

Introduction to Complex Numbers

- 1 Write each of the following in the form ki , where k is a real number or a surd in simplest form.
 - (a) $\sqrt{-49}$ [1]
 - (b) $\sqrt{-12}$ [1]
- 2 Write down the values of i^{10} and i^{23} . [2]
- 3 Write down the real part and the imaginary part of each number.
 - (a) $5 - 9i$ [1]
 - (b) $4i$ [1]
 - (c) 8 [1]
- 4 Solve the equation $3x^2 + 48 = 0$. [2]
- 5 The real numbers a and b satisfy $2a + 3bi = 8 - 12i$. Find a and b . [2]
- 6 State whether each of the following is real, purely imaginary or neither.
 - (a) i^2 [1]
 - (b) $3i^3$ [1]
 - (c) $2 + i^4$ [1]
- 7 Solve the equation $x^2 - 6x + 13 = 0$, giving your answers in the form $a + bi$. [3]
- 8 Simplify each of the following.
 - (a) i^{47} [2]
 - (b) $\frac{1}{i^3}$ [1]
- 9 The real numbers a and b satisfy $(a + b) + (a - b)i = 7 + 3i$. Find a and b . [3]
- 10 Solve $2x^2 + 6x + 7 = 0$, giving your answers in the form $a + bi$. [3]
- 11 A student writes " $\sqrt{-2} \times \sqrt{-8} = \sqrt{16} = 4$ ".
Explain why this is wrong and state the correct value of $\sqrt{-2} \times \sqrt{-8}$. [3]
- 12 Find the quadratic equation with real coefficients which has roots $2 + 3i$ and $2 - 3i$. [3]
- 13 Show that $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$ for every integer n . [2]
- 14 The quadratic equation $x^2 + px + q = 0$, where p and q are real, has a root $1 + 2i$. Find the values of p and q . [4]
- 15 Show that $x = 1$ is a root of $x^3 - 3x^2 + 4x - 2 = 0$, and hence find the other two roots. [5]
- 16 Find the exact value of $i + i^2 + i^3 + \dots + i^{50}$. [4]
- 17 Find the set of values of the real number k for which the equation $x^2 - 2kx + 3k + 4 = 0$ has no real roots. [4]
- 18 The real numbers a and b satisfy $a(1 + 2i) + b(3 - i) = 5 + 3i$. Find the value of a and the value of b . [4]
- 19 Find all the solutions of $x^4 - 16 = 0$ in the set of complex numbers. [4]

- 20** The equation $z^2 + 6z + k = 0$, where k is a real constant, has two non-real roots whose difference is $10i$. Find the roots and the value of k . [4]
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- 21** The cubic equation $z^3 + az^2 + bz + 10 = 0$, where a and b are real, has a root $2 + i$. Find the values of a and b , and the other two roots. [6]
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- 22** A student solves $x^2 + 4x + 13 = 0$ as follows:

$$x = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -4 \pm 6i$$
 (a) Identify the error in the student's working. [2]
 (b) Write down the correct solutions. [3]
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- 23** Prove that $(1 + i)^{4n} = (-4)^n$ for every positive integer n , and hence find the value of $(1 + i)^{20} + (1 - i)^{20}$. [6]
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- 24** The roots of $x^2 + 2x + 5 = 0$ are α and β .
 (a) Solve the equation to find α and β . [2]
 (b) Find a quadratic equation with real coefficients whose roots are α^2 and β^2 . [4]
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- 25** The equation $z^2 + (2 + i)z + (k + 3i) = 0$, where k is a real constant, has a real root. Find the value of k and the other root of the equation. [7]
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- 26** Let n be an integer. Show that $i^n + i^{-n}$ takes only the values 2 , 0 and -2 , and state, in terms of n , when each value occurs. [6]
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- 27** Consider the polynomial $x^4 + 4$.
 (a) Show that $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$. [2]
 (b) Hence solve $x^4 + 4 = 0$, giving the four roots in the form $a + bi$. [4]
 (c) Write $x^4 + 4$ as a product of four linear factors. [2]
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- 28** The equation $z^4 + az^3 + 6z^2 + bz + 5 = 0$, where a and b are real constants, has a root $1 - 2i$. Find the values of a and b and the other three roots. [8]
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- 29** Let p and q be real numbers.
 (a) Prove that if the complex number w is a root of $z^2 + pz + q = 0$ then its conjugate $\bar{w}$ is also a root. [4]
 (b) Hence show that a cubic equation with real coefficients always has at least one real root. [3]
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- 30** Find the exact value of $1 + 2i + 3i^2 + 4i^3 + \dots + 100i^{99}$. [6]

Complex Arithmetic

- 1** Given that $z = 3 + 4i$ and $w = 1 - 2i$, find
 (a) $z + w$ [1]
 (b) $z - w$ [1]
-
- 2** Expand and simplify $(2 + i)(3 - i)$, giving your answer in the form $a + bi$. [2]
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- 3** Find $(1 + 3i)^2$ in the form $a + bi$. [2]
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- 4** Express $\frac{10}{2 - i}$ in the form $a + bi$. [2]

- 5** Simplify $3(2 - i) - i(4 + 2i)$, giving your answer in the form $a + bi$. [2]
- 6** Show that $(3 - 2i)(3 + 2i)$ is a real number and state its value. [2]
- 7** Simplify $(1 + i)(2 - i)(3 + i)$, giving your answer in the form $a + bi$. [3]
- 8** Express $\frac{5 + i}{2 + 3i}$ in the form $a + bi$. [3]
- 9** Given that $z = 1 - 2i$, find z^2 and z^3 in the form $a + bi$. [3]
- 10** The real numbers x and y satisfy $(x + yi)(2 - i) = 5 - 5i$. Find x and y . [4]
- 11** Two components in an electrical circuit have complex impedances $Z_1 = (1 + 2i) \Omega$ and $Z_2 = (2 - i) \Omega$.
 (a) The components are connected in series, so the total impedance is $Z_1 + Z_2$. Find this total. [1]
 (b) The components are connected in parallel, so the total impedance is $\frac{Z_1 Z_2}{Z_1 + Z_2}$. Find this total in the form $a + bi$. [3]
- 12** Express $\frac{(1 + i)^2}{1 - i}$ in the form $a + bi$. [3]
- 13** Simplify $\frac{i}{1 + i} + \frac{1}{1 - i}$, giving your answer in the form $a + bi$. [3]
- 14** Find the real values of λ for which $\frac{\lambda + 6i}{2 + \lambda i}$ is a real number. [4]
- 15** The real number λ is such that $(3 + \lambda i)(2 - i) = p + qi$, where p and q are real. Find the value of λ for which the product is
 (a) a real number, [2]
 (b) purely imaginary. [2]
- 16** Find $(2 - i)^4$ in the form $a + bi$, showing your working. [4]
- 17** Show that $\frac{1 + 2i}{2 - i} = i$. [3]
- 18** The real numbers x and y satisfy $\frac{x}{1 + i} + \frac{y}{2 - i} = \frac{17}{4 + i}$. Find x and y . [4]
- 19** Given that $z = 3 - i$, find the real numbers p and q such that $z^2 = p + qz$. [4]
- 20** Given that $z = 1 + i$, find $\frac{1}{z^2} + \frac{1}{z^4}$ in the form $a + bi$. [4]
- 21** A student simplifies $\frac{3 + i}{2 - i}$ as follows:

$$\frac{3 + i}{2 - i} = \frac{(3 + i)(2 + i)}{(2 - i)(2 + i)} = \frac{6 + 3i + 2i + i^2}{4 - i^2} = \frac{5 + 5i}{3}$$

 (a) Identify the error in the student's working. [2]
 (b) Write down the correct value of $\frac{3 + i}{2 - i}$ and check it by multiplying by $2 - i$. [4]

- 22** Let a and b be real numbers, not both zero.
- (a) Show that $(a + bi)(a - bi) = a^2 + b^2$. [2]
- (b) Hence show that $\frac{1}{a + bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$. [2]
- (c) Hence find $\frac{1}{3 + 4i} - \frac{1}{3 - 4i}$. [2]
-
- 23** Two components with impedances $Z_1 = (2 + 3i) \Omega$ and $Z_2 = (4 - 3i) \Omega$ are connected in parallel. The total impedance Z satisfies $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$.
- (a) Show that $Z = \frac{17}{6} + i \Omega$. [4]
- (b) A voltage of 26 V is applied across the circuit. Use $I = \frac{V}{Z}$ to find the current I in the form $a + bi$. [3]
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- 24** The complex number z is given by $z = x + iy$, where x and y are real. Find, in terms of x and y , the real part and the imaginary part of
- (a) z^2 , [2]
- (b) z^3 , [3]
- (c) $\frac{1}{z}$. [3]
-
- 25** The real number a is such that $(1 + ai)^3 = p + qi$, where p and q are real.
- (a) Show that $p = 1 - 3a^2$ and find q in terms of a . [2]
- (b) Find the values of a for which $(1 + ai)^3$ is real. [2]
- (c) Find the values of a for which $(1 + ai)^3$ is purely imaginary. [2]
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- 26** The complex numbers z and w satisfy the simultaneous equations
 $z + w = 3 + i$,
 $iz + 2w = 4 + 3i$.
 Find z and w in the form $a + bi$. [6]
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- 27** The complex number $z = x + iy$, where x and y are real, is such that $z \neq 1$. The complex number w is given by $w = \frac{z + 2i}{z - 1}$.
- (a) Show that the real part of w is $\frac{x^2 - x + y^2 + 2y}{(x - 1)^2 + y^2}$ and find the imaginary part of w in a similar form. [5]
- (b) Hence find the equation of the line in which z lies when w is real. [2]
- (c) Find the value of z for which $w = i$. [3]
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- 28** Let $u = 1 + \sqrt{3}i$.
- (a) Show that $u^3 = -8$. [3]
- (b) Hence find the exact value of u^{12} . [2]
- (c) Find the smallest positive integer n for which u^n is a positive real number. Justify your answer. [3]
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- 29** The sum S is given by $S = 1 + (1 + i) + (1 + i)^2 + (1 + i)^3 + \dots + (1 + i)^7$.
- (a) Show that $(1 + i)^8 = 16$. [3]
- (b) Use the formula for the sum of a geometric series to find S in the form $a + bi$. [3]
- (c) Hence write down the sum of the real parts and the sum of the imaginary parts of the eight terms. [2]

