

Centres of Mass of Particles

1 (2 marks)

$$\bar{x} = \frac{2 \times 1 + 3 \times 4}{2 + 3}$$

M1

$$= \frac{14}{5} = 2.8 \text{ m}$$

A1

Answer: 2.8 m

2 marks

2 (2 marks)

$$\bar{x} = \frac{1 \times 0 + 2 \times 2 + 3 \times 5}{1 + 2 + 3}$$

M1

$$= \frac{19}{6} = 3.17 \text{ m}$$

A1

Answer: 3.17 m

2 marks

3 (2 marks)

$$\bar{x} = \frac{2 \times 1 + 3 \times 4 + 5 \times (-2)}{10} = 0.4$$

M1

$$\bar{y} = \frac{2 \times 3 + 3 \times 0 + 5 \times 2}{10} = 1.6, \text{ so } (0.4, 1.6)$$

A1

Answer: (0.4, 1.6)

2 marks

4 (2 marks)

$$\text{Distance from A} = \frac{6 \times 2}{4 + 6}$$

M1

$$= 1.2 \text{ m}$$

A1

Answer: 1.2 m

2 marks

5 (2 marks)

$$\bar{x} = \frac{0 + 6 + 0}{3} = 2$$

M1

$$\bar{y} = \frac{0 + 0 + 3}{3} = 1, \text{ so } (2, 1)$$

A1

Answer: (2, 1)

2 marks

6 (2 marks)

$$\text{Distance} = \frac{2m \times 0.9}{m + 2m}$$

M1

$$= 0.6 \text{ m}$$

A1

Answer: 0.6 m

2 marks

7 (3 marks)

$$\frac{2 \times 0 + 3m + 4 \times 6}{2 + m + 4} = 3.5$$

M1

$$3m + 24 = 3.5(6 + m) = 21 + 3.5m$$

M1

$$0.5m = 3, \text{ so } m = 6$$

A1

Answer: 6

3 marks

8 (4 marks)

Take O as origin, OA along the x-axis: O(0,0), A(4,0), B(4,4), C(0,4)

B1

$$\bar{x} = \frac{0 + 2 \times 4 + 3 \times 4 + 0}{10} = 2 \text{ m}$$

M1

$$\bar{y} = \frac{0 + 0 + 3 \times 4 + 4 \times 4}{10} = 2.8 \text{ m}$$

M1 A1

Answer: 2 m from OC, 2.8 m from OA

4 marks

9 (3 marks)

$$\text{Total mass 10 kg: } \frac{3 \times 0 + 5 \times 4 + 2p}{10} = 3$$

M1

$$20 + 2p = 30$$

M1

$$p = 5$$

A1

Answer: 5

3 marks

10 (3 marks)

$$\text{Distance from A} = \frac{2 \times 0 + 3 \times 1.5 + 5 \times 0.5}{2 + 3 + 5}$$

M1 A1

$$= \frac{7}{10} = 0.7 \text{ m}$$

A1

Answer: 0.7 m

3 marks

11 (5 marks)

$$\text{(a) A: } x = 3t, \text{ B: } x = 10 - 2t$$

B1

$$\bar{x} = \frac{2 \times 3t + 3(10 - 2t)}{5} = 6$$

M1 A1

Answer: 6 m

3 marks

$$\text{(b) The centre of mass stays at rest at } x = 6$$

B1

$$\text{The total momentum } 2 \times 3 + 3 \times (-2) = 0$$

B1

Answer: at rest

2 marks

12 (3 marks)

$$\mathbf{r} = \frac{2(2\mathbf{i} + \mathbf{j}) + 3(-\mathbf{i} + 4\mathbf{j}) + 5(3\mathbf{i} - 2\mathbf{j})}{10}$$

M1 A1

$$= (1.6\mathbf{i} + 0.4\mathbf{j}) \text{ m}$$

A1

Answer: (1.6i + 0.4j) m

3 marks

13 (4 marks)

$$\bar{y} = \frac{1 \times 0 + 2 \times 0 + 6m}{3 + m} = 4, \text{ so } 6m = 12 + 4m$$

M1 A1

$$m = 6$$

A1

$$\bar{x} = \frac{2 \times 6}{9} = \frac{4}{3}$$

B1

Answer: m = 6, x = $\frac{4}{3}$

4 marks

14 (5 marks)

$$m + n = 8$$

B1

$$\frac{2m + 5n}{10} = 2.8, \text{ so } 2m + 5n = 28$$

M1 A1

$$\text{Substitute } m = 8 - n: 16 + 3n = 28$$

M1

$$n = 4, m = 4$$

A1

Answer: m = 4, n = 4

5 marks

15 (4 marks)

$$\text{Let the distance be } x. \frac{2 \times 0 + 3 \times 2 + 2x}{7} = 1$$

M1 A1

$$6 + 2x = 7$$

M1

$$x = 0.5 \text{ m}$$

A1

Answer: 0.5 m

4 marks

16 (5 marks)

$$\text{(a) By symmetry, the centre of mass is at O}$$

B1

Answer: O

1 mark

$$\text{(b) Take moments about O: the removed particle was at distance 0.5 m (radius of the hexagon = side)}$$

B1

$$\text{The remaining five particles have total moment about O of } -m \times 0.5 \text{ (sum of all six is zero)}$$

M1

$$\text{Distance} = \frac{0.5m}{5m} = 0.1 \text{ m, on the opposite side of O from the removed particle}$$

M1 A1

Answer: 0.1 m

4 marks

17 (5 marks)

$$(a) \frac{5 \times 0 + 2 \times 1 + 3 \times 3}{10}$$

M1 A1

$$= 1.1 \text{ m}$$

A1

Answer: 1.1 m

3 marks

$$(b) R = \text{total weight} = 10 \times 9.8$$

M1

$$= 98 \text{ N}$$

A1

Answer: 98 N

2 marks

18 (6 marks)

$$(a) \mathbf{v} = \frac{2(3\mathbf{i} + \mathbf{j}) + 3(-\mathbf{i} + 2\mathbf{j}) + 5(\mathbf{i} - \mathbf{j})}{10}$$

M1

$$= (0.8\mathbf{i} + 0.3\mathbf{j}) \text{ m s}^{-1}$$

A1

Answer: $(0.8\mathbf{i} + 0.3\mathbf{j}) \text{ m s}^{-1}$

2 marks

$$(b) t = 0: \frac{2(2\mathbf{i}) + 3(4\mathbf{j}) + 5(-\mathbf{i} + \mathbf{j})}{10} = (-0.1\mathbf{i} + 1.7\mathbf{j}) \text{ m}$$

M1 A1

$$t = 3: \text{add } 3 \times (0.8\mathbf{i} + 0.3\mathbf{j}): (2.3\mathbf{i} + 2.6\mathbf{j}) \text{ m}$$

M1 A1

Answer: $(-0.1\mathbf{i} + 1.7\mathbf{j}) \text{ m}; (2.3\mathbf{i} + 2.6\mathbf{j}) \text{ m}$

4 marks

19 (4 marks)

$$\frac{2 \times 1 + 3 \times 4 + 5a}{10} = 3$$

M1

$$14 + 5a = 30, a = 3.2$$

A1

$$\frac{2 \times 2 + 3 \times (-1) + 5b}{10} = 1$$

M1

$$1 + 5b = 10, b = 1.8$$

A1

Answer: $a = 3.2, b = 1.8$

4 marks

20 (6 marks)

