)^3$: real part $\cos^3 \theta - 3\cos \theta \sin^2 \theta$ M1 A1

$$= \cos^3 \theta - 3\cos \theta (1 - \cos^2 \theta) = 4\cos^3 \theta - 3\cos \theta$$
A1

Answer: $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$ 3 marks

(b) Let $x = \cos \theta$: $8x^3 - 6x = 2(4\cos^3\theta - 3\cos\theta) = 2\cos 3\theta$, so the equation is $\cos 3\theta = \frac{1}{2}$

M1 A1

$$3\theta = \pm \frac{\pi}{3} + 2k\pi, \text{ so } \theta = \pm \frac{\pi}{9} + \frac{2k\pi}{3}$$

M1

$\theta = \frac{\pi}{9}, \frac{7\pi}{9}, \frac{5\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$; other values repeat these cosines. These are three distinct roots of the cubic

A1

Answer: $\cos \frac{\pi}{9}, \cos \frac{5\pi}{9}, \cos \frac{7\pi}{9}$

4 marks

27 (9 marks)

(a) $n = 1$: $z^1 = \cos \theta + i \sin \theta$, true

B1

Assume $z^k = \cos k\theta + i \sin k\theta$

M1

$$z^{k+1} = (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta)$$

M1

$$= (\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta)$$

A1

$$= \cos(k+1)\theta + i \sin(k+1)\theta$$

A1

True for $n = 1$, and true for k implies true for $k + 1$, so true for all $n \geq 1$

E1

Answer: $z^n = \cos n\theta + i \sin n\theta$ for every positive integer n

6 marks

(b) Let $n = -m$ with m a positive integer: $z^{-m} = \frac{1}{z^m} = \frac{1}{\cos m\theta + i \sin m\theta}$

M1

$$= \cos m\theta - i \sin m\theta$$

A1

$$= \cos(-m\theta) + i \sin(-m\theta) = \cos n\theta + i \sin n\theta$$

A1

Answer: The result also holds when n is a negative integer

3 marks

28 (8 marks)

$$(1 + e^{i\theta})^n = \sum_{r=0}^n C(n, r)e^{ir\theta}$$

M1 A1

$$\text{Real part of the right-hand side: } \sum_{r=0}^n C(n, r)\cos r\theta$$

A1

$$1 + e^{i\theta} = e^{i\frac{\theta}{2}}(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}) = 2\cos \frac{\theta}{2} e^{i\frac{\theta}{2}}$$

M1 A1

$$(1 + e^{i\theta})^n = 2^n \cos^n \frac{\theta}{2} e^{i\frac{n\theta}{2}}$$

M1

$$\text{Real part: } 2^n \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2}$$

A1

Equating the two real parts gives the result

A1

Answer: For every positive integer n , $\sum_{r=0}^n C(n, r)\cos r\theta = 2^n \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2}$, where $C(n, r)$ is the binomial coefficient

29 (8 marks)

(a) $(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$; the real part is $c^3 - 3cs^2$ and the imaginary part is $3c^2s - s^3$

M1 A1

$$\tan 3\theta = \frac{3c^2s - s^3}{c^3 - 3cs^2}$$

M1

$$\text{Divide top and bottom by } c^3: \frac{3t - t^3}{1 - 3t^2}$$

A1

$$\text{Answer: } \tan 3\theta = \frac{3t - t^3}{1 - 3t^2}$$

4 marks

(b) $\tan 3\theta = 1$ gives $3t - t^3 = 1 - 3t^2$, i.e. $t^3 - 3t^2 - 3t + 1 = 0$

M1

$$3\theta = \frac{\pi}{4} + k\pi, \text{ so } \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$$

M1 A1

So the roots are $\tan \frac{\pi}{12}, \tan \frac{5\pi}{12}, \tan \frac{3\pi}{4} = -1$; these are $2 - \sqrt{3}, 2 + \sqrt{3}, -1$

A1

Answer: $t = 2 - \sqrt{3}, 2 + \sqrt{3}, -1$

4 marks

30 (8 marks)

$$(a) \cos \frac{3\pi}{5} = 4c^3 - 3c \quad \text{B1}$$

$$\cos \frac{2\pi}{5} = 2c^2 - 1 \quad \text{B1}$$

$$\text{Answer: } \cos \frac{3\pi}{5} = 4c^3 - 3c, \cos \frac{2\pi}{5} = 2c^2 - 1 \quad \text{2 marks}$$

$$(b) \frac{3\pi}{5} = \pi - \frac{2\pi}{5} \quad \text{M1}$$

$$\cos(\pi - x) = -\cos x \quad \text{A1}$$

$$\text{Answer: } \cos(\pi - x) = -\cos x \quad \text{2 marks}$$

$$(c) 4c^3 - 3c = -(2c^2 - 1) \quad \text{M1}$$

$$4c^3 + 2c^2 - 3c - 1 = 0, \text{ which factorises as } (c + 1)(4c^2 - 2c - 1) = 0 \quad \text{M1 A1}$$

$$c = -1 \text{ is impossible, and } 4c^2 - 2c - 1 = 0 \text{ gives } c = \frac{1 + \sqrt{5}}{4} \text{ (the positive root)} \quad \text{A1}$$

$$\text{Answer: } \cos \frac{\pi}{5} = \frac{1 + \sqrt{5}}{4} \quad \text{4 marks}$$

Roots of Complex Numbers**1 (2 marks)**

$$z^4 = 1 \text{ gives } z^2 = \pm 1 \quad \text{M1}$$

$$z = 1, -1, i, -i \quad \text{A1}$$

$$\text{Answer: } z = 1, -1, i, -i \quad \text{2 marks}$$

2 (2 marks)

$$\text{Square root: halve the argument and take the square root of the modulus} \quad \text{M1}$$

$$\sqrt{2}(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) \quad \text{A1}$$

$$\text{Answer: } \sqrt{2}(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) \quad \text{2 marks}$$

3 (3 marks)

$$8 = 8(\cos 2k\pi + i \sin 2k\pi), \text{ so } z = 2(\cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}) \quad \text{M1}$$

$$k = 0: z = 2 \quad \text{B1}$$

$$k = 1, 2: z = -1 \pm \sqrt{3}i \quad \text{A1}$$

$$\text{Answer: } z = 2, -1 + \sqrt{3}i, -1 - \sqrt{3}i \quad \text{3 marks}$$

4 (3 marks)

$$\text{Roots have modulus 1 and arguments } \frac{2k\pi}{5} \quad \text{B1}$$

$$\text{Arguments } 0, \frac{2\pi}{5}, \frac{4\pi}{5}, -\frac{4\pi}{5}, -\frac{2\pi}{5} \quad \text{M1}$$

$$\cos 0 + i \sin 0 = 1 \text{ and } \cos(\pm \frac{2\pi}{5}) + i \sin(\pm \frac{2\pi}{5}), \cos(\pm \frac{4\pi}{5}) + i \sin(\pm \frac{4\pi}{5}) \quad \text{A1}$$

$$\text{Answer: } 1, \cos(\pm \frac{2\pi}{5}) + i \sin(\pm \frac{2\pi}{5}), \cos(\pm \frac{4\pi}{5}) + i \sin(\pm \frac{4\pi}{5}) \quad \text{3 marks}$$

5 (2 marks)

$$i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}, \text{ so a root has argument } \frac{\pi}{4} \quad \text{M1}$$

$$z = \pm(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) \quad \text{A1}$$

$$\text{Answer: } \pm(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) \quad \text{2 marks}$$

6 (2 marks)

$$-4i = 4(\cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2})), \text{ so a root is } 2(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}))$$

M1

$$z = \pm(\sqrt{2} - \sqrt{2}i)$$

A1

$$\text{Answer: } \pm(\sqrt{2} - \sqrt{2}i)$$

2 marks

7 (4 marks)

$$-16 = 16(\cos \pi + i \sin \pi), \text{ so } z = 2(\cos \frac{\pi + 2k\pi}{4} + i \sin \frac{\pi + 2k\pi}{4})$$