- 30** This question shows how complex numbers can be used to write a product of two sums of two squares as another sum of two squares.
- (a) Show that $(2 + i)(2 - i) = 5$ and $(3 + 2i)(3 - 2i) = 13$. [2]
- (b) Expand $(2 + i)(3 + 2i)$ and $(2 - i)(3 - 2i)$, and hence show that $65 = 4^2 + 7^2$. [3]
- (c) Use the product $(2 + i)(3 - 2i)$ to write 65 as a different sum of two squares. [3]

Complex Conjugates

- 1** Write down the complex conjugate of each number.
- (a) $3 + 5i$ [1]
- (b) $-2 - i$ [1]
- (c) $7i$ [1]
- 2** The complex number z is $4 - 3i$.
- (a) Find $z + \bar{z}$. [1]
- (b) Find $z\bar{z}$. [1]
- 3** Find the complex conjugate of $(1 + i)(2 - i)$. [2]
- 4** By multiplying the numerator and the denominator by the complex conjugate of the denominator, write $\frac{1}{3 + i}$ in the form $a + bi$. [2]
- 5** State whether each statement is true or false for a complex number z .
- (a) If z is real then $\bar{z} = z$. [1]
- (b) If z is purely imaginary and non-zero then $\bar{z} = -z$. [1]
- 6** One root of the equation $x^2 - 6x + 25 = 0$ is $3 + 4i$.
- (a) Write down the other root. [1]
- (b) Hence write down the sum and the product of the roots. [1]
- 7** Let $z = a + bi$, where a and b are real.
- (a) Show that $z + \bar{z}$ is real. [2]
- (b) Show that $z\bar{z}$ is real and non-negative. [2]
- 8** Given that $z = \frac{2 + i}{1 - i}$, find $\bar{z}$ in the form $a + bi$. [3]
- 9** The complex number z is $3 - 2i$. Find $z + \bar{z}$ and $z\bar{z}$, and hence use the identity $z^2 + \bar{z}^2 = (z + \bar{z})^2 - 2z\bar{z}$ to find the value of $z^2 + \bar{z}^2$. [3]
- 10** A cubic equation with real coefficients has roots 2 and $1 + i$. Find the third root and write the equation in the form $x^3 + px^2 + qx + r = 0$. [4]
- 11** Find the complex number z that satisfies $z + 2\bar{z} = 6 + 3i$. [3]
- 12** The complex numbers z and w are $2 + i$ and $1 - 3i$. Find the conjugate of zw and the product $\overline{zw}$. What do you notice? [3]
- 13** Given that $z = 1 + 2i$, find $\frac{\bar{z}}{z}$ in the form $a + bi$. [3]
- 14** The cubic equation $z^3 - 5z^2 + 17z - 13 = 0$ has a root $2 + 3i$. Find the other two roots. [4]

- 15** Find the complex numbers z for which $z\bar{z} + 3z = 14 + 6i$. [5]
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- 16** Let n be a positive integer.
 (a) Show that $(3 + i)^n + (3 - i)^n$ is a real number. [2]
 (b) Hence find the value of $(3 + i)^4 + (3 - i)^4$. [2]
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- 17** The complex number w is not real. Show that $\frac{w}{\bar{w}} + \frac{\bar{w}}{w}$ is real. [4]
-
- 18** One root of a quadratic equation with real coefficients is $\frac{3-i}{1+i}$. Find the equation in the form $x^2 + px + q = 0$. [4]
-
- 19** Given that $z = 3 + 4i$, find the complex number w such that $w\bar{z} = 25$. [3]
-
- 20** Solve the equation $2z + i\bar{z} = 4 + 5i$. [4]
-
- 21** The equation $z^2 - (3 + i)z + 2 + 2i = 0$ has a root $1 + i$. A student states that, since the roots of a quadratic come in conjugate pairs, the other root must be $1 - i$.
 (a) Explain why the student's reasoning is not valid here. [1]
 (b) Show that $1 - i$ is not a root of the equation. [2]
 (c) Find the other root. [3]
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- 22** The complex number z satisfies $z\bar{z} = 20$ and $\frac{z}{\bar{z}} = \frac{3}{5} + \frac{4}{5}i$. Find z . [6]
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- 23** Find all the complex numbers z that satisfy $z^2 = \bar{z}$. [6]
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- 24** The quartic equation $z^4 - 6z^3 + 18z^2 - 30z + 25 = 0$ has a root $1 + 2i$. Solve the equation completely. [6]
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- 25** The complex numbers z and w are $z = a + bi$ and $w = c + di$, where a, b, c, d are real.
 (a) Show that $z\bar{w} + \bar{z}w = 2(ac + bd)$. [3]
 (b) Given that $z = 2 + 3i$ and $w = p + 4i$, with p real, and that $z\bar{w} + \bar{z}w = 0$, find p . [2]
 (c) Show that if $z\bar{w} + \bar{z}w = 0$ and $w \neq 0$, then $\frac{z}{w}$ is purely imaginary. [3]
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- 26** The roots of the equation $z^2 - 2z + 5 = 0$ are α and β .
 (a) Write down the value of $\alpha + \beta$ and of $\alpha\beta$, and hence find $\alpha^3 + \beta^3$ without finding α and β . [3]
 (b) Verify your answer by solving the equation and calculating α^3 directly. [3]
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- 27** The complex number z satisfies $z\bar{z} = 1$ and $z \neq -1$. Let $w = \frac{z-1}{z+1}$.
 (a) Show that $\bar{w} = -w$. [4]
 (b) Hence explain why w is purely imaginary or zero. [2]
 (c) Verify the result for $z = \frac{3}{5} + \frac{4}{5}i$ by finding w . [2]
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- 28** The complex numbers z and w satisfy the simultaneous equations
 $z + 2\bar{w} = 4 + 7i$,
 $2\bar{z} + w = 5 - 5i$.
 Find z and w . [7]
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- 29** The complex number $z = a + bi$ is non-zero, where a and b are real.
- (a) Show that $\frac{z}{\bar{z}} = \frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$. [3]
- (b) Hence show that $\frac{z}{\bar{z}}$ is real if and only if z is real or purely imaginary. [2]
- (c) Find all non-zero z for which $\frac{z}{\bar{z}}$ is purely imaginary, and describe their positions on an Argand diagram. [3]
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- 30** You may use the result that $\overline{zw} = \bar{z}\bar{w}$ for all complex numbers z and w .
- (a) Prove by induction that $\overline{z^n} = (\bar{z})^n$ for every positive integer n . [4]
- (b) Hence prove that if the polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, with real coefficients, has a root w , then $\bar{w}$ is also a root. [4]

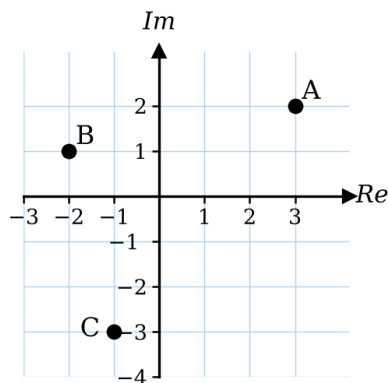
Solving Complex Equations

- 1** Solve the equation $z^2 + 4z + 8 = 0$. [2]
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- 2** Solve the equation $2z - 3i = 4 + z$. [2]
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- 3** Solve $(1 + i)z = 4$, giving z in the form $a + bi$. [2]
-
- 4** The real numbers x and y satisfy $(x + 2y) + (x - y)i = 5 + 2i$. Find x and y . [2]
-
- 5** Show that $z = 1 + i$ is a solution of the equation $z^2 - 2z + 2 = 0$. [2]
-
- 6** Solve $(z - 2)(z^2 + 9) = 0$ over the complex numbers. [3]
-
- 7** Find the two square roots of $5 + 12i$. [4]
-
- 8** Solve the equation $z^2 - (2 + i)z + 1 + i = 0$. [4]
-
- 9** Solve the equation $(z + 2i)(1 - i) = 4 + 2i$, giving z in the form $a + bi$. [3]
-
- 10** Solve $2z^2 + 2z + 5 = 0$, giving the roots in the form $a + bi$. [3]
-
- 11** The complex numbers z and w satisfy $z + w = 4$ and $zw = 5$. Find the two possible pairs of values of z and w . [3]
-
- 12** Show that $z = 2i$ is a root of $z^3 - 2z^2 + 4z - 8 = 0$ and find the other roots. [4]
-
- 13** Solve the equation $iz + 3 = 2z - i$. [3]
-
- 14** Consider the equation $z^2 - 4z + 13 = 0$.
- (a) Solve $z^2 - 4z + 13 = 0$. [2]
- (b) Hence solve $(w + 1)^2 - 4(w + 1) + 13 = 0$. [2]
-
- 15** The equation $z^2 - (5 + i)z + (m + 2i) = 0$, where m is a real constant, has a real root. Find the value of m and both roots of the equation. [5]
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- 16** Solve the equation $\frac{2z + i}{z - 1} = 3 + i$, giving z in the form $a + bi$. [4]
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- 17** Solve the equation $z^4 + 3z^2 - 4 = 0$ over the complex numbers. [4]