Take A as origin, AB along the x-axis: A(0,0), B(6a,0), C(3a, 3a√3)

B1

$$\bar{x} = \frac{3m \times 0 + 2m \times 6a + m \times 3a}{6m} = \frac{15a}{6} = 2.5a$$

M1 A1

$$\bar{y} = \frac{m \times 3a\sqrt{3}}{6m} = \frac{a\sqrt{3}}{2}$$

M1 A1

Centre of mass is $\frac{a\sqrt{3}}{2}$ from AB, and 0.5a from the perpendicular bisector ($x = 3a$)

A1

Answer: $\frac{a\sqrt{3}}{2}$ from AB; 0.5a from the bisector

6 marks

21 (7 marks)

Take A as origin, B = (a, 0), C = $(\frac{a}{2}, \frac{a\sqrt{3}}{2})$

B1

$$\bar{x} = \frac{m \times 0 + 2m \times a + 3m \times \frac{a}{2}}{6m} = \frac{7a}{12}$$

M1 A1

$$\bar{y} = \frac{3m \times \frac{a\sqrt{3}}{2}}{6m} = \frac{a\sqrt{3}}{4}$$

M1 A1

$$\text{Distance}^2 = \frac{49a^2}{144} + \frac{3a^2}{16} = \frac{76a^2}{144}$$

M1

$$\text{Distance} = \frac{a\sqrt{76}}{12} = \frac{a\sqrt{19}}{6}$$

A1

Answer: The distance of the centre of mass from A is $\frac{a\sqrt{19}}{6}$

7 marks

22 (6 marks)

$$\text{Position vector of G} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$$

M1 A1

$$\text{Position vector of M} = \frac{\mathbf{b} + \mathbf{c}}{2}, \text{ so } \mathbf{b} + \mathbf{c} = 2\mathbf{m}$$

B1

$$\mathbf{g} = \frac{\mathbf{a} + 2\mathbf{m}}{3} = \mathbf{a} + \frac{2}{3}(\mathbf{m} - \mathbf{a})$$

M1

$$\text{So G lies on AM, with } AG = \frac{2}{3}AM$$

A1

$$\text{Hence } AG : GM = 2 : 1$$

A1

Answer: G lies on the median AM, with AG : GM = 2 : 1

6 marks

23 (6 marks)

The student has found the mean of the positions and ignored the masses

E1

$$\bar{x} = \frac{2 \times 0 + 3 \times 4 + 5 \times 0}{10}$$

M1

$$= 1.2$$

A1

$$\bar{y} = \frac{2 \times 0 + 3 \times 0 + 5 \times 6}{10}$$

M1

$$= 3, \text{ so the centre of mass is } (1.2, 3)$$

A1

It is closer to the 5 kg particle at (0, 6) than the mean point is

B1

Answer: (1.2, 3)

6 marks

24 (6 marks)

$$\text{Moments about C: } 2g \times 0.4 = mg \times 0.6$$

M1

$$m = \frac{4}{3}$$

A1

$$T = (2 + \frac{4}{3})g = 32.7 \text{ N}$$

M1 A1

$$\text{New position} = \frac{2 \times 0 + 1 \times 0.5 + \frac{4}{3} \times 1}{2 + 1 + \frac{4}{3}}$$

M1

$$= \frac{11}{6} \div \frac{13}{3} = \frac{11}{26} = 0.423 \text{ m}$$

A1

$$\text{Answer: } m = \frac{4}{3}; T = 32.7 \text{ N; } 0.423 \text{ m}$$

6 marks

25 (7 marks)

$$\text{(a) Distance from OC: } \frac{3 \times 6 + k \times 6}{10 + k}$$

M1 A1

$$= \frac{18 + 6k}{10 + k}; \text{ distance from OA: } \frac{k \times 4 + 5 \times 4}{10 + k} = \frac{4k + 20}{10 + k}$$

M1 A1

$$\text{Answer: } \frac{18 + 6k}{10 + k} \text{ m, } \frac{4k + 20}{10 + k} \text{ m}$$

4 marks

$$\text{(b) } 18 + 6k = 4k + 20, \text{ so } k = 1$$

M1 A1

$$\text{Distance} = \frac{24}{11} = 2.18 \text{ m}$$

A1

$$\text{Answer: } k = 1; 2.18 \text{ m}$$

3 marks

26 (6 marks)

$$\text{Distance from the Sun} = \frac{m_J d}{M + m_J}$$

M1

$$= \frac{1.90 \times 10^{27} \times 7.78 \times 10^{11}}{1.99 \times 10^{30} + 1.90 \times 10^{27}}$$

M1 A1

$$= 7.42 \times 10^8 \text{ m}$$

A1

$$7.42 \times 10^8 > 6.96 \times 10^8, \text{ so the centre of mass lies outside the Sun}$$

E1 A1

$$\text{Answer: } 7.42 \times 10^8 \text{ m; outside the Sun}$$

6 marks

27 (10 marks)

$$(a) \text{ Total mass} = \sum_{k=1}^n k = \frac{n(n+1)}{2} \quad \text{B1}$$

$$\text{Moment} = \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{M1 A1}$$

$$\bar{x} = \frac{n(n+1)(2n+1)}{6} \div \frac{n(n+1)}{2} \quad \text{M1}$$

$$= \frac{2n+1}{3} \quad \text{A1}$$

Answer: The x-coordinate of the centre of mass is $\frac{2n+1}{3}$ 5 marks

$$(b) \frac{2n+1}{3} = 7 \quad \text{M1}$$

$$n = 10 \quad \text{A1}$$

Answer: 10 2 marks

$$(c) \text{ Total mass} = \frac{n(n+1)(2n+1)}{6}, \text{ moment} = \sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4} \quad \text{M1 A1}$$

$$\bar{x} = \frac{3n(n+1)}{2(2n+1)} \quad \text{A1}$$

$$\text{Answer: } \frac{3n(n+1)}{2(2n+1)} \quad \text{3 marks}$$

28 (9 marks)

$$(a) \mathbf{g} = \frac{1(\mathbf{i}) + 2(2\mathbf{j}) + 3(3\mathbf{k}) + 4(2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})}{10} \quad \text{M1 A1}$$

$$= \frac{9\mathbf{i} + 12\mathbf{j} + 17\mathbf{k}}{10} = 0.9\mathbf{i} + 1.2\mathbf{j} + 1.7\mathbf{k} \quad \text{A1}$$

Answer: $0.9\mathbf{i} + 1.2\mathbf{j} + 1.7\mathbf{k}$ 3 marks

$$(b) |\mathbf{OG}| = \sqrt{0.81 + 1.44 + 2.89} \quad \text{M1}$$

$$= 2.27 \text{ m} \quad \text{A1}$$

Answer: 2.27 m 2 marks

$$(c) \text{ Total mass } 20 \text{ kg: } 10\mathbf{g} + 10\mathbf{r} = 20(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \text{M1}$$

$$10\mathbf{r} = (20 - 9)\mathbf{i} + (20 - 12)\mathbf{j} + (20 - 17)\mathbf{k} \quad \text{M1 A1}$$

$$\mathbf{r} = 1.1\mathbf{i} + 0.8\mathbf{j} + 0.3\mathbf{k} \quad \text{A1}$$

Answer: $1.1\mathbf{i} + 0.8\mathbf{j} + 0.3\mathbf{k}$ 4 marks

29 (10 marks)

$$(a) \text{ Horizontal: } \frac{m \times 6 + 2m \times 3}{3m} = 4 \text{ m s}^{-1} \quad \text{M1 A1}$$

$$\text{Vertical: } \frac{m \times 8 + 2m \times 12}{3m} = \frac{32}{3} = 10.67 \text{ m s}^{-1} \quad \text{A1}$$

$$\text{Answer: } 4 \text{ m s}^{-1}, \frac{32}{3} \text{ m s}^{-1} \quad \text{3 marks}$$