M1 A1

$$\text{Arguments } \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}$$

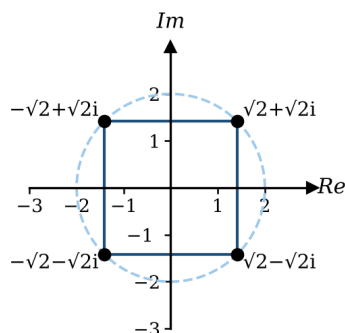
A1

$$z = \pm\sqrt{2} \pm \sqrt{2}i; \text{ the roots lie at the corners of a square inscribed in the circle of radius 2}$$

A1

$$\text{Answer: } z = \pm\sqrt{2} \pm \sqrt{2}i$$

4 marks

**8 (4 marks)**

$$8i = 8(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}), \text{ so } z = 2(\cos(\frac{\pi}{2} + \frac{2k\pi}{3}) + i \sin(\frac{\pi}{2} + \frac{2k\pi}{3}))$$

M1 A1

$$\text{Arguments } \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$$

A1

$$z = \sqrt{3} + i, -\sqrt{3} + i, -2i$$

A1

$$\text{Answer: } z = \sqrt{3} + i, -\sqrt{3} + i, -2i$$

4 marks

9 (4 marks)

$$32i = 32e^{i\frac{\pi}{2}}, \text{ so } |z| = 32^{1/5} = 2$$

B1

$$\arg z = \frac{\frac{\pi}{2} + 2k\pi}{5} = \frac{\pi}{10} + \frac{2k\pi}{5}$$

M1

$$k = 0, 1, 2: z = 2e^{i\frac{\pi}{10}}, 2e^{i\frac{\pi}{2}}, 2e^{i\frac{9\pi}{10}}$$

A1

$$k = -1, -2: z = 2e^{-i\frac{3\pi}{10}}, 2e^{-i\frac{7\pi}{10}}$$

A1

$$\text{Answer: } z = 2e^{i\frac{\pi}{10}}, 2e^{i\frac{\pi}{2}}, 2e^{i\frac{9\pi}{10}}, 2e^{-i\frac{3\pi}{10}}, 2e^{-i\frac{7\pi}{10}}$$

4 marks

10 (4 marks)

$$z = \cos \frac{2k\pi}{6} + i \sin \frac{2k\pi}{6}, k = 0, 1, \dots, 5$$

M1 A1

$$k = 0 \text{ and } k = 3: z = 1 \text{ and } z = -1$$

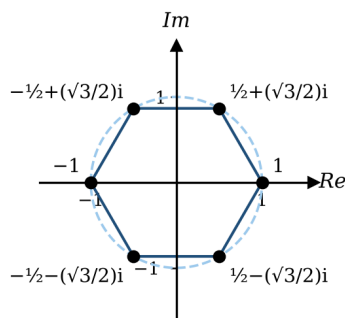
A1

$$k = 1, 2, 4, 5: z = \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

A1

$$\text{Answer: } z = \pm 1, \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

4 marks



11 (4 marks)

$$|z| = 16^{1/4} = 2$$

B1

$$\arg z = \left(\frac{2\pi}{3} + \frac{2k\pi}{4}\right) = \frac{\pi}{6} + \frac{k\pi}{2}$$

M1

$$\text{Arguments } \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}, \text{ so in } (-\pi, \pi]: \frac{\pi}{6}, \frac{2\pi}{3}, -\frac{5\pi}{6}, -\frac{\pi}{3}$$

A1

$$z = 2(\cos \theta + i \sin \theta) \text{ for each of these } \theta$$

A1

$$\text{Answer: } z = 2(\cos \theta + i \sin \theta), \theta = \frac{\pi}{6}, \frac{2\pi}{3}, -\frac{5\pi}{6}, -\frac{\pi}{3}$$

4 marks

12 (3 marks)

$$\text{Sum} = 2(1 + \omega + \omega^2)$$

M1

$$1 + \omega + \omega^2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$$

A1

$$\text{Product} = 8\omega^3 = 8(\cos 2\pi + i \sin 2\pi) = 8$$

A1

$$\text{Answer: The sum of the cube roots is 0 and their product is 8}$$

3 marks

13 (4 marks)

$$1 - i\sqrt{3} = 2\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)$$

B1

$$\text{A square root has modulus } \sqrt{2} \text{ and argument } -\frac{\pi}{6}$$

M1

$$\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right) = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$$

A1

$$\text{The roots are } \pm\left(\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i\right)$$

A1

$$\text{Answer: } \pm\left(\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i\right)$$

4 marks

14 (4 marks)

$$|-4 + 4i| = 4\sqrt{2} = 2^{5/2}, \arg = \frac{3\pi}{4}$$

B1

$$|z| = (2^{5/2})^{1/3} = 2^{5/6}$$

M1

$$\arg z = \left(\frac{3\pi}{4} + \frac{2k\pi}{3}\right) = \frac{\pi}{4} + \frac{2k\pi}{3}$$

M1

$$z = 2^{5/6}(\cos \theta + i \sin \theta), \theta = \frac{\pi}{4}, \frac{11\pi}{12}, -\frac{5\pi}{12}$$

A1

$$\text{Answer: } z = 2^{5/6}(\cos \theta + i \sin \theta), \theta = \frac{\pi}{4}, \frac{11\pi}{12}, -\frac{5\pi}{12}$$

4 marks

15 (4 marks)

$$|8 + 8\sqrt{3}i| = 16 \text{ and } \arg = \frac{\pi}{3}$$

B1

$$|z| = 16^{1/4} = 2$$

B1

$$\arg z = \frac{\pi}{12} + \frac{k\pi}{2}$$

M1

$$\theta = \frac{\pi}{12}, \frac{7\pi}{12}, -\frac{11\pi}{12}, -\frac{5\pi}{12}$$

A1

$$\text{Answer: } z = 2(\cos \theta + i \sin \theta), \theta = \frac{\pi}{12}, \frac{7\pi}{12}, -\frac{11\pi}{12}, -\frac{5\pi}{12}$$

4 marks

16 (4 marks)

$$z + 1 = 2(\cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3})$$

M1

$$z + 1 = 2, -1 + \sqrt{3}i, -1 - \sqrt{3}i$$

A1

$$z = 1$$

A1

$$z = -2 + \sqrt{3}i \text{ or } z = -2 - \sqrt{3}i$$

A1

$$\text{Answer: } z = 1, -2 + \sqrt{3}i, -2 - \sqrt{3}i$$

4 marks

17 (4 marks)

$$\sqrt{3} + i = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$$

B1

$$(\sqrt{3} + i)^3 = 8(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = 8i$$

M1 A1

$$z^2 = 8i, \text{ so } z = \pm 2\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) = \pm(2 + 2i)$$

A1

$$\text{Answer: } z = 2 + 2i \text{ or } z = -2 - 2i$$

4 marks

18 (5 marks)

$$(a) z^4 = -4 = 4(\cos \pi + i \sin \pi), \text{ so } |z| = \sqrt{2} \text{ and } \arg z = \frac{\pi}{4} + \frac{k\pi}{2}$$

M1 A1

$$z = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) = 1 + i, \text{ and similarly the others}$$

A1

$$\text{Answer: } z = 1 + i, -1 + i, -1 - i, 1 - i$$

3 marks

$$(b) (z - 1 - i)(z - 1 + i) = z^2 - 2z + 2 \text{ and } (z + 1 - i)(z + 1 + i) = z^2 + 2z + 2$$

M1

$$z^4 + 4 = (z^2 - 2z + 2)(z^2 + 2z + 2)$$

A1

$$\text{Answer: } (z^2 - 2z + 2)(z^2 + 2z + 2)$$

2 marks

19 (4 marks)

$$z = 2(\cos \frac{k\pi}{3} + i \sin \frac{k\pi}{3}), k = 0, \dots, 5$$

M1 A1

$$\text{Positive imaginary part: } k = 1 \text{ and } k = 2, \text{ giving } 1 + \sqrt{3}i \text{ and } -1 + \sqrt{3}i$$