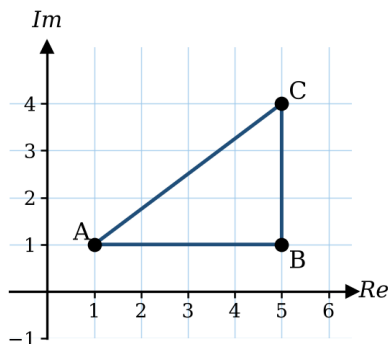
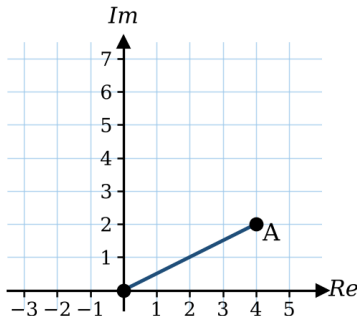
- 18** Solve the equation $z^2 - 2iz + 3 = 0$. [4]
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- 19** Solve the equation $z^3 + 2z^2 + 9z + 18 = 0$. [5]
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- 20** Find the complex number z that satisfies $(2 + i)z + (1 - 3i)\bar{z} = 10 - 5i$. [5]
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- 21** Solve the equation $z^2 - (5 + i)z + 6 + 3i = 0$. [7]
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- 22** Solve the equation $z^2 - 2z - 6 + 24i = 0$. [7]
-
- 23** The cubic equation $z^3 - (2 + 3i)z^2 - (3 - 7i)z + 6 - 2i = 0$ has exactly one real root. Find all three roots of the equation. [6]
-
- 24** Solve the equation $z^4 + 6z^2 + 25 = 0$, giving all four roots in the form $a + bi$. [7]
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- 25** Find all the complex numbers z that satisfy $z\bar{z} + 2(z - \bar{z}) = 13 + 8i$. [6]
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- 26** A student solves $z^2 = 5 + 12i$ as follows: "Let $z = a + bi$. Then $a^2 + b^2i^2 = 5 + 12i$, so $a^2 = 5$ and $b^2 = 12$."
 (a) Explain why the student's working is wrong. [2]
 (b) Solve the equation correctly. [4]
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- 27** Consider the equation $z^4 = -7 + 24i$.
 (a) Show that $(3 + 4i)^2 = -7 + 24i$. [2]
 (b) Hence write down the two possible values of z^2 for the equation $z^4 = -7 + 24i$. [1]
 (c) Solve the equation $z^4 = -7 + 24i$, giving the four roots in the form $a + bi$. [5]
-
- 28** Find all the complex numbers z that satisfy $z^2 + 2\bar{z} + 1 = 0$. [7]
-
- 29** Let $w = z^2 - z$.
 (a) Show that the equation $z^4 - 2z^3 - 3z^2 + 4z - 12 = 0$ can be written as $w^2 - 4w - 12 = 0$. [3]
 (b) Hence solve the equation $z^4 - 2z^3 - 3z^2 + 4z - 12 = 0$, giving non-real roots in the form $a + bi$. [5]
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- 30** $f(z) = z^4 - 4z^3 + 14z^2 - 36z + 45$.
 (a) Show that $z = 3i$ is a root of $f(z) = 0$. [3]
 (b) Solve the equation $f(z) = 0$ completely. [5]
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Argand Diagrams

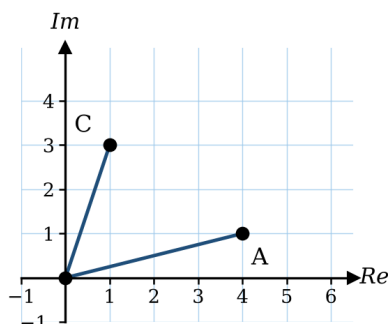
- 1 The points A, B and C on the Argand diagram represent the complex numbers a , b and c .



- (a) Write down a . [1]
 - (b) Write down b . [1]
 - (c) Write down c . [1]
- 2 Draw an Argand diagram and plot the points P, Q and R representing the complex numbers $2 + 3i$, $-1 + i$ and $4 - 2i$. [3]
- 3 The complex number $z = 5 - 2i$ is represented by the point P on an Argand diagram. State the coordinates of P, of the point representing $\bar{z}$ and of the point representing $-z$. [3]
- 4 The complex numbers z and w are $3 + 2i$ and $1 + 4i$. Find the coordinates of the point on the Argand diagram that represents $z + w$. [2]
- 5 The points A and B on an Argand diagram represent $1 + 2i$ and $4 + 6i$. Find the distance AB. [2]
- 6 The complex number $-3i$ is represented by the point P on an Argand diagram.
 - (a) State which axis the point P lies on. [1]
 - (b) Write down the complex number represented by the reflection of P in the real axis. [1]
- 7 The complex numbers z and w are $3 + i$ and $1 + 3i$. On an Argand diagram, O is the origin and the points A and B represent z and w .
 - (a) Draw an Argand diagram showing A, B, the point C representing $z + w$ and the point D representing $z - w$. [3]
 - (b) Describe fully the quadrilateral OACB, giving a reason. [3]
- 8 The points A and B on an Argand diagram represent $2 - 3i$ and $6 + 5i$.
 - (a) Find the complex number represented by the midpoint M of AB. [2]
 - (b) The point P lies on AB with $AP : PB = 1 : 3$. Find the complex number represented by P. [2]
- 9 The points A, B and C represent the complex numbers $1 + i$, $4 + 2i$ and $5 + 5i$. The quadrilateral ABCD is a parallelogram. Find the complex number represented by D. [3]
- 10 The complex number $z = 3 + 2i$ is represented by the point P on an Argand diagram.
 - (a) Find iz . [1]
 - (b) The point Q represents iz . Describe the geometrical relationship between OP and OQ. [2]
- 11 The points A, B and C on an Argand diagram represent $1 + 2i$, $4 + 5i$ and $7 + 8i$. Show that A, B and C are collinear. [3]

- 12** Solve the equation $z^2 - 4z + 13 = 0$ and show the roots on an Argand diagram. State the geometrical relationship between the two points. [4]
-
- 13** The points A, B and C on the Argand diagram represent $1 + i$, $5 + i$ and $5 + 4i$.
- 
- (a) Show that angle ABC is a right angle. [2]
 (b) Find the area of triangle ABC. [2]
-
- 14** OABC is a square on an Argand diagram, labelled anticlockwise, where O is the origin and A represents $4 + 2i$. Find the complex numbers represented by B and C.
- 
- [4]
-
- 15** The points A and B represent $2 + i$ and $6 + 3i$. Triangle ABC has a right angle at A and $AC = AB$, with C above the line AB (so that C is obtained from B by a 90° anticlockwise rotation about A). Find the complex number represented by C. [5]
-
- 16** The complex number $z = 3 + 2i$ is represented by A. The points B, C and D represent $\bar{z}$, $-z$ and $-\bar{z}$. Show that ABCD is a rectangle and find its area. [4]
-
- 17** The point P on an Argand diagram represents $z = 3 + 4i$. The point Q is the reflection of P in the line $y = x$.
 (a) Write down the complex number q represented by Q. [2]
 (b) Show that $q = iz$. [2]
-
- 18** The points A, B and C on an Argand diagram represent $1 + i$, $3 + 2i$ and $2 + 3i$. Show that triangle ABC is isosceles and find its area. [5]
-
- 19** The complex number $z = 3 + i$ is represented by the point A. The points B, C and D represent iz , $-z$ and $-iz$ respectively. Show that ABCD is a square and find its area. [5]
-
- 20** A triangle has vertices A, B and C representing $2 + i$, $5 + 7i$ and $-1 + 4i$.
 (a) Find the complex number represented by the centroid G of the triangle, where $g = \frac{1}{3}(a + b + c)$. [2]
 (b) The point D is such that G is the midpoint of AD. Find the complex number represented by D. [2]
-

- 21** OABC is a parallelogram on an Argand diagram, where O is the origin, A represents $4 + i$ and C represents $1 + 3i$.



- (a) Find the complex number represented by B. [2]
 - (b) Show that the diagonals OB and AC bisect each other. [2]
 - (c) Find the area of the parallelogram. [2]
-
- 22** A student makes the following statement about the points P and Q representing $z = 3 + 2i$ and $w = -2 + 3i$ on an Argand diagram:
 "w = iz, and multiplying by i rotates a point through 90° clockwise about the origin."
 (a) Show that $w = iz$. [2]
 (b) Explain why the student's statement about the rotation is wrong, and state the correct rotation. [3]
-
- 23** The points A, B, C and D on an Argand diagram represent $2 + i$, $5 + 2i$, $6 + 5i$ and $3 + 4i$. Show that ABCD is a parallelogram and decide, with reasons, whether it is a rectangle, a rhombus or a square. [7]
-
- 24** The point A represents $4 + 3i$ and the point P represents $1 + i$. The point B is the image of A under a rotation of 90° anticlockwise about P. Find the complex number represented by B. [5]
-
- 25** The roots of the equation $z^2 - 2z + 10 = 0$ are represented by the points A and B on an Argand diagram, and O is the origin.
 (a) Solve the equation. [2]
 (b) Show that triangle OAB is isosceles. [2]
 (c) Find the area of triangle OAB. [2]
-
- 26** The points A and B on an Argand diagram represent $1 + i$ and $4 + 3i$. The point C represents $p + (p + 2)i$, where p is a real number.
 (a) Find the value of p for which A, B and C are collinear. [3]
 (b) Find the value of p for which OC is perpendicular to AB, where O is the origin. [3]
-
- 27** ABC is an equilateral triangle on an Argand diagram with its centre at the origin O. The point A represents $4 + 2i$, and B is obtained from A by an anticlockwise rotation about O.
 (a) Find the complex numbers represented by B and C, giving exact values. [5]
 (b) The points D, E and F are the midpoints of the sides of triangle ABC. Find the exact area of triangle DEF. [3]
-
- 28** Points A, B, C and D on an Argand diagram represent the complex numbers a, b, c and d.
 (a) Prove that ABCD (in that order) is a parallelogram if and only if $a + c = b + d$. [3]
 (b) A, B and C represent $1 + 2i$, $5 + i$ and $7 + 4i$. Use part (a) to find d. [2]
 (c) Using part (a), prove that the midpoints of the four sides of any quadrilateral ABCD are the vertices of a parallelogram. [4]
-
- 29** ABCD is a square, labelled anticlockwise, on an Argand diagram. The opposite vertices A and C represent $1 + i$ and $7 + 5i$. Find the complex numbers represented by B and D, and verify that AB is perpendicular to BC and that $AB = BC$. [8]

- 30** The points A and B on an Argand diagram represent $z = a + bi$ and $w = c + di$, where a, b, c and d are real, and O is the origin.
- (a) Show that the imaginary part of $\bar{z}w$ is $ad - bc$, and hence prove that the area of triangle OAB is $\frac{1}{2}|\text{Im}(\bar{z}w)|$. [3]
- (b) Use this result to find the area of the triangle with vertices O, $3 + 2i$ and $-1 + 4i$. [2]
- (c) Find the real number k for which the points O, $2 + i$ and $k + 3i$ are collinear. [3]