(b) Each particle has vertical acceleration $-g$, so the centre of mass has vertical acceleration $-g$ M1

$$x = 4t, y = \frac{32}{3}t - 4.9t^2 \quad \text{M1 A1}$$

$$\text{Eliminate } t = \frac{x}{4}: y = \frac{8}{3}x - \frac{4.9}{16}x^2 \quad \text{M1}$$

$$a = \frac{8}{3}, b = \frac{49}{160} = 0.306 \quad \text{A1}$$

$$\text{Answer: } y = \frac{8}{3}x - \frac{49}{160}x^2 \quad \text{5 marks}$$

$$(c) \text{ Greatest height when the vertical velocity is zero: } \frac{\frac{1024}{9}}{19.6} \quad \text{M1}$$

$$= 5.80 \text{ m} \quad \text{A1}$$

Answer: 5.80 m 2 marks

30 (10 marks)**(a)** Write $\mathbf{r}_i - \mathbf{p} = (\mathbf{r}_i - \mathbf{g}) + (\mathbf{g} - \mathbf{p})$

M1

$$|\mathbf{r}_i - \mathbf{p}|^2 = |\mathbf{r}_i - \mathbf{g}|^2 + 2(\mathbf{r}_i - \mathbf{g}) \cdot (\mathbf{g} - \mathbf{p}) + |\mathbf{g} - \mathbf{p}|^2$$

M1

Multiply by m_i and sum: the middle term is $2(\sum m_i \mathbf{r}_i - M\mathbf{g}) \cdot (\mathbf{g} - \mathbf{p})$

M1

and $\sum m_i \mathbf{r}_i = M\mathbf{g}$, so the middle term is zero

A1

Hence the result

A1

Answer: $I(\mathbf{p}) = \sum m_i |\mathbf{r}_i - \mathbf{g}|^2 + M|\mathbf{g} - \mathbf{p}|^2$

5 marks

(b) The first term does not depend on $\mathbf{p}$, and $M|\mathbf{g} - \mathbf{p}|^2 \geq 0$ with equality only when $\mathbf{p} = \mathbf{g}$

E1 A1

Answer: $\mathbf{p} = \mathbf{g}$

2 marks

(c) $\bar{x} = \frac{0 + 2 + 12}{6} = \frac{7}{3}$

M1

Least value $= 1\left(\frac{7}{3}\right)^2 + 2\left(\frac{4}{3}\right)^2 + 3\left(\frac{5}{3}\right)^2 = \frac{49 + 32 + 75}{9} = \frac{52}{3}$

M1 A1

Answer: $\frac{52}{3}$

3 marks

Centres of Mass of Rods and Laminae

1 (2 marks)**(a)** Midpoint: 0.4 m from A

B1

Answer: 0.4 m from A

1 mark

(b) At the centre, 2 cm from AB and 3 cm from BC

B1

Answer: 2 cm from AB, 3 cm from BC

1 mark

2 (2 marks)

Distance $= \frac{1}{3} \times 9$

M1

$= 3 \text{ cm}$

A1

Answer: 3 cm

2 marks

3 (2 marks)

Distance $= \frac{4r}{3\pi} = \frac{4 \times 6}{3\pi}$

M1

$= 2.55 \text{ cm}$

A1

Answer: 2.55 cm

2 marks

4 (2 marks)

Masses are proportional to lengths: $\bar{x} = \frac{4 \times 2 + 3 \times 0}{7} = \frac{8}{7}$

M1

$\bar{y} = \frac{4 \times 0 + 3 \times 1.5}{7} = \frac{9}{14}$, so $(\frac{8}{7}, \frac{9}{14})$

A1

Answer: $(\frac{8}{7}, \frac{9}{14}) \text{ cm}$

2 marks

5 (2 marks)

Distance $= \frac{2r}{\pi} = \frac{10}{\pi}$

M1

$= 3.18 \text{ cm}$

A1

Answer: 3.18 cm

2 marks

6 (2 marks)

(a) $\frac{3r}{8} = \frac{3 \times 8}{8} = 3 \text{ cm}$

B1

Answer: 3 cm

1 mark

(b) $\frac{h}{4} = \frac{12}{4} = 3 \text{ cm}$ from the base

B1

Answer: 3 cm

1 mark

7 (4 marks)

Lengths 4, 3, 5 with centres (2, 0), (0, 1.5), (2, 1.5); total length 12

M1

$$\bar{x} = \frac{4 \times 2 + 3 \times 0 + 5 \times 2}{12} = 1.5$$

M1 A1

$$\bar{y} = \frac{4 \times 0 + 3 \times 1.5 + 5 \times 1.5}{12} = 1$$

A1

Answer: (1.5, 1) cm

4 marks

8 (3 marks)Half-angle $\alpha = 60^\circ = \frac{\pi}{3}$ radians

B1

$$\text{Distance} = \frac{2r \sin \alpha}{3\alpha} = \frac{2 \times 10 \times \sin 60^\circ}{3 \times \frac{\pi}{3}}$$

M1

= 5.51 cm

A1

Answer: 5.51 cm

3 marks

9 (5 marks)

$$\text{Area} = \int_0^2 x^2 dx = \frac{8}{3}$$

B1

$$\bar{x} \times \frac{8}{3} = \int_0^2 x \cdot x^2 dx = 4, \text{ so } \bar{x} = 1.5$$

M1 A1

$$\bar{y} \times \frac{8}{3} = \frac{1}{2} \int_0^2 x^4 dx = \frac{16}{5}, \text{ so } \bar{y} = 1.2$$

M1 A1

Answer: (1.5, 1.2)

5 marks

10 (3 marks)

The centre of mass is at the mean of the vertices

M1

$$\bar{x} = \frac{1 + 7 + 4}{3} = 4$$

A1

$$\bar{y} = \frac{1 + 2 + 8}{3} = \frac{11}{3}$$

A1

Answer: (4, $\frac{11}{3}$)

3 marks

11 (4 marks)Half-angle $\alpha = \frac{\pi}{4}$; distance from O along the bisector = $\frac{2r \sin \alpha}{3\alpha} = 5.40$

M1 A1

$$\text{Distance from each edge} = 5.40 \times \sin 45^\circ = \frac{4r}{3\pi} = \frac{4 \times 9}{3\pi}$$

M1

= 3.82 cm

A1

Answer: 3.82 cm

4 marks

12 (3 marks)Half-angle $\alpha = 30^\circ = \frac{\pi}{6}$ radians

B1

$$\text{Distance} = \frac{r \sin \alpha}{\alpha} = \frac{12 \sin 30^\circ}{\frac{\pi}{6}}$$

M1

= 11.5 cm

A1

Answer: 11.5 cm

3 marks

13 (3 marks)