M1

$$\text{Sum} = 2\sqrt{3}i$$

A1

$$\text{Answer: } 2\sqrt{3}i$$

4 marks

20 (5 marks)

The vertices are at 90° intervals about O, so each is obtained by multiplying by i

M1

$$i(2 + i) = -1 + 2i$$

A1

$$i(-1 + 2i) = -2 - i \text{ and } i(-2 - i) = 1 - 2i$$

A1

$$c = (2 + i)^4 = (3 + 4i)^2$$

M1

$$= -7 + 24i$$

A1

$$\text{Answer: } -1 + 2i, -2 - i, 1 - 2i; c = -7 + 24i$$

5 marks

21 (7 marks)

(a) $32 = 32(\cos 2k\pi + i \sin 2k\pi)$

M1

$$z = 2\left(\cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$$

A1

Equivalently, arguments $0, \pm \frac{2\pi}{5}, \pm \frac{4\pi}{5}$

A1

Answer: $z = 2\left(\cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}\right)$

3 marks

(b) The roots are equally spaced on the circle $|z| = 2$: a regular pentagon

B1

Answer: Regular pentagon

1 mark

(c) The pentagon is five congruent triangles with two sides 2 and included angle $\frac{2\pi}{5}$

M1

$$\text{Area} = 5 \times \frac{1}{2} \times 2^2 \times \sin \frac{2\pi}{5}$$

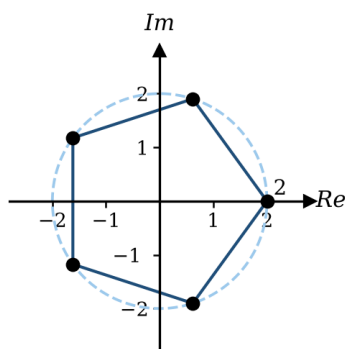
M1

$$= 9.51$$

A1

Answer: 9.51

3 marks

**22 (8 marks)**

$z = 1$ is not a solution, so divide by $(z - 1)^4$: $\left(\frac{z+1}{z-1}\right)^4 = 1$

M1 A1

$\frac{z+1}{z-1} = w$, where $w^4 = 1$, so $w = 1, i, -1$ or $-i$

M1

$w = 1$: $z + 1 = z - 1$ has no solution

A1

$$z = \frac{w+1}{w-1}$$

M1

$$w = i: z = \frac{i+1}{i-1} = \frac{(1+i)(-1-i)}{(-1+i)(-1-i)} = \frac{-2i}{2} = -i$$

A1

$$w = -1: z = 0$$

A1

$$w = -i: z = i. \text{ Solutions } z = 0, i, -i$$

A1

Answer: $z = 0, i, -i$

8 marks

23 (7 marks)

(a) Only one of the three cube roots was found; the argument $\pi + 2k\pi$ must be used

E1

A cubic equation has three roots, with arguments $\frac{\pi + 2k\pi}{3}$

E1

Answer: Missing roots: must use $\pi + 2k\pi$

2 marks

(b) Arguments $\frac{\pi}{3}, \pi, -\frac{\pi}{3}$

M1

$$z = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) = 1 + \sqrt{3}i, z = -2$$

A1

$$z = 1 - \sqrt{3}i$$

A1

Answer: $z = -2, 1 + \sqrt{3}i, 1 - \sqrt{3}i$

3 marks

(c) $(1 + \sqrt{3}i) + (1 - \sqrt{3}i) + (-2)$

M1

$$= 0$$

A1

Answer: The three roots sum to zero

2 marks

24 (6 marks)

(a) $(3 + 3i)^2 = 18i$

M1

$w = (18i)^2 = -324$

A1

Answer: -324

2 marks

(b) The roots are equally spaced at $\frac{\pi}{2}$ intervals: multiply by i repeatedly

M1

$-3 + 3i, -3 - 3i, 3 - 3i$

A1

Answer: -3 + 3i, -3 - 3i, 3 - 3i

2 marks

(c) Side length = $|(3 + 3i) - (-3 + 3i)| = 6$

M1

Area = 36

A1

Answer: 36

2 marks

25 (7 marks)

$(z^3 + 8)(z^3 - 1) = 0$

M1 A1

$z^3 = 1: z = \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}$, so arguments $0, \pm \frac{2\pi}{3}$, modulus 1

M1 A1

$z^3 = -8: |z| = 2$ and $\arg z = \frac{\pi + 2k\pi}{3}$

M1

Arguments $\frac{\pi}{3}, \pi, -\frac{\pi}{3}$

A1

Roots: $1, \cos(\pm \frac{2\pi}{3}) + i \sin(\pm \frac{2\pi}{3}), 2(\cos(\pm \frac{\pi}{3}) + i \sin(\pm \frac{\pi}{3})), -2$

A1

Answer: $1, \cos \frac{2\pi}{3} \pm i \sin \frac{2\pi}{3}, 2(\cos \frac{\pi}{3} \pm i \sin \frac{\pi}{3}), -2$

7 marks

26 (6 marks)

$|1 - i| = \sqrt{2}$ and $\arg(1 - i) = -\frac{\pi}{4}$

B1

$|z| = (\sqrt{2})^{1/3} = 2^{1/6}$

M1

$\arg z = (-\frac{\pi}{4} + \frac{2k\pi}{3})$

M1

Arguments $-\frac{\pi}{12}, \frac{7\pi}{12}, -\frac{3\pi}{4}$

A1

Smallest argument $-\frac{3\pi}{4}: z = 2^{1/6}(\cos(-\frac{3\pi}{4}) + i \sin(-\frac{3\pi}{4}))$

M1

$= -0.794 - 0.794i$

A1

Answer: $2^{1/6}(\cos(-\frac{\pi}{12}) + i \sin(-\frac{\pi}{12})), 2^{1/6}(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12}), 2^{1/6}(\cos(-\frac{3\pi}{4}) + i \sin(-\frac{3\pi}{4})); -0.794 - 0.794i$

6 marks

27 (8 marks)**(a)** ω is a root of $z^5 - 1 = 0, \omega \neq 1$

B1

$z^5 - 1 = (z - 1)(1 + z + z^2 + z^3 + z^4)$

M1

So $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

A1

Answer: $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

3 marks

(b) ω^3 and ω^4 are the conjugates of ω^2 and ω , so the real parts satisfy $1 + 2\cos \frac{2\pi}{5} + 2\cos \frac{4\pi}{5} = 0$

M1

$\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$

A1

Answer: $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$

2 marks

(c) Let $c = \cos \frac{2\pi}{5}: c + 2c^2 - 1 = -\frac{1}{2}$, so $4c^2 + 2c - 1 = 0$

M1 A1

$c = \frac{-1 + \sqrt{5}}{4}$ (the positive root, as $\frac{2\pi}{5}$ is acute)

A1

Answer: $\frac{\sqrt{5} - 1}{4}$

3 marks

28 (9 marks)

(a) $-64 = 64(\cos \pi + i \sin \pi)$

M1

$$z = 2\left(\cos \frac{\pi + 2k\pi}{6} + i \sin \frac{\pi + 2k\pi}{6}\right)$$

A1

$$= 2\left(\cos\left(\frac{\pi}{6} + \frac{k\pi}{3}\right) + i \sin\left(\frac{\pi}{6} + \frac{k\pi}{3}\right)\right), k = 0, \dots, 5$$

A1

Answer: $z = 2\left(\cos\left(\frac{\pi}{6} + \frac{k\pi}{3}\right) + i \sin\left(\frac{\pi}{6} + \frac{k\pi}{3}\right)\right)$

3 marks

(b) The roots lie on $|z| = 2$ with consecutive arguments differing by $\frac{\pi}{3}$