Modulus and Argument

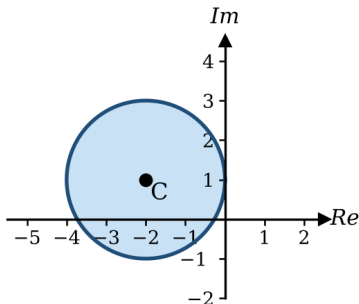
- 1** Find the modulus of each complex number.
- (a) $3 + 4i$ [1]
 (b) $5 - 12i$ [1]
- 2** Write down the modulus of each complex number.
- (a) -7 [1]
 (b) $4i$ [1]
 (c) $1 + i$ [1]
- 3** Find the principal argument of each complex number, giving your answer in radians as a multiple of π .
- (a) $1 + i$ [1]
 (b) $-1 + i$ [1]
 (c) $-1 - i$ [1]
- 4** Write down the principal argument of each complex number, in radians.
- (a) -5 [1]
 (b) $5i$ [1]
 (c) $-2i$ [1]
- 5** Find the argument of $\sqrt{3} + i$, giving your answer as a multiple of π . [2]
- 6** Given that $z = (1 + 2i)(3 - i)$, find $|z|$ in the form $k\sqrt{2}$. [2]
- 7** Find the principal argument of $-2 + 3i$ in radians, correct to 3 significant figures. [3]
- 8** Find the exact modulus and the principal argument of $-3 - 3\sqrt{3}i$. [4]
- 9** The complex number z has modulus 4 and argument $\frac{2\pi}{3}$. Write z in the form $a + bi$, giving exact values of a and b . [3]
- 10** The complex numbers z and w are $1 + \sqrt{3}i$ and $2 - 2i$. Find $|zw|$ and $|\frac{z}{w}|$, giving exact answers. [4]
- 11** $z = 2 + i$ and $zw = -5 + 5i$. Find w in the form $a + bi$, and find $|w|$ and $\arg w$, giving the argument in radians to 3 significant figures. [4]
- 12** Show that $|(2 + i)^3| = 5\sqrt{5}$. [3]
- 13** The complex number z has $|z| = 13$ and $\text{Re}(z) = 5$, and the imaginary part of z is negative. Find z , and find $\arg z$ correct to 3 significant figures. [4]
- 14** Given that $z = (-1 + i)^3$, find $|z|$ and $\arg z$. [4]

- 15** Prove that $|zw| = |z||w|$ for all complex numbers z and w . You may use the facts that $|u|^2 = u\bar{u}$ and that $\overline{zw} = \bar{z}\bar{w}$. [4]
-
- 16** The complex numbers z and w are $-3 + i$ and $4 + 2i$.
Find $\arg z - \arg w$, in radians correct to 3 significant figures, and hence show that $\arg\left(\frac{z}{w}\right) = \frac{3\pi}{4}$. [5]
-
- 17** The complex number z is $2 + 2i$. Find the modulus and the argument of each of the following.
(a) z^2 [2]
(b) $\frac{1}{z}$ [2]
(c) $\bar{z}$ [2]
-
- 18** In an alternating current circuit, two voltages are represented by the complex numbers $V_1 = (6 + 2i)$ volts and $V_2 = (-2 + 6i)$ volts. The total voltage is $V = V_1 + V_2$. The modulus of V is the amplitude of the total voltage and the argument of V is its phase angle.
Find the amplitude and the phase angle of the total voltage, giving the phase angle in radians to 3 significant figures. [5]
-
- 19** The complex number z is $3 - 4i$. Find $|z|$ and $\arg z$, and find, correct to 3 significant figures, the principal arguments of $-z$ and iz . [5]
-
- 20** The complex number z has modulus 3 and argument $\frac{\pi}{6}$, and w has modulus 2 and argument $\frac{\pi}{3}$. Find the modulus and argument of zw and of $\frac{z}{w}$, and hence write zw and $\frac{z}{w}$ in the form $a + bi$, with exact values. [6]
-
- 21** A student finds the argument of $-1 - i$ as follows: " $\tan \theta = \frac{-1}{-1} = 1$, so $\theta = \tan^{-1}1 = \frac{\pi}{4}$."
(a) Explain why the student's answer is wrong. [2]
(b) Write down the correct principal argument of $-1 - i$. [2]
(c) Find the principal argument of $-\sqrt{3} + i$. [2]
-
- 22** Let $z = a + bi$ and $w = c + di$, where a, b, c and d are real.
(a) Show that $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$. [4]
(b) Verify the result for $z = 3 + i$ and $w = 1 - 2i$. [2]
-
- 23** The complex number z satisfies $|z| = 5$. Given that z^2 is purely imaginary with a negative imaginary part, find the possible values of z in the form $a + bi$, giving exact values. [6]
-
- 24** The complex number z is given by $z = \frac{a + i}{1 + ai}$, where a is a real constant.
(a) Show that $|z| = 1$ for every real value of a . [2]
(b) Show that $z = \frac{2a + (1 - a^2)i}{1 + a^2}$. [2]
(c) Find the values of a for which z is real. [2]
(d) Find the value of a for which $\arg z = \frac{\pi}{4}$. [3]
-
- 25** The complex numbers z and w satisfy $|z| = |w|$ and $z \neq w$. Show that $\frac{z + w}{z - w}$ is purely imaginary or zero. [6]
-

- 26** The points A and B on an Argand diagram represent $a = 1$ and $b = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$. The point C represents $a + b$, and O is the origin.
- (a) Show that OACB is a rhombus. [2]
- (b) Explain why $\arg(a + b) = \frac{\pi}{8}$. [2]
- (c) Hence show that $\tan \frac{\pi}{8} = \sqrt{2} - 1$. [3]
-
- 27** Let z and w be complex numbers.
- (a) Prove that $|z + w| \leq |z| + |w|$. You may use the fact that $\operatorname{Re}(u) \leq |u|$ for every complex number u . [5]
- (b) Given that $|z| = 5$, use part (a) to find the greatest possible value of $|z - 3 + 4i|$, and state a value of z for which it is attained. [3]
-
- 28** The complex numbers z and w satisfy $|z| = |w| = 1$, and $zw \neq -1$.
- (a) Show that $\frac{z + w}{1 + zw}$ is real. [6]
- (b) Verify the result for $z = \frac{3 + 4i}{5}$ and $w = \frac{5 - 12i}{13}$. [2]
-
- 29** This question uses arguments of products of complex numbers.
- (a) Show that $(1 + i)(2 + i)(3 + i) = 10i$. [3]
- (b) Hence show that $\tan^{-1} 1 + \tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3}) = \frac{\pi}{2}$. [3]
- (c) By considering $(1 + 2i)(1 + 3i)$, show that $\tan^{-1} 2 + \tan^{-1} 3 = \frac{3\pi}{4}$. [2]
-
- 30** The complex numbers z and w satisfy $|z| = |w| = 2$ and $z + w = 2 + 2i$. Find all the possible pairs (z, w) . [7]
-

Complex Number Loci

- 1** Describe the locus of the point z on an Argand diagram given by each equation.
- (a) $|z| = 4$ [1]
- (b) $|z - 2i| = 3$ [1]
-
- 2** Write down the Cartesian equation of the locus $|z - 3| = 2$. [2]
-
- 3** Describe the locus of the point z given by $\arg z = \frac{\pi}{3}$. [2]
-
- 4** Describe the locus of the point z given by $|z - 2| = |z + 2|$. [2]
-
- 5** Describe the locus of z given by each equation.
- (a) $\operatorname{Re}(z) = 3$ [1]
- (b) $\operatorname{Im}(z) = -2$ [1]
-
- 6** Write down the centre and the radius of the circle given by each equation.
- (a) $|z - 1 - 2i| = 5$ [1]
- (b) $|z + 3i| = 2$ [1]
-
- 7** Sketch on an Argand diagram the locus of z given by $|z - 2 - i| = 3$. [3]
-
- 8** Sketch on an Argand diagram the locus of z given by $\arg(z + 1 - 2i) = \frac{3\pi}{4}$. [3]
-

- 9** Find the Cartesian equation of the locus of z given by $|z - 3| = |z - i|$. [4]
- 10** On an Argand diagram, shade the region of points z that satisfy $|z + 2| < |z - 2|$. [3]
- 11** Find the Cartesian equation of the locus of z given by $\arg(z - 1 + i) = \frac{\pi}{4}$, stating the range of values of x . [3]
- 12** The shaded region in the diagram, including its boundary, is a disc with centre C at the point $(-2, 1)$ and radius 2.
Write down an inequality of the form $|z - a| \leq r$ that describes the shaded region. [3]
- 
- 13** Find the complex numbers z that satisfy both $|z| = 5$ and $\operatorname{Re}(z) = -3$. [3]
- 14** Show that the locus of the point z given by $|z - 4| = 2|z - 1|$ is a circle, and state its centre and radius. [4]
- 15** The point P represents the complex number z on an Argand diagram, where z satisfies $z\bar{z} + 2(z + \bar{z}) - 5 = 0$. Show that the locus of P is a circle, and state its centre and radius. [4]
- 16** On a single Argand diagram sketch the locus $|z - 2| = 2$ and the locus $\arg z = \frac{\pi}{4}$. Find the complex number represented by the point where the two loci meet. [5]
- 17** On an Argand diagram, shade the region R defined by $1 \leq |z - 2i| \leq 3$, and find the exact area of R . [4]
- 18** On an Argand diagram, shade the region R defined by $|z| \leq 4$ and $-\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4}$, and find the exact area of R . [5]
- 19** Two lighthouses stand at the points A and B , represented on an Argand diagram by 0 and 8 (distances in kilometres). A boat at the point z is always the same distance from A as from B .
(a) Write down the locus of z in the form $|z - a| = |z - b|$ and find its Cartesian equation. [2]
(b) At a certain moment the boat is also exactly 5 km from A . Find the two possible positions of the boat as complex numbers. [3]
- 20** Find the complex number z that satisfies both $|z - 1| = |z - 3 - 4i|$ and $\arg z = \frac{\pi}{4}$. [4]
- 21** The half-line with equation $\arg z = \alpha$, where $\alpha > 0$, touches the circle with equation $|z - 4| = 2$.
(a) Find the value of α . [3]
(b) Find the complex number represented by the point where the half-line touches the circle, in the form $a + bi$. [3]