Rod: mass 2 at 0.6 m from A; particle: mass 1 at 1.2 m

B1

$$\text{Distance} = \frac{2 \times 0.6 + 1 \times 1.2}{3}$$

M1

= 0.8 m

A1

Answer: 0.8 m

3 marks

14 (5 marks)

Arc: length πr , distance from the diameter $\frac{2r}{\pi}$ B1

Diameter: length $2r$, distance 0 B1

$$\text{Distance} = \frac{\pi r \times \frac{2r}{\pi} + 2r \times 0}{\pi r + 2r}$$
M1

$$= \frac{2r^2}{r(\pi + 2)} = \frac{2r}{\pi + 2}$$
A1

$$r = 5: \frac{10}{\pi + 2} = 1.94 \text{ cm}$$
A1

Answer: 1.94 cm 5 marks

15 (7 marks)

(a) $\text{Area} = \int_0^{\pi} \sin x \, dx = 2$ M1 A1

By symmetry about $x = \frac{\pi}{2}$, $\bar{x} = \frac{\pi}{2}$ B1

Answer: area 2; $x = \frac{\pi}{2}$ 3 marks

(b) $\bar{y} \times 2 = \frac{1}{2} \int_0^{\pi} \sin^2 x \, dx = \frac{1}{4} \int_0^{\pi} (1 - \cos 2x) \, dx$ M1 A1

$$= \frac{1}{4} [x - \frac{1}{2} \sin 2x]_0^{\pi} = \frac{\pi}{4}$$
M1

$$\bar{y} = \frac{\pi}{8}$$
A1

Answer: The y-coordinate of the centre of mass is $\frac{\pi}{8}$ 4 marks

16 (6 marks)

$$\text{Area} = \int_0^4 x^{1/2} \, dx = \frac{2}{3} [x^{3/2}]_0^4 = \frac{16}{3}$$
M1 A1

$$\bar{x} \times \frac{16}{3} = \int_0^4 x^{3/2} \, dx = \frac{2}{5} \times 32 = \frac{64}{5}$$
M1

$$\bar{x} = \frac{12}{5} = 2.4$$
A1

$$\bar{y} \times \frac{16}{3} = \frac{1}{2} \int_0^4 x \, dx = 4, \text{ so } \bar{y} = \frac{3}{4}$$
M1 A1

Answer: (2.4, 0.75) 6 marks

17 (5 marks)

Height of A above BC = $\sqrt{13^2 - 5^2} = 12 \text{ cm}$ M1 A1

AB and AC have centres 6 cm above BC; BC has centre on BC B1

$$\text{Distance} = \frac{10 \times 0 + 13 \times 6 + 13 \times 6}{36}$$
M1

$$= \frac{156}{36} = \frac{13}{3} = 4.33 \text{ cm}$$
A1

Answer: $\frac{13}{3} = 4.33 \text{ cm}$ 5 marks

18 (5 marks)

(a) $V = \pi \int_0^3 y^2 \, dx = \pi \int_0^3 x^4 \, dx$ M1

$$= \pi [\frac{x^5}{5}]_0^3 = \frac{243\pi}{5}$$
A1

Answer: The volume of the solid is $\frac{243\pi}{5}$ 2 marks

(b) $\bar{x} V = \pi \int_0^3 x \cdot x^4 \, dx = \pi [\frac{x^6}{6}]_0^3 = \frac{729\pi}{6}$ M1 A1

$$\bar{x} = \frac{729}{6} \div \frac{243}{5} = 2.5 \text{ units}$$
A1

Answer: 2.5**3 marks****19 (6 marks)**

$$\text{Area} = \int_1^3 \frac{1}{x} dx = \ln 3$$

M1 A1

$$\bar{x} \ln 3 = \int_1^3 x \cdot \frac{1}{x} dx = 2, \text{ so } \bar{x} = 1.82$$

M1 A1

$$\bar{y} \ln 3 = \frac{1}{2} \int_1^3 \frac{1}{x^2} dx = \frac{1}{2} \left(1 - \frac{1}{3}\right) = \frac{1}{3}$$

M1

$$\bar{y} = 0.303$$

A1**Answer: (1.82, 0.303)****6 marks****20 (6 marks)**

$$\text{Arc length} = 10 \times \frac{2\pi}{3} = 20.9 \text{ cm, centre of mass at } \frac{r \sin \alpha}{\alpha} = 8.27 \text{ cm from O}$$

M1 A1

Each radius has length 10 cm, with centre of mass $5 \cos 60^\circ = 2.5$ cm from O measured along the bisector

M1 A1

$$\text{Distance} = \frac{20.9 \times 8.27 + 20 \times 2.5}{20.9 + 20}$$

M1

$$= 5.45 \text{ cm}$$

A1**Answer: 5.45 cm****6 marks****21 (7 marks)**

Take the centre O as origin, the lamina being the region $-r \leq x \leq r, 0 \leq y \leq \sqrt{r^2 - x^2}$

B1

By symmetry the centre of mass lies on the y-axis

B1

$$\text{Area} = \frac{\pi r^2}{2}$$

B1

$$\text{Moment about the x-axis: } \frac{1}{2} \int_{-r}^r (r^2 - x^2) dx$$

M1

$$= \frac{1}{2} \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r = \frac{1}{2} \times \frac{4r^3}{3} = \frac{2r^3}{3}$$

M1 A1

$$\bar{y} = \frac{2r^3}{3} \div \frac{\pi r^2}{2} = \frac{4r}{3\pi}$$

A1**Answer: $\frac{4r}{3\pi}$** **7 marks****22 (6 marks)**

Rotate $y = \frac{rx}{h}, 0 \leq x \leq h$, about the x-axis (vertex at the origin)

B1

$$V = \pi \int_0^h \frac{r^2 x^2}{h^2} dx = \frac{\pi r^2 h}{3}$$

M1 A1

$$\bar{x} V = \pi \int_0^h x \frac{r^2 x^2}{h^2} dx = \frac{\pi r^2}{h^2} \times \frac{h^4}{4} = \frac{\pi r^2 h^2}{4}$$

M1 A1

$$\bar{x} = \frac{\pi r^2 h^2}{4} \div \frac{\pi r^2 h}{3} = \frac{3h}{4} \text{ from the vertex, i.e. } \frac{h}{4} \text{ from the base}$$

A1**Answer: $\frac{h}{4}$ from the base****6 marks****23 (6 marks)**

$$V = \pi \int_0^r (r^2 - x^2) dx = \pi \left(r^3 - \frac{r^3}{3} \right) = \frac{2\pi r^3}{3}$$

M1 A1

$$\bar{x} V = \pi \int_0^r x(r^2 - x^2) dx$$

M1

$$= \pi \left[\frac{r^2 x^2}{2} - \frac{x^4}{4} \right]_0^r = \frac{\pi r^4}{4}$$

M1 A1

$$\bar{x} = \frac{\pi r^4}{4} \div \frac{2\pi r^3}{3} = \frac{3r}{8}$$

A1**Answer: $\frac{3r}{8}$** **6 marks**

24 (7 marks)**(a)** Intersections at $x = 0$ and $x = 1$ **B1**

$$\text{Area} = \int_0^1 (x - x^2) dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

M1 A1

$$\text{Answer: } \frac{1}{6}$$

3 marks

$$\text{(b)} \bar{x} \times \frac{1}{6} = \int_0^1 x(x - x^2) dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \text{ so } \bar{x} = \frac{1}{2}$$