M1

Six equilateral triangles of side 2

A1

$$\text{Area} = 6 \times \frac{\sqrt{3}}{4} \times 2^2 = 6\sqrt{3}$$

A1

Answer: $6\sqrt{3}$

3 marks

(c) The sum of the arguments is $6 \times \frac{\pi}{6} + \frac{\pi}{3}(0 + 1 + 2 + 3 + 4 + 5) = \pi + 5\pi = 6\pi$

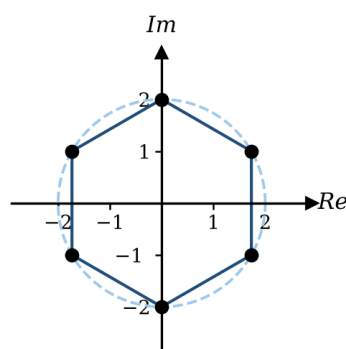
M1 A1

$$\text{Product} = 2^6(\cos 6\pi + i \sin 6\pi) = 64$$

A1

Answer: 64

3 marks



29 (8 marks)

(a) $z = \cos \frac{k\pi}{3} + i \sin \frac{k\pi}{3}, k = 0, \dots, 5$

M1

$$z = \pm 1, \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

A1

Answer: $z = \pm 1, \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

2 marks

(b) Complex roots occur in conjugate pairs: $(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i)(z - \frac{1}{2} + \frac{\sqrt{3}}{2}i) = z^2 - z + 1$

M1 A1

$$z^6 - 1 = (z - 1)(z + 1)(z^2 - z + 1)(z^2 + z + 1)$$

A1

Answer: $(z - 1)(z + 1)(z^2 - z + 1)(z^2 + z + 1)$

3 marks

(c) $z^6 - 1 = (z^2 - 1)(z^4 + z^2 + 1)$

M1

$$\text{So } z^4 + z^2 + 1 = (z^2 - z + 1)(z^2 + z + 1)$$

A1

$$z = \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

A1

Answer: $z = \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

3 marks

30 (9 marks)

(a) $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$

B1

$$w = \cos\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right) + i \sin\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right)$$

M1

$$w = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}, \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}, \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)$$

A1

Answer: $w = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}, \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}, \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)$

3 marks

(b) $w(z + 1) = z - 1$, so $wz + w = z - 1$

M1

$$z(1 - w) = -1 - w, \text{ giving } z = \frac{1 + w}{1 - w}$$

A1

Answer: If $w = \frac{z - 1}{z + 1}$ then $z = \frac{1 + w}{1 - w}$

2 marks

(c) $1 + w = e^{i\frac{\varphi}{2}} \times 2\cos\frac{\varphi}{2}$ and $1 - w = -e^{i\frac{\varphi}{2}} \times 2i \sin\frac{\varphi}{2}$

M1

$$\text{Quotient} = \frac{2\cos\frac{\varphi}{2}}{-2i \sin\frac{\varphi}{2}} = i \cot\frac{\varphi}{2}$$

A1

Answer: If $w = \cos \varphi + i \sin \varphi$ then $\frac{1 + w}{1 - w} = i \cot\frac{\varphi}{2}$

2 marks

(d) $z = i \cot\frac{\pi}{12} = (2 + \sqrt{3})i$ and $z = i \cot\frac{5\pi}{12} = (2 - \sqrt{3})i$

M1

$$z = i \cot(-\frac{\pi}{4}) = -i$$

A1

Answer: $z = (2 + \sqrt{3})i, (2 - \sqrt{3})i, -i$

2 marks

Roots of Unity

1 (2 marks)

The roots of $z^5 = 1$ all have modulus 1

B1

They are equally spaced, so consecutive roots differ in argument by $\frac{2\pi}{5}$

B1

Answer: Modulus 1; angle $\frac{2\pi}{5}$

2 marks

2 (2 marks)

$$i^2 = -1$$

M1

$$i^4 = (-1)^2 = 1$$

A1

Answer: $z = i$ is a fourth root of unity

2 marks

3 (2 marks)

The roots are $\cos\frac{2k\pi}{8} + i \sin\frac{2k\pi}{8}$

M1

$$\theta = 0, \pm\frac{\pi}{4}, \pm\frac{\pi}{2}, \pm\frac{3\pi}{4}, \pi$$

A1

Answer: $\theta = 0, \pm\frac{\pi}{4}, \pm\frac{\pi}{2}, \pm\frac{3\pi}{4}, \pi$

2 marks

4 (2 marks)

$$\omega^2 = \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}$$

M1

$$= \cos(-\frac{2\pi}{3}) + i \sin(-\frac{2\pi}{3})$$

A1

Answer: $\cos(-\frac{2\pi}{3}) + i \sin(-\frac{2\pi}{3})$

2 marks

5 (2 marks)

(a) $\omega^3 = 1$

B1

Answer: 1

1 mark

(b) The sum of the cube roots of unity is zero

B1

Answer: 0

1 mark

6 (2 marks)

The roots are $\cos\frac{2k\pi}{7} + i \sin\frac{2k\pi}{7}$

B1

The imaginary part $\sin\frac{2k\pi}{7}$ is zero only for $k = 0$, so only $z = 1$ is real

B1

Answer: One: $z = 1$

2 marks

7 (3 marks)

Roots have modulus 1: they lie on the unit circle

B1

Arguments $0, \pm \frac{2\pi}{5}, \pm \frac{4\pi}{5}$: equally spaced with one root at 1

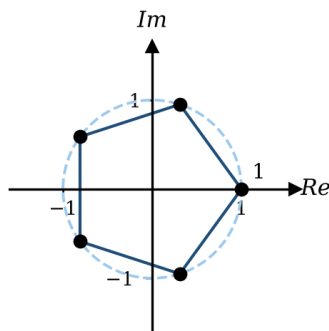
B1

They are the vertices of a regular pentagon

B1

Answer: Regular pentagon inscribed in the unit circle

3 marks

**8 (3 marks)**

$$z = \cos \frac{k\pi}{4} + i \sin \frac{k\pi}{4}, k = 0, \dots, 7$$

M1

$$k = 0, 2, 4, 6: z = 1, i, -1, -i$$

A1

$$k = 1, 3, 5, 7: z = \pm \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$$

A1

Answer: $z = \pm 1, \pm i, \pm \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$

3 marks

9 (4 marks)

$$\omega^3 = 1, \text{ so } \omega^{10} = \omega^9 \times \omega = \omega$$

M1 A1

$$\omega^{-4} = \omega^{-4} \times \omega^6 = \omega^2$$

A1

$$\omega^{100} = (\omega^3)^{33} \times \omega = \omega$$

A1

Answer: $\omega^{10} = \omega, \omega^{-4} = \omega^2, \omega^{100} = \omega$

4 marks

10 (3 marks)

$$(1 - \omega)(1 - \omega^2) = 1 - \omega - \omega^2 + \omega^3$$

M1

$$\omega^3 = 1 \text{ and } \omega + \omega^2 = -1$$

A1

$$= 1 + 1 + 1 = 3$$

A1

Answer: $(1 - \omega)(1 - \omega^2) = 3$

3 marks

11 (3 marks)

$$1 - \omega = 1 - \cos \frac{2\pi}{5} - i \sin \frac{2\pi}{5}$$

B1

$$|1 - \omega|^2 = (1 - \cos \frac{2\pi}{5})^2 + \sin^2 \frac{2\pi}{5} = 2 - 2\cos \frac{2\pi}{5}$$

M1

$$= 4 \sin^2 \frac{\pi}{5}, \text{ so } |1 - \omega| = 2 \sin \frac{\pi}{5}$$

A1

Answer: $|1 - \omega| = 2 \sin \frac{\pi}{5}$

3 marks

12 (3 marks)

By De Moivre, the left side is $\cos \frac{n\pi}{7} + i \sin \frac{n\pi}{7}$

M1

This equals 1 when $\frac{n\pi}{7}$ is a multiple of 2π , so n is a multiple of 14