- 22** The region R on an Argand diagram is defined by $|z - 3| \leq 3$ and $|z| \leq |z - 3|$.
 (a) Sketch and shade R. [4]
 (b) Find the exact area of R. [4]
-
- 23** Find the complex number z that satisfies both $\arg(z - 3i) = \frac{\pi}{6}$ and $|z| = \sqrt{13}$, giving z in the form $a + bi$ with exact values. [6]
-
- 24** A student sketches the locus $|z - 1 + 2i| = 2$ as a circle with centre $(1, 2)$ and radius 2.
 (a) Explain the student's error. [2]
 (b) Write down the correct Cartesian equation of the locus. [2]
 (c) Find the greatest value of $\text{Im}(z)$ on the locus. [2]
-
- 25** The loci $|z - 1| = 2$ and $|z + 1| = 2$ meet at the points P and Q.
 (a) Find the complex numbers represented by P and Q. [3]
 (b) The centres of the two circles are A $(1, 0)$ and B $(-1, 0)$. Show that triangle ABP is equilateral. [3]
-
- 26** The point z moves on the locus $|z - 5 - 12i| = 4$.
 (a) Find the greatest and the least values of $|z|$. [3]
 (b) Find the greatest and the least values of $\arg z$, giving your answers in radians to 3 significant figures. [4]
-
- 27** The locus of the point z on an Argand diagram is given by $|z - 3| = 2|z + 3i|$.
 (a) Show that the locus is a circle, and find its centre and its radius. [6]
 (b) Find the greatest value of $|z|$ on the locus, in exact form. [2]
-
- 28** The region R on an Argand diagram is defined by $|z - 1| \leq 1$ and $|z - i| \leq 1$.
 (a) Sketch and shade R. [3]
 (b) Find the two points where the boundaries of the discs meet. [2]
 (c) Show that the area of R is $\frac{\pi}{2} - 1$. [3]
-
- 29** The complex number $z = x + iy$, where x and y are real, is such that $z \neq -1$.
 (a) Show that $\frac{z - 1}{z + 1} = \frac{x^2 + y^2 - 1 + 2yi}{(x + 1)^2 + y^2}$. [3]
 (b) Hence show that the locus $\arg\left(\frac{z - 1}{z + 1}\right) = \frac{\pi}{2}$ is part of a circle, and describe it. [3]
 (c) Find the equation of the locus $\arg\left(\frac{z - 1}{z + 1}\right) = \frac{\pi}{4}$, and describe it. [4]
-
- 30** The point z moves on the locus $|z - 5i| = 3$.
 (a) Sketch the locus. [2]
 (b) Find the greatest value of $\arg z$ in radians to 3 significant figures, and find the complex number z for which it occurs, in the form $a + bi$. [6]

Modulus-Argument Form

- 1** Write $2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$ in the form $a + bi$, giving exact values of a and b . [2]
-
- 2** Express $1 + i$ in modulus-argument form. [2]
-
- 3** Express -3 in modulus-argument form. [2]
-
- 4** Express $5i$ in modulus-argument form. [2]

- 5 Express $\sqrt{3} - i$ in modulus-argument form. [3]
- 6 $z = 4(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$ and $w = 3(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$. Write zw in modulus-argument form. [2]
- 7 Express $1 - \sqrt{3}i$ in modulus-argument form, with the argument in the interval $(-\pi, \pi]$. [3]
- 8 $z = 8(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$ and $w = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$. Find $\frac{z}{w}$ in the form $a + bi$. [3]
- 9 $z = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ and $w = 2\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$. Find zw in modulus-argument form and in the form $a + bi$, and check your answer by expanding $(1 + i)(2 + 2i)$. [4]
- 10 Express $z = 2(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) + 2(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$ in modulus-argument form. [3]
- 11 The complex number $z = 3(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5})$. Write each of the following in modulus-argument form.
 (a) $\bar{z}$ [1]
 (b) $\frac{1}{z}$ [2]
- 12 A student writes the complex number $2(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3})$ and says that its modulus is 2 and its argument is $\frac{\pi}{3}$. Explain the error and write the number correctly in modulus-argument form. [3]
- 13 Let $z = \cos \theta + i \sin \theta$. Show that $iz = \cos(\theta + \frac{\pi}{2}) + i \sin(\theta + \frac{\pi}{2})$, and hence describe the effect on the point z of multiplying by i . [3]
- 14 Express $\frac{1 + i}{\sqrt{3} - i}$ in modulus-argument form. [4]
- 15 Write $-3(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ in modulus-argument form with a positive modulus and the principal argument. [3]
- 16 z has modulus 2 and argument $\frac{3\pi}{4}$, and w has modulus 3 and argument $\frac{2\pi}{3}$. Write zw in modulus-argument form, with the argument in the interval $(-\pi, \pi]$. [4]
- 17 Show that for any real θ , $\frac{1}{\cos \theta + i \sin \theta} = \cos \theta - i \sin \theta$. [3]
- 18 Consider the product $(1 + i\sqrt{3})(1 + i)$.
 (a) Write $1 + i\sqrt{3}$ and $1 + i$ in modulus-argument form, and hence express $(1 + i\sqrt{3})(1 + i)$ in modulus-argument form. [3]
 (b) Expand $(1 + i\sqrt{3})(1 + i)$ and hence find exact values of $\cos \frac{7\pi}{12}$ and $\sin \frac{7\pi}{12}$. [3]
- 19 $z = 2(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8})$ and $w = 3(\cos \frac{3\pi}{8} + i \sin \frac{3\pi}{8})$. Find zw and $\frac{z}{w}$, giving each answer in the form $a + bi$ with exact values. [4]
- 20 The real number α satisfies $0 < \alpha < \frac{\pi}{2}$. Show that $1 + i \tan \alpha = \sec \alpha(\cos \alpha + i \sin \alpha)$, and state the modulus and the argument of $1 + i \tan \alpha$. [4]

- 21** The real number α satisfies $0 < \alpha < \pi$. Show that $1 - \cos \alpha + i \sin \alpha = 2 \sin \frac{\alpha}{2} (\cos(\frac{\pi}{2} - \frac{\alpha}{2}) + i \sin(\frac{\pi}{2} - \frac{\alpha}{2}))$, and state its modulus and argument. [6]
-
- 22** A student calculates the product of $z = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$ and $w = 3(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$ as follows:
 $zw = 6(\cos \frac{\pi}{3} \times \frac{\pi}{6} + i \sin \frac{\pi}{3} \times \frac{\pi}{6}) = 6(\cos \frac{\pi}{18} + i \sin \frac{\pi}{18})$.
 (a) Identify the error in the student's method. [2]
 (b) Find zw correctly, in modulus-argument form and in the form $a + bi$. [3]
 (c) Find $\frac{z}{w}$ in the form $a + bi$. [2]
-
- 23** Let $z = r(\cos \theta + i \sin \theta)$, where $r > 0$.
 (a) Show that $\frac{z}{z} = \cos 2\theta + i \sin 2\theta$. [3]
 (b) Hence, for $z = 3 + 4i$, find the exact value of $\cos 2\theta$, where $\theta = \arg z$. [3]
-
- 24** The points A and B on an Argand diagram represent the complex numbers 4 and $4(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$, and O is the origin. Show that triangle OAB is equilateral and write B in the form $a + bi$. [5]
-
- 25** Let $z = 1 + 2i$ and $w = 2 + i$.
 (a) Write z and w in modulus-argument form, giving the arguments to 3 significant figures. [3]
 (b) Find zw and hence show that $\tan^{-1} 2 + \tan^{-1}(\frac{1}{2}) = \frac{\pi}{2}$. [3]
-
- 26** Two alternating voltages are represented by the complex numbers $V_1 = 5(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$ and $V_2 = 12(\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3}))$ (in volts). Show that $V_2 = -12i(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$, and hence find the amplitude (modulus) and the phase angle (argument) of the total voltage $V_1 + V_2$, giving the phase angle in radians to 3 significant figures. [6]
-
- 27** Let θ be a real number with $-\pi < \theta < \pi$, and let $z = 1 + \cos \theta + i \sin \theta$.
 (a) Show that $z = 2\cos \frac{\theta}{2} (\cos \frac{\theta}{2} + i \sin \frac{\theta}{2})$, and state the modulus and argument of z . [4]
 (b) Hence show that $\frac{1 + \cos \theta + i \sin \theta}{1 + \cos \theta - i \sin \theta} = \cos \theta + i \sin \theta$. [2]
 (c) Use part (a) to write $1 + \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$ in the form $a + bi$. [2]
-
- 28** The real numbers α and β satisfy $\alpha > \beta$.
 (a) Prove that $(\cos \alpha + i \sin \alpha) + (\cos \beta + i \sin \beta) = 2\cos \frac{\alpha - \beta}{2} (\cos \frac{\alpha + \beta}{2} + i \sin \frac{\alpha + \beta}{2})$. [5]
 (b) Use this result with $\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{6}$ to show that $i + \frac{\sqrt{3}}{2} + \frac{1}{2}i = \frac{\sqrt{3}}{2} + \frac{3}{2}i$, by writing the left-hand side in modulus-argument form. [2]
-
- 29** The real number α satisfies $0 < \alpha < \pi$. Consider the equation $z^2 - 2z \cos \alpha + 1 = 0$.
 (a) Show that the roots are $\cos \alpha \pm i \sin \alpha$. [3]
 (b) Write each root in modulus-argument form. [2]
 (c) For $\alpha = \frac{2\pi}{5}$, the sum of the roots is $2\cos \frac{2\pi}{5}$. Use the product of the roots to state a relationship that the roots satisfy, and show that it holds. [2]

- 30** The real number α satisfies $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and $w = \frac{1 + i \tan \alpha}{1 - i \tan \alpha}$.
- (a) Show that $w = \cos 2\alpha + i \sin 2\alpha$. [4]
- (b) Find the value of $\tan \alpha$ for which $w = i$. [2]
- (c) Hence find $\frac{1 + i \tan \frac{\pi}{12}}{1 - i \tan \frac{\pi}{12}}$ in the form $a + bi$, with exact values. [2]