M1 A1

$$\bar{y} \times \frac{1}{6} = \frac{1}{2} \int_0^1 (x^2 - x^4) dx = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15}$$

M1

$$\bar{y} = \frac{2}{5}$$

A1

$$\text{Answer: } \left(\frac{1}{2}, \frac{2}{5} \right)$$

4 marks**25 (6 marks)** $\frac{4r}{3\pi}$ is the result for a semicircular lamina (area), not for a wire (arc)**E1**

$$\text{For an arc of half-angle } \alpha = \frac{\pi}{2}: \text{distance} = \frac{r \sin \alpha}{\alpha} = \frac{2r}{\pi}$$

M1 A1

$$= \frac{12}{\pi} = 3.82 \text{ cm}$$

A1

$$\text{Lamina: } \frac{4 \times 6}{3\pi} = 2.55 \text{ cm, which is closer to the diameter}$$

M1 A1**Answer: 3.82 cm for the wire (lamina: 2.55 cm)****6 marks****26 (6 marks)**

$$V = \pi \int_0^4 x^2 dy = \pi \int_0^4 y dy$$

M1

$$= \pi \left[\frac{y^2}{2} \right]_0^4 = 8\pi$$

A1

$$\bar{y} V = \pi \int_0^4 y \cdot y dy = \pi \left[\frac{y^3}{3} \right]_0^4 = \frac{64\pi}{3}$$

M1 A1

$$\bar{y} = \frac{64\pi}{3} \div 8\pi = \frac{8}{3} = 2.67 \text{ units from the vertex}$$

A1

(on the axis, by symmetry)

B1

$$\text{Answer: } V = 8\pi; \frac{8}{3} \text{ from the vertex}$$

6 marks**27 (9 marks)**

$$\text{(a) Area} = \int_{-2}^2 (4 - x^2) dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}$$

M1 A1

By symmetry the centre of mass lies on the y-axis

B1

$$\bar{y} \times \frac{32}{3} = \frac{1}{2} \int_{-2}^2 (4 - x^2)^2 dx = \frac{1}{2} \int_{-2}^2 (16 - 8x^2 + x^4) dx$$

M1

$$= \frac{1}{2} \times \frac{512}{15} = \frac{256}{15}$$

A1

$$\bar{y} = \frac{256}{15} \div \frac{32}{3} = \frac{8}{5}$$

A1

$$\text{Answer: } \left(0, \frac{8}{5} \right)$$

6 marks**(b)** The centre of mass G is vertically below A, so angle OAG is the angle between the base AO and the vertical**M1**

$$\tan(\text{OAG}) = \frac{OG}{OA} = \frac{\frac{8}{5}}{2} = 0.8$$

M1

$$\text{Angle} = 38.7^\circ \approx 39^\circ$$

A1**Answer: 39°****3 marks**

28 (8 marks)

Let the shell be $0 \leq z \leq r$, with z measured from the centre O along the axis

B1

A thin band between heights z and $z + dz$ has area $2\pi r dz$

M1 A1

$$\text{Total area} = 2\pi r \int_0^r dz = 2\pi r^2$$

M1

$$\text{Moment about the plane } z = 0: \int_0^r z \times 2\pi r dz = 2\pi r \times \frac{r^2}{2} = \pi r^3$$

M1 A1

$$\text{Distance} = \frac{\pi r^3}{2\pi r^2} = \frac{r}{2}$$

A1

The distance from O is $r/2$, independent of the shape of the strips: each band's height is measured directly

E1

$$\text{Answer: } \frac{r}{2}$$

8 marks

29 (12 marks)

$$(a) V = \pi \int_0^{\pi/2} y^2 dx = \pi \int_0^{\pi/2} \cos^2 x dx$$

M1

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \text{ so } V = \pi \left[\frac{x}{2} + \frac{\sin 2x}{4} \right]_0^{\pi/2}$$

M1 A1

$$= \pi \times \frac{\pi}{4} = \frac{\pi^2}{4}$$

A1

$$\text{Answer: } V = \frac{\pi^2}{4}$$

4 marks

$$(b) \bar{x} V = \pi \int_0^{\pi/2} x \cos^2 x dx = \frac{\pi}{2} \int_0^{\pi/2} x(1 + \cos 2x) dx$$

M1 A1

$$\int x \cos 2x dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$$

M1 A1

$$\int_0^{\pi/2} x(1 + \cos 2x) dx = \left[\frac{x^2}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \right]_0^{\pi/2} = \frac{\pi^2}{8} - \frac{1}{2}$$

M1 A1

$$\bar{x} = \frac{\pi}{2} \left(\frac{\pi^2}{8} - \frac{1}{2} \right) \times \frac{4}{\pi^2} = \frac{\pi}{4} - \frac{1}{\pi}$$

M1

$$= 0.467$$

A1

$$\text{Answer: } \frac{\pi}{4} - \frac{1}{\pi} = 0.467$$

8 marks

30 (9 marks)

$$\text{Area} = \int_1^e \ln x dx = [x \ln x - x]_1^e = 1$$

M1 A1

$$\bar{x} \times 1 = \int_1^e x \ln x dx = \left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{x}{2} dx$$

M1 A1

$$= \frac{e^2}{2} - \frac{1}{4}(e^2 - 1) = \frac{e^2 + 1}{4}$$

A1

$$\bar{y} \times 1 = \frac{1}{2} \int_1^e (\ln x)^2 dx = \frac{1}{2} [x(\ln x)^2 - 2x \ln x + 2x]_1^e$$

M1 A1

$$= \frac{1}{2} [(e - 2e + 2e) - 2] = \frac{e - 2}{2}$$

A1

$$\text{Coordinates } \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2} \right) = (2.10, 0.359)$$

A1

$$\text{Answer: } \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2} \right) = (2.10, 0.359)$$

9 marks

Composite Bodies

1 (3 marks)

Rectangle: area 24 at (3, 2); square: area 4 at (5, 3)

$$\bar{x} = \frac{24 \times 3 + 4 \times 5}{20} = 2.6$$

M1 A1

$$\bar{y} = \frac{24 \times 2 + 4 \times 3}{20} = 1.8, \text{ so } (2.6, 1.8)$$

A1

Answer: (2.6, 1.8)

3 marks

2 (3 marks)

Vertical rectangle: area 12 at (1, 3); horizontal rectangle: area 8 at (4, 1)

$$\bar{x} = \frac{12 \times 1 + 8 \times 4}{20} = 2.2$$

M1 A1

$$\bar{y} = \frac{12 \times 3 + 8 \times 1}{20} = 2.2$$

A1

Answer: (2.2, 2.2)

3 marks

3 (3 marks)

Masses $\propto$ volumes: $9\pi \times 4 = 36\pi$ and $\pi \times 2 = 2\pi$, at heights 2 and 5

B1

$$\text{Height} = \frac{36 \times 2 + 2 \times 5}{38}$$

M1

$$= 2.16 \text{ cm}$$

A1

Answer: 2.16 cm

3 marks

4 (3 marks)

Bar: area 12, centre 1 cm below the top; stem: area 8, centre $2 + 2 = 4$ cm below the top

B1

$$\text{Distance} = \frac{12 \times 1 + 8 \times 4}{20}$$

M1

$$= 2.2 \text{ cm}$$

A1

Answer: 2.2 cm

3 marks

5 (3 marks)