A1

$$n = 14$$

A1

Answer: $n = 14$

3 marks

13 (3 marks)

$$z^3 - 1 = (z - 1)(z^2 + z + 1) = 0, \text{ so } z^3 = 1$$

M1 A1

$$z^{50} = (z^3)^{16} z^2 = z^2 \text{ and } z^{25} = (z^3)^8 z = z, \text{ so the expression is } z^2 + z + 1 = 0$$

A1

Answer: 0

3 marks

14 (4 marks)

$$\text{Geometric series: sum} = \frac{1 - \omega^n}{1 - \omega}$$

M1

$$\omega^n = \cos 2\pi + i \sin 2\pi = 1 \text{ and } \omega \neq 1, \text{ so the sum is } 0$$

A1

$$\text{Real part of } \omega^k \text{ is } \cos \frac{2k\pi}{n}$$

M1

Taking real parts of the sum gives the result

A1

$$\text{Answer: } 1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0; \cos 0 + \cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos \frac{2(n-1)\pi}{n} = 0$$

4 marks

15 (4 marks)

$$\text{Combine: } \frac{(1 + \omega^2) + (1 + \omega)}{(1 + \omega)(1 + \omega^2)} = \frac{2 + \omega + \omega^2}{1 + \omega + \omega^2 + \omega^3}$$

M1 A1

$$\text{Numerator: } 2 + (-1) = 1$$

A1

$$\text{Denominator: } 0 + 1 = 1, \text{ so the value is } 1$$

A1

Answer: 1

4 marks

16 (4 marks)

$$\text{Real part: } 1 + \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{8\pi}{7} + \cos \frac{10\pi}{7} + \cos \frac{12\pi}{7} = 0$$

M1 A1

$$\cos \frac{8\pi}{7} = \cos \frac{6\pi}{7}, \cos \frac{10\pi}{7} = \cos \frac{4\pi}{7}, \cos \frac{12\pi}{7} = \cos \frac{2\pi}{7}$$

M1

$$1 + 2(\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7}) = 0, \text{ giving the result}$$

A1

$$\text{Answer: } \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}$$

4 marks

17 (4 marks)

Roots 1, i, -1, -i: the vertices of a square on the unit circle

B1

$$\text{The diagonals have length 2, so the area} = \frac{1}{2} \times 2 \times 2 = 2$$

M1 A1

$$z^4 - 1 = (z - 1)(z + 1)(z^2 + 1)$$

A1

Answer: Square of area 2; $(z - 1)(z + 1)(z^2 + 1)$

4 marks

18 (4 marks)

$$\omega^4 = \omega^{-1} \text{ is the conjugate of } \omega$$

B1

$$\text{Sum of roots} = \omega + \omega^{-1} = 2\cos \frac{2\pi}{5}$$

M1

$$\text{Product of roots} = \omega^5 = 1, \text{ so the equation is } z^2 - 2\cos \frac{2\pi}{5} z + 1 = 0$$

A1

$$\text{Roots } \omega^2, \omega^3: z^2 - 2\cos \frac{4\pi}{5} z + 1 = 0$$

A1

$$\text{Answer: } z^2 - 2\cos \frac{4\pi}{5} z + 1 = 0$$

4 marks

19 (5 marks)

$$\text{(a) } z = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

M1

$$\cos \frac{2\pi}{3} = -\frac{1}{2}, \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \text{ so the roots are } \omega \text{ and } \omega^2 = \bar{\omega}$$

A1

$$\text{Answer: The roots of } z^2 + z + 1 = 0 \text{ are } \omega \text{ and } \omega^2, \text{ where } \omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$$

2 marks

$$\text{(b) } z^3 = \omega \text{ or } z^3 = \omega^2, \text{ i.e. } z^3 = \cos(\pm \frac{2\pi}{3}) + i \sin(\pm \frac{2\pi}{3})$$

M1

$$\arg z = \pm \frac{2\pi}{9} + \frac{2k\pi}{3} \quad \text{M1}$$

$$\theta = \pm \frac{2\pi}{9}, \pm \frac{4\pi}{9}, \pm \frac{8\pi}{9} \quad \text{A1}$$

Answer: $\theta = \pm \frac{2\pi}{9}, \pm \frac{4\pi}{9}, \pm \frac{8\pi}{9}$ 3 marks

20 (5 marks)

$$|a + b\omega|^2 = (a + b\omega)(a + b\bar{\omega}) \quad \text{M1}$$

$$= a^2 + ab(\omega + \bar{\omega}) + b^2\omega\bar{\omega} \quad \text{M1}$$

$$\omega + \bar{\omega} = 2\cos\frac{2\pi}{3} = -1 \text{ and } \omega\bar{\omega} = 1 \quad \text{A1}$$

$$|a + b\omega|^2 = a^2 - ab + b^2 \quad \text{A1}$$

$$|3 + 2\omega|^2 = 9 - 6 + 4 = 7, \text{ so } |3 + 2\omega| = \sqrt{7} \quad \text{A1}$$

Answer: $|3 + 2\omega| = \sqrt{7}$ 5 marks

21 (10 marks)

(a) $z^n - 1 = (z - 1)(z - \omega) \dots (z - \omega^{n-1})$ since these are its n roots M1 A1

$z^n - 1 = (z - 1)(1 + z + \dots + z^{n-1})$, so dividing by $z - 1$ gives the first result A1

Put $z = 1$: $1 + 1 + \dots + 1 = n$ on the left M1

So $(1 - \omega)(1 - \omega^2) \dots (1 - \omega^{n-1}) = n$ A1

Answer: $1 + z + z^2 + \dots + z^{n-1} = (z - \omega)(z - \omega^2) \dots (z - \omega^{n-1})$; $(1 - \omega)(1 - \omega^2) \dots (1 - \omega^{n-1}) = n$ 5 marks

(b) $n = 4$: $\omega = i$, so $(1 - i)(1 - i^2)(1 - i^3) = (1 - i)(2)(1 + i)$ M1

$= 2 \times 2 = 4$ A1

Answer: $(1 - i)(1 - i^2)(1 - i^3) = 4$ for $n = 4$ 2 marks

(c) Product of the moduli: $2\sin\frac{\pi}{5} \times 2\sin\frac{2\pi}{5} \times 2\sin\frac{3\pi}{5} \times 2\sin\frac{4\pi}{5} = 5$ M1

$\sin\frac{3\pi}{5} = \sin\frac{2\pi}{5}$ and $\sin\frac{4\pi}{5} = \sin\frac{\pi}{5}$, so $16 \sin^2\frac{\pi}{5} \sin^2\frac{2\pi}{5} = 5$ M1

$\sin\frac{\pi}{5} \sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$ A1

Answer: $\sin\frac{\pi}{5} \sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$ 3 marks

22 (6 marks)

(a) The sum of all five fifth roots of unity is 0 E1

The student has left out ω^4 E1

Answer: ω^4 is missing 2 marks

(b) $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ M1

So $1 + \omega + \omega^2 + \omega^3 = -\omega^4$ A1

Answer: $-\omega^4$ 2 marks

(c) $|\omega^4| = |\omega|^4$ M1

$= 1$ A1

Answer: 1 2 marks

23 (8 marks)

$$(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) = x^2 + \omega^2 xy + \omega xz + \omega xy + \omega^3 y^2 + \omega^2 yz + \omega^2 xz + \omega^4 yz + \omega^3 z^2 \quad \text{M1 A1}$$

Using $\omega^3 = 1$, $\omega^4 = \omega$ and $\omega + \omega^2 = -1$: $= x^2 + y^2 + z^2 - xy - xz - yz$ M1 A1

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + y^3 + z^3 - 3xyz \quad \text{M1 A1}$$

With $x, y, z = 1, 2, 3$: the second and third factors are conjugates, so the product $= |1 + 2\omega + 3\omega^2|^2$ M1

$1 + 2\omega + 3\omega^2 = -2 - \omega$, with $|-2 - \omega|^2 = 4 + 2(-1) + 1 = 3$, so the expression is $6 \times 3 = 18$ A1