Exponential Form and Euler's Formula

- 1** Use Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ to write down the value of each of the following.
- (a) $e^{i\frac{\pi}{2}}$ [1]
- (b) $e^{i\pi}$ [1]
- (c) $e^{-i\frac{\pi}{2}}$ [1]
- 2** Write $2e^{i\frac{\pi}{3}}$ in the form $a + bi$. [2]
- 3** Express $1 + i$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]
- 4** Express -4 in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]
- 5** $z = 3e^{i\frac{\pi}{6}}$ and $w = 2e^{i\frac{\pi}{3}}$. Write zw in the form $re^{i\theta}$ and in the form $a + bi$. [2]
- 6** $z = 8e^{i\frac{5\pi}{6}}$ and $w = 2e^{i\frac{\pi}{3}}$. Write $\frac{z}{w}$ in the form $re^{i\theta}$ and in the form $a + bi$. [2]
- 7** Express $\sqrt{3} - i$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]
- 8** Write $e^{i\frac{2\pi}{3}}$ in the form $a + bi$, with exact values of a and b . [3]
- 9** Express $\frac{(1 - i)^5}{(\sqrt{3} + i)^2}$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]
- 10** Write $e^{2 + i\frac{\pi}{3}}$ in the form $re^{i\theta}$ and in the form $a + bi$. Give exact values. [3]
- 11** Use the identities for $\cos(\theta + \varphi)$ and $\sin(\theta + \varphi)$ to show that $e^{i\theta} \times e^{i\varphi} = e^{i(\theta + \varphi)}$. [3]
- 12** Find $(2e^{i\frac{\pi}{4}})^3$ in the form $a + bi$, with exact values of a and b . [3]
- 13** Find the value of θ in the interval $[0, 2\pi)$ for which $e^{i\theta} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$. [3]
- 14** Starting from $e^{i\theta} = \cos \theta + i \sin \theta$, show that $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$. [4]
- 15** Express $\frac{\sqrt{3} + i}{1 - i}$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]
- 16** For a complex number $z = re^{i\theta}$, with $r > 0$ and $-\pi < \theta \leq \pi$, the principal value of $\ln z$ is $\ln r + i\theta$. Find the principal value of $\ln(-2 + 2i)$ in the form $a + bi$, giving a and b to 3 significant figures. [4]

- 17** Solve the equation $e^z = -e$, giving all solutions for z in the form $x + iy$ with x and y real. [4]
-
- 18** Show that $1 + e^{i\theta} = 2\cos\frac{\theta}{2}e^{i\frac{\theta}{2}}$. [4]
-
- 19** Use the exponential forms of cosine and sine to evaluate each of the following.
- (a) $e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}$ [2]
- (b) $e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}}$ [2]
-
- 20** $z = 4e^{i\frac{\pi}{3}}$ and $w = 2e^{-i\frac{\pi}{6}}$. Find $\frac{z^2}{w}$ in the form $re^{i\theta}$ and in the form $a + bi$, with exact values. [5]
-
- 21** The convergent series C and S are given by $C = 1 + \frac{1}{3}\cos\theta + \frac{1}{9}\cos 2\theta + \frac{1}{27}\cos 3\theta + \dots$ and $S = \frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \frac{1}{27}\sin 3\theta + \dots$
- (a) Show that $C + iS = \frac{3}{3 - e^{i\theta}}$. [3]
- (b) Hence show that $C = \frac{3(3 - \cos\theta)}{10 - 6\cos\theta}$ and find a similar expression for S . [3]
-
- 22** A student argues: " $e^{2\pi i} = 1 = e^0$, so $2\pi i = 0$."
- (a) Explain why the student's conclusion is false. [2]
- (b) Find all the complex numbers z for which $e^z = 1$. [2]
- (c) Find all the complex numbers z for which $e^z = 2i$. [3]
-
- 23** A point P moves in the Argand diagram so that, at time t seconds, it is at the point representing $z = 3e^{i(2t + \frac{\pi}{6})}$.
- (a) Write down the distance of P from the origin, and find the position of P when $t = 0$ in the form $a + bi$. [2]
- (b) Find the first time at which P lies on the positive imaginary axis. [2]
- (c) Find the position of P , in the form $a + bi$, when $t = \frac{\pi}{4}$. [2]
-
- 24** Show that $\frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i \tan \frac{\theta}{2}$, and hence verify the result when $\theta = \frac{\pi}{2}$ by direct calculation. [6]
-
- 25** By writing $e^{i\frac{\pi}{12}} = e^{i\frac{\pi}{3}} \times e^{-i\frac{\pi}{4}}$, find exact values for $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$. [6]
-
- 26** Use the exponential form of a complex number.
- (a) Show that $(e^{i\frac{\pi}{4}})^8 = 1$. [1]
- (b) Hence find the exact value of $(e^{i\frac{\pi}{4}})^{2026}$. [3]
- (c) Given that $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$, find $(1 + i)^{2026}$ in the form $2^n i$. [2]
-
- 27** The expression i^i is defined as $e^{i \ln i}$, where $\ln i$ is any logarithm of i , so that $\ln i = i(\frac{\pi}{2} + 2k\pi)$ for integer k .
- (a) Show that i^i takes the values $e^{-(\frac{\pi}{2} + 2k\pi)}$ for integer k . [3]
- (b) Explain why all the values of i^i are real and positive. [2]
- (c) Find the principal value of i^i ($k = 0$), correct to 3 significant figures. [2]

- 28** Let $S = 1 + e^{i\theta} + e^{2i\theta} + \dots + e^{(n-1)i\theta}$, where θ is not a multiple of 2π .
- (a) Show that $S = \frac{e^{in\theta} - 1}{e^{i\theta} - 1}$. [2]
- (b) Hence show that $S = \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} e^{i\frac{(n-1)\theta}{2}}$. [4]
- (c) Deduce that $1 + \cos \theta + \cos 2\theta + \dots + \cos(n-1)\theta = \frac{\sin \frac{n\theta}{2} \cos \frac{(n-1)\theta}{2}}{\sin \frac{\theta}{2}}$, and check the result for $n = 4$, $\theta = \frac{\pi}{2}$. [3]

29 Find $\int e^{2x} \cos x \, dx$ by considering the real part of $\int e^{(2+i)x} \, dx$. [8]

30 Find all the solutions of the equation $e^{2z} - 2e^z + 2 = 0$, giving z in the form $x + iy$. [7]

De Moivre's Theorem

- 1** Use De Moivre's theorem to evaluate $(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})^3$. [2]
- 2** Use De Moivre's theorem to show that $(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5})^5 = -1$. [2]
- 3** Find $[2(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12})]^6$ in the form $a + bi$. [2]
- 4** Simplify $(\cos \theta + i \sin \theta)^5 (\cos \theta + i \sin \theta)^{-2}$, giving your answer in the form $\cos k\theta + i \sin k\theta$. [2]
- 5** Write $(\cos \theta - i \sin \theta)^4$ in the form $\cos k\theta + i \sin k\theta$. [2]
- 6** Write $1 + i$ in modulus-argument form and hence use De Moivre's theorem to show that $(1 + i)^4 = -4$. [3]
- 7** Find $(1 + i\sqrt{3})^5$ in the form $a + bi$, with exact values of a and b . [4]
- 8** Use De Moivre's theorem to show that $(\sqrt{3} - i)^6 = -64$. [4]
- 9** Find the value of $(1 - i)^8$. [3]
- 10** Find the exact value of $(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8})^{-4}$. [3]
- 11** Find $\frac{1}{(1 + i)^6}$ in the form $a + bi$. [4]
- 12** Simplify $\frac{(\cos 2\theta + i \sin 2\theta)^3 (\cos 3\theta - i \sin 3\theta)^2}{(\cos \theta + i \sin \theta)^4}$, giving your answer in the form $\cos k\theta + i \sin k\theta$. [3]
- 13** Find the smallest positive integer n for which $(\sqrt{3} + i)^n$ is purely imaginary, and find its value. [3]
- 14** By considering the expansion of $(\cos \theta + i \sin \theta)^3$, show that $\cos 3\theta = 4\cos^3\theta - 3\cos \theta$ and $\sin 3\theta = 3\sin \theta - 4\sin^3\theta$. [5]

- 15** Find $(\sqrt{3} - i)^{-5}$ in the form $a + bi$, with exact values of a and b . [4]
- 16** Use De Moivre's theorem to show that $(\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n = 2\cos n\theta$, and hence find the value of $(1 + i)^8 + (1 - i)^8$. [4]
- 17** Given that $z = \cos \theta + i \sin \theta$, show that $1 + z^2 = 2\cos \theta \times z$, and hence show that $\frac{(1 + z^2)^n}{z^n} = 2^n \cos^n \theta$. [4]
- 18** Use De Moivre's theorem to show that $\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$. [5]
- 19** Let $z = \cos \theta + i \sin \theta$. Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$, and find the value of $z^2 + \frac{1}{z^2}$ given that $z + \frac{1}{z} = 1$. [5]
- 20** Show that $(1 + i \tan \theta)^n = \sec^n \theta (\cos n\theta + i \sin n\theta)$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, and hence find $(1 + i \tan \frac{\pi}{6})^3$ in the form $a + bi$. [5]
- 21** Let $z = \cos \theta + i \sin \theta$. Use the binomial expansion of $(z + z^{-1})^5$ to show that $\cos^5 \theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$. [7]
- 22** Use De Moivre's theorem and the substitution $z = \cos \theta + i \sin \theta$.
 (a) Show that $\sin^4 \theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$. [5]
 (b) Hence find the exact value of $\int_0^{\frac{\pi}{2}} \sin^4 \theta \, d\theta$. [3]
- 23** A student writes: " $1 + i = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$, so $(1 + i)^8 = \cos 2\pi + i \sin 2\pi = 1$."
 (a) Identify the error in the student's working. [2]
 (b) Calculate $(1 + i)^8$ correctly using De Moivre's theorem. [3]
 (c) Verify your answer by squaring repeatedly. [1]
- 24** Use De Moivre's theorem to show that $\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$. [7]
- 25** Let $z = \cos \theta + i \sin \theta$.
 (a) By considering the binomial expansion of $(z + z^{-1})^4$, show that $8\cos^4 \theta = \cos 4\theta + 4\cos 2\theta + 3$. [4]
 (b) Hence solve the equation $\cos 4\theta + 4\cos 2\theta + 1 = 0$ for $0 \leq \theta \leq \pi$. [3]
- 26** The cubic equation $8x^3 - 6x - 1 = 0$ is to be solved by a trigonometric substitution.
 (a) Show that $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$. [3]
 (b) Hence show that the roots of $8x^3 - 6x - 1 = 0$ are $\cos \frac{\pi}{9}$, $\cos \frac{5\pi}{9}$ and $\cos \frac{7\pi}{9}$. [4]
- 27** The complex number $z = \cos \theta + i \sin \theta$.
 (a) Prove by induction that $z^n = \cos n\theta + i \sin n\theta$ for every positive integer n . [6]
 (b) Hence prove that the result also holds when n is a negative integer. [3]
- 28** Prove that for every positive integer n , $\sum_{r=0}^n C(n, r) \cos r\theta = 2^n \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2}$, where $C(n, r)$ is the binomial coefficient, by considering the real part of $(1 + e^{i\theta})^n$. [8]

29 Let $t = \tan \theta$.

(a) Use De Moivre's theorem to show that $\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}$. [4]