Areas: 25π (whole disc) and 4π (hole): remaining area 21π

B1

$$\text{Distance} = \frac{4\pi \times 3}{21\pi}$$

M1

$$= \frac{4}{7} = 0.571 \text{ cm from O, on the opposite side to the hole}$$

A1

Answer: $\frac{4}{7} = 0.571$ cm

3 marks

6 (3 marks)

Lamina: mass 6 at 3 cm from OC; particle: mass 1 at 6 cm

B1

$$\text{Distance} = \frac{6 \times 3 + 1 \times 6}{7} = \frac{24}{7} = 3.43 \text{ cm}$$

M1 A1

Answer: 3.43 cm

3 marks

7 (4 marks)

Rectangle: area 48 at 4 cm; semicircle: area $\frac{9\pi}{2}$ at $8 + \frac{4 \times 3}{3\pi} = 9.27 \text{ cm}$

B1 B1

$$\text{Distance} = \frac{48 \times 4 + \frac{9\pi}{2} \times 9.273}{48 + \frac{9\pi}{2}}$$

M1

$$= 5.20 \text{ cm}$$

A1

Answer: 5.20 cm

4 marks

8 (4 marks)

Hemisphere: $V = \frac{2}{3}\pi \times 6^3 = 144\pi$, centre $\frac{3r}{8} = 2.25$ cm below the plane face

B1

Cone: $V = \frac{1}{3}\pi \times 6^2 \times 12 = 144\pi$, centre $\frac{h}{4} = 3$ cm above the plane face

B1

Distance above the face = $\frac{144 \times 3 - 144 \times 2.25}{288}$

M1

= 0.375 cm, in the cone

A1

Answer: 0.375 cm above the plane face, in the cone

4 marks**9 (4 marks)**

Square: area 64 at 4 cm; triangle: area 24 at $8 + \frac{6}{3} = 10$ cm

B1 B1

Distance = $\frac{64 \times 4 + 24 \times 10}{88}$

M1

= $\frac{496}{88} = 5.64$ cm

A1

Answer: 5.64 cm

4 marks**10 (4 marks)**

Rectangle: area 60 at 3 cm; semicircle: area 2π at $\frac{4 \times 2}{3\pi} = 0.849$ cm

B1 B1

Distance = $\frac{60 \times 3 - 2\pi \times \frac{8}{3\pi}}{60 - 2\pi}$

M1

= 3.25 cm

A1

Answer: 3.25 cm

4 marks**11 (4 marks)**

Cylinder: $V = 16\pi \times 10 = 160\pi$, centre 5 cm from the undrilled end

B1

Hole: $V = 4\pi \times 5 = 20\pi$, centre $10 - 2.5 = 7.5$ cm from the undrilled end

B1

Distance = $\frac{160 \times 5 - 20 \times 7.5}{140}$

M1

= 4.64 cm

A1

Answer: 4.64 cm

4 marks**12 (3 marks)**

AB and CD each have length 4 and centre 2 cm from BC; BC has length 6 and centre on BC

B1

Distance = $\frac{4 \times 2 + 4 \times 2 + 6 \times 0}{14}$

M1

= $\frac{8}{7} = 1.14$ cm

A1

Answer: 1.14 cm

3 marks**13 (4 marks)**

Square: area 16 at 2 cm from AB; triangle: area $\frac{1}{2} \times 4 \times 3 = 6$ at $\frac{3}{3} = 1$ cm from AB

B1 B1

Distance = $\frac{16 \times 2 - 6 \times 1}{10}$

M1

= 2.6 cm

A1

Answer: 2.6 cm

4 marks

14 (6 marks)

Large semicircle: area $\frac{9\pi a^2}{2}$, centre $\frac{4(3a)}{3\pi} = \frac{4a}{\pi}$ from O B1 B1

Removed semicircle: area $\frac{4\pi a^2}{2} = 2\pi a^2$, centre $\frac{4(2a)}{3\pi} = \frac{8a}{3\pi}$ from O B1

Lamina area = $\frac{9\pi a^2}{2} - 2\pi a^2 = \frac{5\pi a^2}{2}$ M1

Moment: $\frac{9\pi a^2}{2} \times \frac{4a}{\pi} - 2\pi a^2 \times \frac{8a}{3\pi} = 18a^3 - \frac{16a^3}{3} = \frac{38a^3}{3}$ M1

Distance = $\frac{38a^3}{3} \div \frac{5\pi a^2}{2} = \frac{76a}{15\pi} = 1.61 a$ A1

Answer: $\frac{76a}{15\pi} = 1.61 a$ 6 marks

15 (6 marks)

Cylinder: mass $\propto \pi r^2 h$, centre $\frac{h}{2}$ from the common face B1

Hemisphere: mass $\propto \frac{2}{3}\pi r^3$, centre $\frac{3r}{8}$ from the common face on the other side B1

Moments about the common face: $\pi r^2 h \times \frac{h}{2} = \frac{2}{3}\pi r^3 \times \frac{3r}{8}$ M1 A1

$\frac{h^2}{2} = \frac{r^2}{4}$ M1

$h^2 = \frac{r^2}{2}$, so $h = \frac{r}{\sqrt{2}}$ A1

Answer: $h = \frac{r}{\sqrt{2}}$ 6 marks

16 (5 marks)

Cylinder: $V = 96\pi$, centre 3 cm from the undrilled end B1

Cone removed: $V = \frac{1}{3}\pi \times 16 \times 6 = 32\pi$, centre $\frac{6}{4} = 1.5$ cm from its base, so 4.5 cm from the undrilled end M1 A1

Distance = $\frac{96 \times 3 - 32 \times 4.5}{64}$ M1

= 2.25 cm A1

Answer: 2.25 cm 5 marks

17 (6 marks)

Volumes: $\frac{2}{3}\pi r^3$ and $\frac{2}{3}\pi(\frac{r}{2})^3 = \frac{1}{8}$ of the larger B1

Centres: $\frac{3r}{8}$ and $\frac{3}{8} \times \frac{r}{2} = \frac{3r}{16}$ from O B1

Distance = $\frac{V \times \frac{3r}{8} - \frac{V}{8} \times \frac{3r}{16}}{V - \frac{V}{8}}$ M1 A1

= $\frac{\frac{3r}{8} - \frac{3r}{128}}{\frac{7}{8}}$ M1

= $\frac{45r}{128} \times \frac{8}{7} = \frac{45r}{112}$ A1

Answer: The distance of the centre of mass of the remaining solid from O is $\frac{45r}{112}$ 6 marks

18 (6 marks)

Triangle: area $3r^2$, centre $\frac{3r}{3} = r$ below the diameter

B1

Semicircle: area $\frac{\pi r^2}{2}$, centre $\frac{4r}{3\pi}$ above the diameter

B1

$$\text{Distance above the diameter} = \frac{\frac{\pi r^2}{2} \times \frac{4r}{3\pi} - 3r^2 \times r}{3r^2 + \frac{\pi r^2}{2}}$$

M1

$$= \frac{-\frac{7}{3}}{3 + \frac{\pi}{2}} r = -0.510 r$$

M1 A1

So the centre of mass is 0.510 r below the diameter, in the triangle

A1

Answer: 0.510r below the diameter, in the triangle

6 marks

19 (6 marks)

(a) Volumes are proportional to the heights (same base): 4 and 10

B1

Smaller cone: centre $\frac{4}{4} = 1$ cm from O on its side; larger cone: centre $\frac{10}{4} = 2.5$ cm from O on the other side