Answer: 18 8 marks

24 (6 marks)

If r is a multiple of n , $\omega^r = \cos 2m\pi + i \sin 2m\pi = 1$ ($r = mn$) M1

Each term $\omega^{kr} = 1$, so the sum is n A1

Otherwise $\omega^r \neq 1$, and the sum is a geometric series with ratio ω^r M1

$$\text{Sum} = \frac{1 - (\omega^r)^n}{1 - \omega^r}$$
A1

$$(\omega^r)^n = (\omega^n)^r = 1$$
M1

So the sum is 0 A1

Answer: $\sum_{k=0}^{n-1} \omega^{kr} = n$ when r is a multiple of n , and 0 otherwise 6 marks

25 (6 marks)

(a) $V(1 + \omega + \omega^2)$ M1

$= 0$, as $1 + \omega + \omega^2 = 0$ A1

Answer: $V_0 + V_1 + V_2 = 0$ 2 marks

(b) $|V_1 - V_0| = V|\omega - 1|$ M1

$|\omega - 1|^2 = \left(-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 3$, so the voltage is $\sqrt{3}V$ A1

Answer: $|V_1 - V_0| = \sqrt{3}V$ 2 marks

(c) $\sqrt{3} \times 230$ M1

$= 398$ volts A1

Answer: 398 V 2 marks

26 (7 marks)

(a) Coefficient of z^3 : $-2c_1 - 2c_2 = 1$ M1 A1

Coefficient of z^2 : $1 + 4c_1c_2 + 1 = 1$ M1

So $c_1 + c_2 = -\frac{1}{2}$ and $c_1c_2 = -\frac{1}{4}$ A1

Answer: $c_1 + c_2 = -\frac{1}{2}$ and $c_1c_2 = -\frac{1}{4}$ 4 marks

(b) c_1 and c_2 are the roots of $x^2 + \frac{1}{2}x - \frac{1}{4} = 0$, i.e. $4x^2 + 2x - 1 = 0$ M1

$x = \frac{-1 \pm \sqrt{5}}{4}$ A1

$c_1 > 0 > c_2$: $\cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$, $\cos \frac{4\pi}{5} = -\frac{\sqrt{5} + 1}{4}$ A1

Answer: $\cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$, $\cos \frac{4\pi}{5} = -\frac{1 + \sqrt{5}}{4}$ 3 marks

27 (9 marks)

(a) $\omega + \omega^2 + \dots + \omega^6 = -1$, because $1 + \omega + \dots + \omega^6 = 0$ M1

$\alpha + \beta$ contains exactly these six terms: $\alpha + \beta = -1$ A1

Answer: $\alpha + \beta = -1$ 2 marks

(b) $\alpha\beta = \omega^4 + \omega^6 + \omega^7 + \omega^5 + \omega^7 + \omega^8 + \omega^7 + \omega^9 + \omega^{10}$ M1

Using $\omega^7 = 1$: ω^7 appears three times, $\omega^8 = \omega$, $\omega^9 = \omega^2$, $\omega^{10} = \omega^3$ M1

So $\alpha\beta = 3 + (\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6)$ A1

$\alpha\beta = 3 - 1 = 2$ A1

Answer: $\alpha\beta = 2$ 4 marks

(c) α and β are the roots of $x^2 + x + 2 = 0$ M1

$x = \frac{-1 \pm \sqrt{-7}}{2}$ A1

$\alpha = -\frac{1}{2} + \frac{\sqrt{7}}{2}i$, $\beta = -\frac{1}{2} - \frac{\sqrt{7}}{2}i$ A1

Answer: $\alpha = -\frac{1}{2} + \frac{\sqrt{7}}{2}i$, $\beta = -\frac{1}{2} - \frac{\sqrt{7}}{2}i$

3 marks

28 (8 marks)

(a) $|1 - \omega^k|^2 = (1 - \cos \frac{2k\pi}{n})^2 + \sin^2 \frac{2k\pi}{n} = 2 - 2\cos \frac{2k\pi}{n}$

M1 A1

$= 4 \sin^2 \frac{k\pi}{n}$, and $\sin \frac{k\pi}{n} > 0$ for these k

A1

Answer: $|1 - \omega^k| = 2 \sin \frac{k\pi}{n}$ for $k = 1, 2, \dots, n-1$

3 marks

(b) Take the modulus of the given product: $\sum_{k=1}^{n-1} |1 - \omega^k| = n$

M1

$2^{n-1} \sin \frac{\pi}{n} \dots \sin \frac{(n-1)\pi}{n} = n$

A1

Divide by 2^{n-1}

A1

Answer: $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \dots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$

3 marks

(c) $\sin \frac{\pi}{6} \sin \frac{\pi}{3} \sin \frac{\pi}{2} \sin \frac{2\pi}{3} \sin \frac{5\pi}{6} = \frac{1}{2} \times \frac{\sqrt{3}}{2} \times 1 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{3}{16}$

M1

$\frac{6}{2^5} = \frac{3}{16}$

A1

Answer: Both equal $\frac{3}{16}$

2 marks

29 (7 marks)

(a) $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$

M1

The nine ninth roots ω^k , $k = 0, \dots, 8$, satisfy $z^9 = 1$

M1

$k = 0, 3, 6$ satisfy $z^3 = 1$, so the other six are roots of $z^6 + z^3 + 1 = 0$

A1

Answer: ω^k , for $k = 1, 2, 4, 5, 7, 8$, are the roots of $z^6 + z^3 + 1 = 0$

3 marks

(b) The sum of the roots of $z^6 + z^3 + 1 = 0$ is 0, as the z^5 coefficient is zero

M1 A1

Real parts: $2(\cos \frac{2\pi}{9} + \cos \frac{4\pi}{9} + \cos \frac{8\pi}{9} + \dots)$ with $\cos \frac{10\pi}{9} = \cos \frac{8\pi}{9}$, $\cos \frac{14\pi}{9} = \cos \frac{4\pi}{9}$, $\cos \frac{16\pi}{9} =$

M1

$\cos \frac{2\pi}{9}$

So $2(\cos \frac{2\pi}{9} + \cos \frac{4\pi}{9} + \cos \frac{8\pi}{9}) = 0$ and the result follows

A1

Answer: $\cos \frac{2\pi}{9} + \cos \frac{4\pi}{9} + \cos \frac{8\pi}{9} = 0$

4 marks

30 (8 marks)

(a) $|\omega| = 1$, so $\omega\bar{\omega} = 1$

B1

$\frac{1}{1 - \omega} = \frac{1 - \bar{\omega}}{(1 - \omega)(1 - \bar{\omega})}$

M1

Denominator: $1 - \omega - \bar{\omega} + 1 = 2 - (\omega + \bar{\omega})$

A1

Numerator real part: $1 - \text{Re}(\omega)$, denominator $= 2(1 - \text{Re}(\omega))$, so the real part is $\frac{1}{2}$

A1

Answer: The real part of $\frac{1}{1 - \omega}$ is $\frac{1}{2}$

4 marks

(b) The terms for ω^k and ω^{n-k} are conjugates, so the imaginary parts cancel

M1 A1

Each of the $n - 1$ terms has real part $\frac{1}{2}$

M1

Sum $= \frac{n-1}{2}$

A1

Answer: $\frac{n-1}{2}$

4 marks

Geometrical Transformations Using Complex Numbers

1 (2 marks)

$$w = (1 + i) + 3 - 2i = 4 - i$$

B1

T is a translation by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

B1

Answer: $4 - i$; translation by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