(b) Hence solve $\tan 3\theta = 1$ for $0 \leq \theta < \pi$, and show that the roots of $t^3 - 3t^2 - 3t + 1 = 0$ are $\tan \frac{\pi}{12}$, $\tan \frac{5\pi}{12}$ and $\tan \frac{3\pi}{4}$. [4]

30 Let $c = \cos \frac{\pi}{5}$.

(a) Show that $\cos 2\theta = 2\cos^2\theta - 1$, and use the result $\cos 3\theta = 4\cos^3\theta - 3\cos \theta$ to write $\cos \frac{3\pi}{5}$ and $\cos \frac{2\pi}{5}$ in terms of c . [2]

(b) Explain why $\cos \frac{3\pi}{5} = -\cos \frac{2\pi}{5}$. [2]

(c) Hence show that $4c^3 + 2c^2 - 3c - 1 = 0$, and find the exact value of $\cos \frac{\pi}{5}$. [4]

Roots of Complex Numbers

1 Solve $z^4 = 1$ over the complex numbers. [2]

2 $z = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$. Find a square root of z in modulus-argument form. [2]

3 Show that the roots of $z^3 = 8$ are $z = 2(\cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3})$ for $k = 0, 1, 2$, and write them in the form $a + bi$. [3]

4 One root of the equation $z^5 = 1$ is $\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$. Write down all five roots in modulus-argument form, with arguments in the interval $(-\pi, \pi]$. [3]

5 Find the two square roots of i in the form $a + bi$. [2]

6 Find the two square roots of $-4i$ in the form $a + bi$. [2]

7 Solve the equation $z^4 = -16$, giving the roots in the form $a + bi$, and show them on an Argand diagram. [4]

8 Solve the equation $z^3 = 8i$, giving each root in the form $a + bi$. [4]

9 Solve the equation $z^5 = 32i$, giving the roots in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

10 Find the six sixth roots of unity, that is the solutions of $z^6 = 1$, in the form $a + bi$. [4]

11 Solve $z^4 = 16(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})$, giving the roots in modulus-argument form with arguments in $(-\pi, \pi]$. [4]

12 The three cube roots of 8 are 2, 2ω and $2\omega^2$, where $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Show that their sum is 0 and their product is 8. [3]

13 Find the two square roots of $1 - i\sqrt{3}$, giving your answers in the form $a + bi$, with exact values of a and b . [4]

- 14** Solve the equation $z^3 = -4 + 4i$, giving the roots in modulus-argument form with arguments in $(-\pi, \pi]$. [4]
- 15** Solve the equation $z^4 = 8 + 8\sqrt{3}i$, giving the roots in modulus-argument form with arguments in $(-\pi, \pi]$. [4]
- 16** Solve the equation $(z + 1)^3 = 8$, giving the solutions in the form $a + bi$. [4]
- 17** Solve the equation $z^2 = (\sqrt{3} + i)^3$, giving the answers in the form $a + bi$. [4]
- 18** Consider the equation $z^4 + 4 = 0$.
 (a) Solve the equation using modulus-argument form, giving the roots in the form $a + bi$. [3]
 (b) Hence write $z^4 + 4$ as the product of two real quadratic factors. [2]
- 19** The roots of the equation $z^6 = 64$ are $z_0, z_1, \dots, z_5$. Find the roots and find the exact sum of those roots which have a positive imaginary part. [4]
- 20** A square has its centre at the origin of an Argand diagram and one vertex at the point representing $2 + i$. Find the complex numbers representing the other three vertices, and find a value c such that all four vertices satisfy $z^4 = c$. [5]
- 21** Consider the equation $z^5 = 32$.
 (a) Solve the equation, giving the roots in modulus-argument form. [3]
 (b) Sketch the roots on an Argand diagram and describe the shape formed when they are joined in order. [1]
 (c) Find the area of the shape, correct to 3 significant figures. [3]
- 22** Solve the equation $(z + 1)^4 = (z - 1)^4$. [8]
- 23** A student solves $z^3 = -8$ and writes: " $-8 = 8(\cos \pi + i \sin \pi)$, so $z = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$, and this is the only root."
 (a) Explain the errors in the student's solution. [2]
 (b) Find all three roots in the form $a + bi$. [3]
 (c) Check that the three roots sum to zero. [2]
- 24** The equation $z^4 = w$ has a root $3 + 3i$.
 (a) Find w . [2]
 (b) Find the other three roots of the equation. [2]
 (c) The four roots are the vertices of a square. Find its area. [2]
- 25** Solve the equation $z^6 + 7z^3 - 8 = 0$, giving the roots in modulus-argument form with arguments in $(-\pi, \pi]$. [7]
- 26** Find the three cube roots of $1 - i$. Give the roots in modulus-argument form with arguments in $(-\pi, \pi]$, and give the root with the smallest argument in the form $a + bi$, with a and b correct to 3 significant figures. [6]
- 27** Let $\omega = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$.
 (a) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. [3]
 (b) By taking real parts, show that $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$. [2]
 (c) Hence, using $\cos \frac{4\pi}{5} = 2\cos^2 \frac{2\pi}{5} - 1$, find the exact value of $\cos \frac{2\pi}{5}$. [3]

- 28** Consider the equation $z^6 = -64$.
 (a) Solve the equation, giving the roots in modulus-argument form. [3]
 (b) Show that the roots are the vertices of a regular hexagon and find its area. [3]
 (c) Find the product of the six roots. [3]
-
- 29** Consider the equation $z^6 = 1$.
 (a) Solve the equation, giving the roots in the form $a + bi$. [2]
 (b) Hence factorise $z^6 - 1$ into two real linear factors and two real quadratic factors. [3]
 (c) Hence solve $z^4 + z^2 + 1 = 0$. [3]
-
- 30** Consider the equation $\left(\frac{z-1}{z+1}\right)^3 = i$.
 (a) Solve the equation $w^3 = i$, giving the roots in modulus-argument form. [3]
 (b) Show that if $w = \frac{z-1}{z+1}$ then $z = \frac{1+w}{1-w}$. [2]
 (c) Show that if $w = \cos \varphi + i \sin \varphi$ then $\frac{1+w}{1-w} = i \cot \frac{\varphi}{2}$. [2]
 (d) Hence find the three solutions of the original equation in exact form. [2]
-

Roots of Unity

- 1** State the modulus of each of the fifth roots of unity, and the angle between consecutive roots when they are plotted on an Argand diagram. [2]
-
- 2** Show that $z = i$ is a fourth root of unity. [2]
-
- 3** Write down the eight eighth roots of unity in the form $\cos \theta + i \sin \theta$, where $-\pi < \theta \leq \pi$. [2]
-
- 4** $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Write ω^2 in the form $\cos \theta + i \sin \theta$, with $-\pi < \theta \leq \pi$. [2]
-
- 5** ω is a complex cube root of unity.
 (a) Write down the value of ω^3 . [1]
 (b) Write down the value of $1 + \omega + \omega^2$. [1]
-
- 6** How many of the seven seventh roots of unity are real? Give a reason. [2]
-
- 7** Sketch the fifth roots of unity on an Argand diagram and describe the shape formed when they are joined in order. [3]
-
- 8** Solve $z^8 = 1$, giving the solutions in the form $a + bi$. [3]
-
- 9** $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Simplify ω^{10} , ω^{-4} and ω^{100} , giving each as 1, ω or ω^2 . [4]
-
- 10** ω is a complex cube root of unity. Show that $(1 - \omega)(1 - \omega^2) = 3$. [3]
-
- 11** $\omega = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$. Show that $|1 - \omega| = 2 \sin \frac{\pi}{5}$. [3]
-
- 12** Find the smallest positive integer n for which $(\cos \frac{\pi}{7} + i \sin \frac{\pi}{7})^n = 1$. [3]
-
- 13** The complex number z satisfies $z^2 + z + 1 = 0$. Show that $z^3 = 1$, and hence find the value of $z^{50} + z^{25} + 1$. [3]
-

- 14** $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, where $n \geq 2$ is an integer. Show that $1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0$, and deduce that $\cos 0 + \cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos \frac{2(n-1)\pi}{n} = 0$. [4]
- 15** ω is a complex cube root of unity with $\omega \neq 1$. Find the exact value of $\frac{1}{1+\omega} + \frac{1}{1+\omega^2}$. [4]
- 16** Let $\omega = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$. By considering the real part of $1 + \omega + \omega^2 + \dots + \omega^6 = 0$, show that $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}$. [4]
- 17** Write down the fourth roots of unity and show them on an Argand diagram. Find the area of the shape formed by joining them, and write $z^4 - 1$ as a product of two real linear factors and one real quadratic factor. [4]
- 18** $\omega = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$. Show that the quadratic equation with roots ω and ω^4 is $z^2 - 2\cos \frac{2\pi}{5}z + 1 = 0$, and write down the quadratic equation with roots ω^2 and ω^3 . [4]
- 19** The complex number $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$.
- (a) Show that the roots of $z^2 + z + 1 = 0$ are ω and ω^2 , where $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. [2]
- (b) Hence solve $z^6 + z^3 + 1 = 0$, giving the roots in the form $\cos \theta + i \sin \theta$, with $-\pi < \theta \leq \pi$. [3]
- 20** $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$, and a and b are real numbers. Show that $|a + b\omega|^2 = a^2 - ab + b^2$, and hence find $|3 + 2\omega|$. [5]
- 21** $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, $n \geq 2$. The roots of $z^n = 1$ are $1, \omega, \dots, \omega^{n-1}$.
- (a) Show that $1 + z + z^2 + \dots + z^{n-1} = (z - \omega)(z - \omega^2) \dots (z - \omega^{n-1})$, and deduce that $(1 - \omega)(1 - \omega^2) \dots (1 - \omega^{n-1}) = n$. [5]
- (b) Verify the result for $n = 4$. [2]
- (c) Use the identity $|1 - \omega^k| = 2 \sin \frac{k\pi}{n}$ with $n = 5$ to show that $\sin \frac{\pi}{5} \sin \frac{2\pi}{5} = \frac{\sqrt{5}}{4}$. [3]
- 22** A student states that, for $\omega = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$, $1 + \omega + \omega^2 + \omega^3 = 0$ because "the roots of unity add up to zero".
- (a) Explain the error in the student's statement. [2]
- (b) Write down the correct value of $1 + \omega + \omega^2 + \omega^3$ in terms of ω . [2]
- (c) Hence find the modulus of $1 + \omega + \omega^2 + \omega^3$. [2]
- 23** $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Show that $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$ for all complex numbers x, y and z , and use it to find the value of $1^3 + 2^3 + 3^3 - 3(1)(2)(3)$ in the form $(1 + 2 + 3) \times |1 + 2\omega + 3\omega^2|^2$. [8]
- 24** $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, $n \geq 2$, and r is a positive integer. Show that $\sum_{k=0}^{n-1} \omega^{kr} = n$ when r is a multiple of n , and 0 otherwise. [6]