B1

Moments about O, measuring towards the larger cone: $\frac{10 \times 2.5 - 4 \times 1}{14}$

M1

$$= \frac{21}{14} = 1.5 \text{ cm towards the larger cone, so it lies in the larger cone}$$

A1

Answer: 1.5 cm from O, in the larger cone

4 marks

(b) Distance = 10 – 1.5

M1

= 8.5 cm

A1

Answer: 8.5 cm

2 marks

20 (6 marks)

Take A as origin with AB along the x axis and AD along the y axis. Rectangle: area 96, centre (6, 4); hole: area 4π , centre (3, 4)

B1 B1

$$\bar{x} = \frac{96 \times 6 - 4\pi \times 3}{96 - 4\pi} = 6.45 \text{ cm, } \bar{y} = 4 \text{ cm by symmetry}$$

M1 A1

G is vertically below A, so the angle between AB and the vertical is the angle GAB: $\tan = \frac{4}{6.452}$

M1

Angle = 31.8°

A1

Answer: 31.8°

6 marks

21 (7 marks)

Square OABC: area 144 at 6 cm from OC; removed square: area a^2 at $(12 - \frac{a}{2})$ cm

B1 B1

$$\frac{144 \times 6 - a^2(12 - \frac{a}{2})}{144 - a^2} = 5.5$$

M1

$$864 - 12a^2 + \frac{a^3}{2} = 792 - 5.5a^2$$

M1

$$\frac{a^3}{2} - 6.5a^2 + 72 = 0, \text{ so } a^3 - 13a^2 + 144 = 0$$

A1

$a = 4$ is a root: $64 - 208 + 144 = 0$, and the factorisation $(a - 4)(a^2 - 9a - 36) = (a - 4)(a - 12)(a + 3)$

M1

$a < 12$ and $a > 0$, so $a = 4$

A1

Answer: $a = 4$

7 marks

22 (6 marks)

(a) Cone: $\frac{1}{3}\pi r^2 h$, centre $\frac{h}{4}$ above the face B1

Hemisphere: $\frac{2}{3}\pi r^3$, centre $\frac{3r}{8}$ below the face B1

Equal moments: $\frac{\pi r^2 h}{3} \times \frac{h}{4} = \frac{2\pi r^3}{3} \times \frac{3r}{8}$ M1

$\frac{h^2}{12} = \frac{r^2}{4}$, so $h^2 = 3r^2$, $h = \sqrt{3} r$ A1

Answer: The centre of mass of the solid lies on the common plane face if $h = \sqrt{3} r$ 4 marks

(b) The cone's moment $\frac{\pi r^2 h^2}{12}$ then exceeds the hemisphere's, so the centre of mass is above the face, in the cone E1 A1

Answer: in the cone 2 marks

23 (6 marks)

The hole has been added instead of removed: the remaining area is $25\pi - 4\pi = 21\pi$ E1

Taking moments about O: $21\pi \times d = 25\pi \times 0 - 4\pi \times 3$ M1

$d = -\frac{12}{21} = -\frac{4}{7}$ A1

Distance = $\frac{4}{7} = 0.571$ cm from O A1

The negative sign shows that it lies on the opposite side of O from H B1

Check: the removed piece pulls the centre of mass away from the hole B1

Answer: 0.571 cm from O, on the side away from H 6 marks

24 (6 marks)

Lengths: AB = CD = 6, DA = 4, BC = 4, arc = 2π B1

Distances from AD: AB, CD: 3; DA: 0; BC: 6; arc: $6 + \frac{2 \times 2}{\pi} = 7.27$ B1 B1

Distance = $\frac{6 \times 3 + 6 \times 3 + 4 \times 0 + 4 \times 6 + 2\pi \times 7.273}{26.28}$ M1 A1

= 4.02 cm A1

Answer: 4.02 cm 6 marks

25 (9 marks)

(a) Take A as origin, AB along the x axis and AD along the y axis. The folded shape is the square with triangle AEF removed, plus a second layer of triangle AEF reflected in EF B1

Square: area 144, centre (6, 6). Triangle AEF: area $\frac{1}{2} \times 6 \times 6 = 18$, centroid (2, 2) B1

Reflection of (2, 2) in the line $x + y = 6$ is (4, 4): the folded triangle has centroid (4, 4) M1 A1

$\bar{x} = \frac{144 \times 6 - 18 \times 2 + 18 \times 4}{144} = 6.25$ cm, and $\bar{y} = 6.25$ cm by symmetry M1 A1

Answer: 6.25 cm from AB and 6.25 cm from AD 6 marks

(b) G is at (6.25, 6.25) and B is at (12, 0), so BG = (-5.75, 6.25) M1

BG is vertical and BC points along the y direction: $\tan(\text{angle}) = \frac{5.75}{6.25}$ M1

Angle = 42.6° A1

Answer: 42.6° 3 marks

26 (6 marks)

Hemisphere: mass $\frac{2}{3}\pi r^3\rho$, centre $\frac{3r}{8}$ below the face B1

Cone: $V = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$, mass $\frac{2}{3}\pi r^3 k\rho$, centre $\frac{2r}{4} = \frac{r}{2}$ above the face B1

Moments about the face: $\frac{2}{3}\pi r^3 k\rho \times \frac{r}{2} = \frac{2}{3}\pi r^3\rho \times \frac{3r}{8}$ M1 A1

$$\frac{k}{2} = \frac{3}{8} \quad \text{M1}$$

$$k = \frac{3}{4} \quad \text{A1}$$

Answer: $\frac{3}{4}$ 6 marks

27 (10 marks)

(a) Rectangle: area $2ab$, centre $\frac{b}{2}$ from AB B1

Semicircle: area $\frac{\pi a^2}{2}$, centre $\frac{4a}{3\pi}$ from AB on the other side B1

Centre of mass on AB: $2ab \times \frac{b}{2} = \frac{\pi a^2}{2} \times \frac{4a}{3\pi}$ M1 A1

$$ab^2 = \frac{2a^3}{3}, \text{ so } b = a\sqrt{\frac{2}{3}} \quad \text{A1}$$

Answer: The centre of mass of the lamina lies on AB if $b = a\sqrt{\frac{2}{3}}$ 5 marks

(b) Rectangle: area 72 at 3 above AB; semicircle: area 18π at $\frac{4 \times 6}{3\pi} = \frac{8}{\pi}$ below AB B1

$$\text{Distance above AB} = \frac{72 \times 3 - 18\pi \times \frac{8}{\pi}}{72 + 18\pi} \quad \text{M1 A1}$$

$$= \frac{216 - 144}{72 + 56.55} = 0.560 \text{ cm above AB (on the rectangle's side)} \quad \text{A1}$$

Answer: 0.560 cm on the rectangle's side 4 marks

(c) The rectangle's moment about AB, ab^2 , then exceeds the semicircle's $\frac{2a^3}{3}$ E1

Answer: $ab^2 > \frac{2a^3}{3}$ 1 mark

28 (10 marks)