2 marks

2 (2 marks)

$$w = 2(3 - i) = 6 - 2i$$

B1

An enlargement with scale factor 2 and centre the origin

B1

Answer: $6 - 2i$; enlargement, scale factor 2, centre O

2 marks

3 (2 marks)

$$w = i(2 + 3i) = -3 + 2i$$

B1

A rotation through $\frac{\pi}{2}$ anticlockwise about the origin

B1

Answer: $-3 + 2i$; rotation $\frac{\pi}{2}$ anticlockwise about O

2 marks

4 (2 marks)

$$w = 4 - 5i$$

B1

A reflection in the real axis

B1

Answer: $4 - 5i$; reflection in the real axis

2 marks

5 (2 marks)

$$a = 3(\cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2})) = -3i$$

M1

$$w = -3i(2 + i) = -6i + 3 = 3 - 6i$$

A1

Answer: $a = -3i$; the image is $3 - 6i$

2 marks

6 (3 marks)

Image of 1 is $1 + i$

B1

Image of i is $i(1 + i) = -1 + i$

B1

$|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \frac{\pi}{4}$: enlargement scale factor $\sqrt{2}$ and rotation through $\frac{\pi}{4}$ anticlockwise

B1

Answer: $1 + i$ and $-1 + i$; enlargement $\sqrt{2}$ and rotation $\frac{\pi}{4}$

3 marks

7 (3 marks)

$$w - (1 - i) = 2z$$

M1

$$|w - (1 - i)| = 2|z| = 2$$

A1

Circle with centre $1 - i$ and radius 2

A1

Answer: $|w - 1 + i| = 2$

3 marks

8 (3 marks)

$$u = x + 1 \text{ and } v = y$$

B1

$$x = u - 1 \text{ and } y = v: v = 2(u - 1)$$

M1

$$v = 2u - 2$$

A1

Answer: $v = 2u - 2$

3 marks

9 (3 marks)

$$w = i(x + iy) = -y + ix, \text{ so } u = -y \text{ and } v = x$$

M1 A1

$$y = x \text{ becomes } -u = v, \text{ i.e. } v = -u$$

A1

Answer: $v = -u$

3 marks

10 (3 marks)

$$|2 + i| = \sqrt{5}, \text{ so the scale factor is } \sqrt{5}$$

B1

$$\arg(2 + i) = \tan^{-1}\left(\frac{1}{2}\right)$$

M1

Rotation through 0.464 radians anticlockwise

A1

Answer: Enlargement $\sqrt{5}$ and rotation 0.464 rad anticlockwise

3 marks

11 (4 marks)

$$(a) z = 3z - 4$$

M1

$$z = 2$$

A1

$$\text{Answer: } z = 2$$

2 marks

$$(b) z(1 - i) = 1$$

M1

$$z = \frac{1}{1 - i} = \frac{1 + i}{2}$$

A1

$$\text{Answer: } z = \frac{1 + i}{2}$$

2 marks

12 (3 marks)

$$\text{Rotation about } c: w - c = i(z - c) \text{ with } c = 1 + i$$

M1

$$w = iz + c(1 - i) = iz + (1 + i)(1 - i)$$

A1

$$= iz + 2$$

A1

Answer: T can be written as $w = iz + 2$

3 marks

13 (4 marks)

$$(1 + i)^2 = 2i, (2i)^2 = -4, (-3)^2 = 9$$

B1 B1

$$|w| = |z|^2$$

M1

$$|z| = 2 \text{ gives } |w| = 4$$

A1

Answer: 2i, -4, 9; $|w| = 4$

4 marks

14 (5 marks)

$$w = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

M1

$$u = \frac{x}{x^2 + y^2} \text{ and } v = -\frac{y}{x^2 + y^2}$$

A1

$$u^2 + v^2 = \frac{x^2 + y^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2}$$

M1

$$\text{For } x = 1: u = \frac{1}{1 + y^2}, \text{ so } u^2 + v^2 = u$$

A1

$u > 0$ so the origin is excluded: the image is the circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$, without the origin

A1

Answer: $u^2 + v^2 = u$ (origin excluded)

5 marks

15 (5 marks)For real z , $|z - i| = |z + i|$ ($z - i$ and $z + i$ are conjugates)

M1 A1

$$\text{So } |w| = 1$$

A1

$$z = 0: w = \frac{-i}{i} = -1$$

B1

$$z = 1: w = \frac{1 - i}{1 + i} = \frac{(1 - i)^2}{2} = -i$$

A1

Answer: $|w| = 1$; images -1 and $-i$

5 marks

16 (5 marks)

$$w = (x + iy)^2 = x^2 - y^2 + 2xyi$$

M1

$$u = x^2 - y^2, v = 2xy$$

A1

$$x = 1: u = 1 - y^2 \text{ and } v = 2y$$

A1

$$y = \frac{v}{2}, \text{ so } u = 1 - \frac{v^2}{4}$$

M1

$$u = 1 - \frac{v^2}{4}, \text{ a parabola}$$

A1

$$\text{Answer: } u = 1 - \frac{v^2}{4}$$

5 marks

17 (4 marks)

$$|w| = 2 \text{ gives } |z + 3| = 2|z - 1|$$

M1

$$(x + 3)^2 + y^2 = 4[(x - 1)^2 + y^2], \text{ where } z = x + iy$$

M1

$$3x^2 + 3y^2 - 14x - 5 = 0$$

A1

$$(x - \frac{7}{3})^2 + y^2 = \frac{64}{9}: \text{ a circle with centre } (\frac{7}{3}, 0) \text{ and radius } \frac{8}{3}$$

A1

$$\text{Answer: } (x - \frac{7}{3})^2 + y^2 = \frac{64}{9}: \text{ circle, centre } (\frac{7}{3}, 0), \text{ radius } \frac{8}{3}$$

4 marks

18 (5 marks)

$$0 \rightarrow 2, 1 \rightarrow 3 + i$$

B1 B1

$$1 + i \rightarrow (1 + i)^2 + 2 = 2 + 2i, i \rightarrow i(1 + i) + 2 = 1 + i$$

B1

$$|1 + i| = \sqrt{2} \text{ and } \arg(1 + i) = \frac{\pi}{4}$$

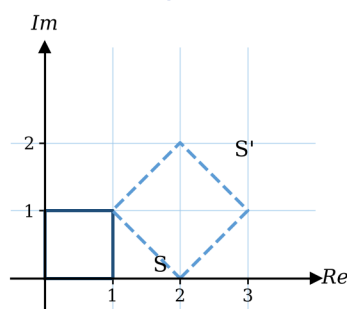
M1

$$\text{Enlargement scale factor } \sqrt{2}, \text{ rotation } \frac{\pi}{4} \text{ anticlockwise (both about O), then translation by } \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

A1

$$\text{Answer: Images } 2, 3 + i, 2 + 2i, 1 + i$$

5 marks

**19 (4 marks)**

$$a + b = 3 \text{ and } ai + b = 1 + 2i$$

M1

$$\text{Subtract: } a(1 - i) = 2 - 2i$$

M1

$$a = 2$$

A1

$$b = 1$$

A1

$$\text{Answer: } a = 2, b = 1$$

4 marks

20 (4 marks)

$$a = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$$

M1

$$= 1 + \sqrt{3}i$$

A1

$$w = (1 + \sqrt{3}i)(1 + i) = 1 + i + \sqrt{3}i - \sqrt{3}$$

M1

$$= (1 - \sqrt{3}) + (1 + \sqrt{3})i$$

A1

$$\text{Answer: } a = 1 + \sqrt{3}i, \text{ image } (1 - \sqrt{3}) + (1 + \sqrt{3})i$$

4 marks

21 (7 marks)

(a) $a(2 + i) = (2 - i)(2 + c + i)$ M1

$(2 - i)(2 + c + i) = 2(2 + c) + 1 + (2 - (2 + c))i = (2c + 5) - ci$ M1

Real parts: $2a = 2c + 5$. Imaginary parts: $a = -c$ A1

$-2c = 2c + 5$, so $c = -\frac{5}{4}$ and $a = \frac{5}{4}$ A1

Answer: $a = \frac{5}{4}, c = -\frac{5}{4}$ 4 marks

(b) $w = z$ gives $\frac{5z}{4z - 5} = z$ M1

$5z = z(4z - 5)$, so $z(4z - 10) = 0$ A1

$z = 0$ or $z = \frac{5}{2}$ A1

Answer: $z = 0$ and $z = \frac{5}{2}$ 3 marks

22 (7 marks)