- 25** In a three-phase electrical supply the voltages in the three wires are represented by the complex numbers $V_0 = V$, $V_1 = V\omega$ and $V_2 = V\omega^2$, where V is a positive real number and $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$.
- (a) Show that $V_0 + V_1 + V_2 = 0$. [2]
- (b) The voltage between wires 0 and 1 is $|V_1 - V_0|$. Show that $|V_1 - V_0| = \sqrt{3}V$. [2]
- (c) Given $V = 230$ volts, find the voltage between two wires to the nearest volt. [2]

- 26** The fifth roots of unity are $1, \omega, \omega^2, \omega^3, \omega^4$, where $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Let $c_1 = \cos\frac{2\pi}{5}$ and $c_2 = \cos\frac{4\pi}{5}$. It is given that $z^4 + z^3 + z^2 + z + 1 = (z^2 - 2c_1z + 1)(z^2 - 2c_2z + 1)$.
- (a) By comparing coefficients, show that $c_1 + c_2 = -\frac{1}{2}$ and $c_1c_2 = -\frac{1}{4}$. [4]
- (b) Hence find the exact values of c_1 and c_2 . [3]

- 27** $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. Let $\alpha = \omega + \omega^2 + \omega^4$ and $\beta = \omega^3 + \omega^5 + \omega^6$.
- (a) Show that $\alpha + \beta = -1$. [2]
- (b) Show that $\alpha\beta = 2$. [4]
- (c) Hence find α and β in the form $a + bi$, given that the imaginary part of α is positive. [3]

- 28** $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, $n \geq 2$. You may use the result $(1 - \omega)(1 - \omega^2) \dots (1 - \omega^{n-1}) = n$.
- (a) Show that $|1 - \omega^k| = 2 \sin\frac{k\pi}{n}$ for $k = 1, 2, \dots, n - 1$. [3]
- (b) Hence prove that $\sin\frac{\pi}{n} \sin\frac{2\pi}{n} \dots \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$. [3]
- (c) Check the result for $n = 6$. [2]

- 29** $\omega = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$.
- (a) Show that ω^k , for $k = 1, 2, 4, 5, 7, 8$, are the roots of $z^6 + z^3 + 1 = 0$. [3]
- (b) Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$. [4]

- 30** ω is an n th root of unity, $\omega \neq 1$, where $n \geq 2$.
- (a) Show that the real part of $\frac{1}{1 - \omega}$ is $\frac{1}{2}$. [4]
- (b) Hence find the value of $\sum_{k=1}^{n-1} \frac{1}{1 - \omega^k}$. [4]

Geometrical Transformations Using Complex Numbers

- 1** The transformation T maps z to $w = z + 3 - 2i$. Find the image of the point $1 + i$, and describe T geometrically. [2]
- 2** The transformation T maps z to $w = 2z$. Find the image of $3 - i$, and describe T geometrically. [2]
- 3** The transformation T maps z to $w = iz$. Find the image of $2 + 3i$, and describe T geometrically. [2]
- 4** The transformation T maps z to $w = \bar{z}$. Find the image of $4 + 5i$, and describe T geometrically. [2]

- 5** A rotation through $\frac{\pi}{2}$ clockwise about the origin, followed by an enlargement with scale factor 3 and centre the origin, can be written as $w = az$. Find a , and find the image of the point $2 + i$. [2]
- 6** The transformation T maps z to $w = (1 + i)z$. Find the images of 1 and of i , and describe T as a combination of an enlargement and a rotation about the origin. [3]
- 7** The transformation T maps z to $w = 2z + 1 - i$. Find the equation of the image of the circle $|z| = 1$ under T . [3]
- 8** The transformation T maps $z = x + iy$ to $w = u + iv$, where $w = z + 1$. Find the Cartesian equation of the image of the line $y = 2x$. [3]
- 9** The transformation T maps $z = x + iy$ to $w = u + iv$, where $w = iz$. Find the Cartesian equation of the image of the line $y = x$. [3]
- 10** The transformation T maps z to $w = (2 + i)z$. Describe T as an enlargement followed by a rotation about the origin, giving the angle of rotation correct to 3 significant figures. [3]
- 11** Find the invariant points of each transformation.
 (a) $w = 3z - 4$ [2]
 (b) $w = iz + 1$ [2]
- 12** The transformation T is a rotation through $\frac{\pi}{2}$ anticlockwise about the point $1 + i$. Show that T can be written as $w = iz + 2$. [3]
- 13** The transformation T maps z to $w = z^2$. Find the images of the points $1 + i$, $2i$ and -3 , and show that the image of the circle $|z| = 2$ is the circle $|w| = 4$. [4]
- 14** The transformation T maps $z = x + iy$ to $w = u + iv$, where $w = \frac{1}{z}$. Show that T maps the line $x = 1$ to part of the circle with equation $u^2 + v^2 = u$. [5]
- 15** The transformation T maps z to $w = \frac{z - i}{z + i}$. Show that T maps the real axis onto part of the circle $|w| = 1$, and find the images of $z = 0$ and $z = 1$. [5]
- 16** The transformation T maps $z = x + iy$ to $w = u + iv$, where $w = z^2$. Find the Cartesian equation of the image of the line $x = 1$ under T . [5]
- 17** The transformation T maps z to $w = \frac{z + 3}{z - 1}$, $z \neq 1$. The locus of z is mapped by T onto the circle $|w| = 2$. Find a Cartesian equation of the locus of z , and describe it. [4]
- 18** The transformation T maps z to $w = (1 + i)z + 2$. The square S has vertices 0 , 1 , $1 + i$ and i . Find the images of the vertices of S , and describe T as a combination of an enlargement, a rotation and a translation. [5]
- 19** The transformation T maps z to $w = az + b$, where a and b are complex constants. T maps 1 to 3 and i to $1 + 2i$. Find a and b . [4]
- 20** A transformation P is a rotation through $\frac{\pi}{3}$ anticlockwise about the origin, followed by an enlargement with scale factor 2 and centre the origin. Write P in the form $w = az$, giving a in the form $p + qi$ with exact values, and find the image of $1 + i$. [4]

- 21** The transformation T is given by $w = \frac{az}{z+c}$, $z \neq -c$, where a and c are real constants. T maps the point $2 + i$ to the point $2 - i$.
- (a) Find the value of a and the value of c . [4]
(b) Find the invariant points of T . [3]
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- 22** The transformation T maps z to $w = \frac{z-1}{z+1}$, $z \neq -1$.
- (a) Show that T maps the imaginary axis onto the unit circle $|w| = 1$ (excluding one point). [4]
(b) Show that T maps the unit circle $|z| = 1$, $z \neq -1$, onto the imaginary axis. [3]
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- 23** A student is asked to find the image of the circle $|z| = 3$ under the transformation $w = z^2$, and writes: " $|w| = 2 \times 3 = 6$, so the image is the circle $|w| = 6$."
- (a) Identify the error in the student's working. [2]
(b) Find the correct image of the circle $|z| = 3$. [2]
(c) Explain why every point of the image circle comes from two different points of the original circle. [2]
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- 24** The region R of the z -plane is defined by $|z| \leq 2$ and $0 \leq \arg z \leq \frac{\pi}{4}$. The transformation T maps z to $w = z^2$.
- (a) Find the area of R . [2]
(b) Describe the image of R under T , and find its area. [5]
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- 25** A point on an Argand diagram is rotated about the centre C , which represents $3 + 2i$. The rotation is described by $w - c = e^{i\alpha}(z - c)$, where $c = 3 + 2i$.
- (a) Find the image of the point $5 + 2i$ under a rotation through $\frac{\pi}{2}$ anticlockwise about C . [3]
(b) Find the image of the point $1 + 5i$ under a rotation through $\frac{\pi}{2}$ clockwise about C . [3]
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- 26** The transformation T maps z to $w = (1 - i)z + 2$.
- (a) Describe T geometrically, giving the scale factor and the angle of rotation of the linear part, and the translation. [2]
(b) Find the image of the circle $|z - i| = 1$ under T . [3]
(c) Find the area of the image of the disc $|z - i| \leq 1$. [2]
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- 27** The transformation T maps z to $w = \frac{z+2}{z-4i}$, $z \neq 4i$.
- (a) Given that $|w| = 1$, show that z lies on the line with equation $x + 2y = 3$. [4]
(b) Verify that $z = 1 + i$ lies on the line, and show that its image has modulus 1. [2]
(c) Find the equation of the locus of z for which w is real. [3]
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- 28** The transformation T maps $z = x + iy$ to $w = u + iv$, where $w = \frac{1}{z}$.
- (a) Show that T maps the circle $|z - 2| = 2$ to the line $u = \frac{1}{4}$. [4]
(b) Show that T maps the circle $|z - 3| = 1$ to the circle with centre $(\frac{3}{8}, 0)$ and radius $\frac{1}{8}$. [5]
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- 29** The transformation T maps z to $w = e^{i\frac{\pi}{3}}z + 2$.
- (a) Show that T has a single invariant point and find it in the form $a + bi$. [4]
(b) Show that T is a rotation about the invariant point, and state the angle of rotation. [2]
(c) A rotation of $\frac{\pi}{2}$ anticlockwise about 0 is followed by a rotation of $\frac{\pi}{2}$ anticlockwise about 1. Show that the combined transformation is $w = -z + 1 - i$, and find its invariant point. [3]

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The transformation T maps z to $w = z + \frac{1}{z}$, $z \neq 0$. A point $z = x + iy$ lies on the circle $|z| = r$, where $r > 0$ and $r \neq 1$, and $w = u + iv$.

- (a) Show that $u = (r + \frac{1}{r})\cos \theta$ and $v = (r - \frac{1}{r})\sin \theta$, where $z = r(\cos \theta + i \sin \theta)$, and hence show that the image of the circle is an ellipse. [4]
- (b) Show that the foci of the ellipse are at $w = \pm 2$, whatever the value of r . [2]
- (c) Describe the image of the circle $|z| = 1$. [2]
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