(a) $z = iy$: $|iy - 1| = |iy + 1|$, so $|w| = 1$ M1 A1

$w = \frac{(-1 + iy)(1 - iy)}{1 + y^2} = \frac{y^2 - 1}{y^2 + 1} + \frac{2y}{y^2 + 1}i$ M1

 $w = 1$ would need $y^2 - 1 = y^2 + 1$, which is impossible: the point 1 is excluded A1**Answer:** $|w| = 1$, excluding $w = 1$ 4 marks

(b) For $|z| = 1$: $\bar{z} = \frac{1}{z}$, so $\bar{w} = \frac{\bar{z} - 1}{\bar{z} + 1} = \frac{\frac{1}{z} - 1}{\frac{1}{z} + 1} = \frac{1 - z}{1 + z} = -w$ M1 A1

So $w + \bar{w} = 0$, i.e. $\text{Re}(w) = 0$, so w lies on the imaginary axis ($z = -1$ is excluded because w is undefined there) A1**Answer:** $\text{Re}(w) = 0$, so the image lies on the imaginary axis 3 marks**23 (6 marks)**

(a) The modulus of z^2 is $|z|^2$, not $2|z|$ E1

 $(2|z|)$ is the modulus of $2z$ E1

Answer: $|z^2| = |z|^2$ 2 marks

(b) $|w| = |z|^2 = 9$ M1

The circle $|w| = 9$ A1

Answer: $|w| = 9$ 2 marks

(c) $w = z^2$ gives $z = \pm\sqrt{w}$, so z and $-z$ have the same image M1

Both z and $-z$ lie on $|z| = 3$ E1**Answer:** z and $-z$ have the same image 2 marks**24 (7 marks)**

(a) R is a sector of radius 2 and angle $\frac{\pi}{4}$ M1

$\text{Area} = \frac{1}{2} \times 4 \times \frac{\pi}{4} = \frac{\pi}{2}$ A1

Answer: $\frac{\pi}{2}$ 2 marks

(b) $|w| = |z|^2 \leq 4$ M1 A1

$\arg w = 2 \arg z$, so $0 \leq \arg w \leq \frac{\pi}{2}$ M1 A1

The image is a quarter-disc of radius 4 with area $\frac{1}{4}\pi \times 16 = 4\pi$ A1**Answer:** Quarter-disc $|w| \leq 4, 0 \leq \arg w \leq \frac{\pi}{2}$; area 4π 5 marks

25 (6 marks)

(a) $z - c = 2$

B1

$w - c = i \times 2 = 2i$

M1

$w = 3 + 4i$

A1

Answer: 3 + 4i

3 marks

(b) $z - c = (1 + 5i) - (3 + 2i) = -2 + 3i$

B1

Clockwise: multiply by $-i$: $-i(-2 + 3i) = 3 + 2i$

M1

$w = c + (3 + 2i) = 6 + 4i$

A1

Answer: 6 + 4i

3 marks

26 (7 marks)

(a) $|1 - i| = \sqrt{2}$, $\arg(1 - i) = -\frac{\pi}{4}$

B1

Enlargement scale factor $\sqrt{2}$, rotation $\frac{\pi}{4}$ clockwise (both about O), then translation by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

B1

Answer: Enlargement $\sqrt{2}$, rotation $\frac{\pi}{4}$ clockwise, translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

2 marks

(b) Centre i maps to $(1 - i)i + 2 = 3 + i$

M1 A1

Radius is multiplied by $\sqrt{2}$: circle $|w - 3 - i| = \sqrt{2}$

A1

Answer: $|w - 3 - i| = \sqrt{2}$

3 marks

(c) Image radius $\sqrt{2}$, so area $= \pi \times 2$

M1

 $= 2\pi$ (the area scale factor is 2)

A1

Answer: 2π

2 marks

27 (9 marks)

(a) $|z + 2| = |z - 4i|$

M1

$(x + 2)^2 + y^2 = x^2 + (y - 4)^2$

M1 A1

$4x + 4 = -8y + 16$, so $x + 2y = 3$

A1

Answer: $x + 2y = 3$

4 marks

(b) $1 + 2 = 3$, so the point is on the line

B1

$w = \frac{3 + i}{1 - 3i} = \frac{(3 + i)(1 + 3i)}{10} = i$, with $|w| = 1$

B1

Answer: $w = i$

2 marks

(c) w is real when $(z + 2)\overline{(z - 4i)}$ is real: $(x + 2 + iy)(x - (y - 4)i)$

M1

Imaginary part: $xy - (x + 2)(y - 4) = 4x - 2y + 8 = 0$

A1

$y = 2x + 4$ (excluding $z = 4i$)

A1

Answer: $y = 2x + 4$

3 marks

28 (9 marks)

(a) $|z - 2| = 2$ gives $(x - 2)^2 + y^2 = 4$, so $x^2 + y^2 = 4x$

M1 A1

$w = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}$, so $u = \frac{x}{x^2 + y^2}$

M1

$u = \frac{x}{4x} = \frac{1}{4}$

A1

Answer: $u = \frac{1}{4}$

4 marks

(b) $|z - 3| = 1$ gives $x^2 + y^2 - 6x + 8 = 0$

B1

$w = \frac{1}{z}$ gives $x = \frac{u}{u^2 + v^2}$ and $x^2 + y^2 = \frac{1}{u^2 + v^2}$

M1 A1

$1 - 6u + 8(u^2 + v^2) = 0$, so $u^2 + v^2 - \frac{3}{4}u + \frac{1}{8} = 0$

M1

$(u - \frac{3}{8})^2 + v^2 = \frac{1}{64}$

A1

Answer: $(u - \frac{3}{8})^2 + v^2 = \frac{1}{64}$

5 marks

29 (9 marks)

(a) $z = e^{i\frac{\pi}{3}}z + 2$, so $z(1 - e^{i\frac{\pi}{3}}) = 2$

M1

$$1 - e^{i\frac{\pi}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{-i\frac{\pi}{3}}$$

A1

$$z = 2e^{i\frac{\pi}{3}}$$

M1

$$= 1 + \sqrt{3}i$$

A1

Answer: $z = 1 + \sqrt{3}i$

4 marks

(b) $w - c = e^{i\frac{\pi}{3}}(z - c)$, with $c = 1 + \sqrt{3}i$

M1

A rotation through $\frac{\pi}{3}$ anticlockwise about $1 + \sqrt{3}i$

A1

Answer: Rotation through $\frac{\pi}{3}$ about $1 + \sqrt{3}i$

2 marks

(c) First: $w_1 = iz$. Second: $w = i(w_1 - 1) + 1$

M1

$$w = i(iz - 1) + 1 = -z + 1 - i$$

A1

Invariant: $2z = 1 - i$, so $z = \frac{1-i}{2}$: a half-turn about this point

A1

Answer: $z = \frac{1-i}{2}$

3 marks

30 (8 marks)

(a) $\frac{1}{z} = \frac{1}{r}(\cos \theta - i \sin \theta)$

B1

$$w = (r + \frac{1}{r})\cos \theta + i(r - \frac{1}{r})\sin \theta$$

M1 A1

$$\frac{u^2}{(r + \frac{1}{r})^2} + \frac{v^2}{(r - \frac{1}{r})^2} = \cos^2 \theta + \sin^2 \theta = 1, \text{ an ellipse}$$

A1

Answer: Ellipse

4 marks

(b) Semi-axes $a = r + \frac{1}{r}$ and $b = |r - \frac{1}{r}|$

B1

$a^2 - b^2 = 4$, so the foci are at distance 2 from the centre, on the real axis

A1

Answer: Foci ± 2

2 marks

(c) $r = 1$: $v = 0$ and $u = 2\cos \theta$

M1

The image is the line segment from -2 to 2 on the real axis

A1

Answer: The segment $-2 \leq u \leq 2$, $v = 0$

2 marks