(a) Rectangle with sides a and h : area ah , centre $\frac{h}{2}$ from DC B1

Triangle with base $(b - a)$ and height h : area $\frac{1}{2}(b - a)h$, centre $\frac{h}{3}$ from DC B1

$$\text{Total area} = \frac{1}{2}h(a + b) \quad \text{B1}$$

$$\text{Moment of the two parts about DC: } ah \times \frac{h}{2} + \frac{1}{2}(b - a)h \times \frac{h}{3} = \frac{h^2(2a + b)}{6} \quad \text{M1}$$

$$\text{Distance} = \frac{h^2(2a + b)}{6} \div \frac{h(a + b)}{2} = \frac{h(2a + b)}{3(a + b)} \quad \text{M1 A1}$$

Answer: The distance of its centre of mass from DC is $\frac{h(b + 2a)}{3(a + b)}$ 6 marks

$$(b) \frac{6(10 + 8)}{3(14)} \quad \text{M1}$$

$$= \frac{108}{42} = \frac{18}{7} = 2.57 \text{ cm} \quad \text{A1}$$

Answer: $\frac{18}{7} = 2.57 \text{ cm}$ 2 marks

$$(c) a = b: \frac{3bh}{6b} = \frac{h}{2} \text{ (a rectangle)} \quad \text{B1}$$

$$a = 0: \frac{bh}{3b} = \frac{h}{3} \text{ (a triangle with base DC)} \quad \text{B1}$$

Answer: $\frac{h}{2}; \frac{h}{3}$ **2 marks****29 (8 marks)**

Parabolic region: area = $\int_{-2}^2 (4 - x^2) dx = \frac{32}{3}$

M1 A1

$$\bar{y} \times \frac{32}{3} = \frac{1}{2} \int_{-2}^2 (16 - x^4) dx = \frac{1}{2} [16x - \frac{x^5}{5}]_{-2}^2 = \frac{1}{2} \times \frac{256}{5} = \frac{128}{5}$$

M1 A1

So $\bar{y} = \frac{128}{5} \div \frac{32}{3} = \frac{12}{5}$

A1

Hole: area $\frac{\pi}{4}$ at $y = 3$

B1

$$\text{Distance} = \frac{\frac{32}{3} \times \frac{12}{5} - \frac{\pi}{4} \times 3}{\frac{32}{3} - \frac{\pi}{4}}$$

M1

$$= \frac{25.6 - 2.356}{10.667 - 0.785} = 2.35 \text{ cm}$$

A1**Answer: 2.35 cm****8 marks****30 (9 marks)**

The removed cone is similar, with height h and base radius $\frac{R}{3}$

B1

Large cone: $V = \frac{1}{3} \pi R^2 (3h) = \pi R^2 h$, with centre $\frac{1}{4}$ of the height from the base: $\frac{3h}{4}$

M1 A1

Small cone: $V = \frac{1}{3} \pi (\frac{R}{3})^2 h = \frac{\pi R^2 h}{27}$, centre $\frac{h}{4}$ above its base, which is $2h$ above the large base: $\frac{9h}{4}$

M1 A1

Frustum: $V = \pi R^2 h (1 - \frac{1}{27}) = \frac{26\pi R^2 h}{27}$

B1

$$\text{Distance} = \frac{\pi R^2 h \times \frac{3h}{4} - \frac{\pi R^2 h}{27} \times \frac{9h}{4}}{\frac{26\pi R^2 h}{27}} = \frac{\frac{3}{4} - \frac{1}{12}}{\frac{26}{27}} h$$

M1

$$= \frac{2}{3} \times \frac{27}{26} h = \frac{9h}{13}$$

A1

$h = 13: 9 \text{ cm}$

B1**Answer: $\frac{9h}{13}; 9 \text{ cm}$** **9 marks****Equilibrium of Rigid Bodies****1 (2 marks)**

By symmetry, $R_A = R_B$, and $R_A + R_B = 20g$

M1

$R_A = R_B = 98 \text{ N}$

A1**Answer: 98 N each****2 marks****2 (2 marks)**

Moments about A: $4R_C = 30g \times 3$

M1

$R_C = 220 \text{ N}$

A1**Answer: 220 N****2 marks****3 (2 marks)**

Moments about the pivot: $30g \times d = 40g \times 1.5$

M1

$d = 2 \text{ m}$

A1**Answer: 2 m****2 marks**

4 (2 marks)

Resolving vertically: the smooth wall gives no vertical force

M1

$$R = 15 \times 9.8 = 147 \text{ N}$$

A1

Answer: 147 N

2 marks

5 (2 marks)

$$\text{Moments about A: } T \times AB = 4g \times \frac{AB}{2}$$

M1

$$T = 2g = 19.6 \text{ N}$$

A1

Answer: 19.6 N

2 marks

6 (2 marks)Moments about the pivot at distance x from A: $3gx = 5g(1.2 - x)$

M1

$$8x = 6, \text{ so } x = 0.75 \text{ m}$$

A1

Answer: 0.75 m

2 marks

7 (4 marks)

$$\text{Moments about A: } 4R_C = 40g \times 2.5 + 30g \times 5$$

M1 A1

$$R_C = 612 \text{ N}$$

A1

$$R_A = 70g - R_C = 73.5 \text{ N}$$

A1

Answer: $R_A = 73.5 \text{ N}$, $R_C = 612 \text{ N}$

4 marks

8 (5 marks)

$$\text{Moments about A: } T \sin 35^\circ \times 1.6 = 6g \times 0.8$$

M1 A1

$$T = \frac{3g}{\sin 35^\circ} = 51.3 \text{ N}$$

A1

$$\text{Horizontal: } H = T \cos 35^\circ = 42.0 \text{ N (away from the wall)}$$

M1

$$\text{Vertical: } V = 6g - T \sin 35^\circ = 29.4 \text{ N (upwards)}$$

A1

Answer: $T = 51.3 \text{ N}$, $H = 42.0 \text{ N}$, $V = 29.4 \text{ N}$

5 marks

9 (5 marks)

$$\text{Moments about A: } S \times 5 \sin 60^\circ = 20g \times 2.5 \cos 60^\circ$$

M1 A1

$$S = 56.6 \text{ N, and friction } F = S$$

A1

$$R = 20g = 196 \text{ N, } \mu \geq \frac{F}{R} = 0.289$$

M1 A1

Answer: $F = 56.6 \text{ N}$, $\mu \geq 0.289$

5 marks

10 (4 marks)

$$\text{Moments about A: } 4T_B = 50g \times 2 + 30g \times 1$$

M1 A1

$$T_B = 318 \text{ N}$$

A1

$$T_A = 80g - T_B = 466 \text{ N}$$

A1

Answer: $T_A = 466 \text{ N}$, $T_B = 318 \text{ N}$

4 marks

11 (5 marks)

$$\text{Moments about A: } 1.5T = 5g \times 1 + 2g \times 2 = 9g$$

M1 A1

$$T = 6g = 58.8 \text{ N}$$

A1

Resolving vertically: $V + T = 7g$, so $V = g = 9.8 \text{ N}$, upwards (the force is vertical as there is no horizontal force)

M1 A1

Answer: $T = 58.8 \text{ N}$; 9.8 N upwards

5 marks

12 (3 marks)

$$\text{Moments about A: } 2R_B = 30 \times 0.8$$

M1

$$R_B = 12 \text{ N}$$

A1

$$R_A = 30 - 12 = 18 \text{ N}$$

A1

Answer: $R_A = 18 \text{ N}$, $R_B = 12 \text{ N}$

3 marks

13 (4 marks)

CD = 4 – 0.5 – 1 = 2.5 m; the centre of mass is 1.5 m from C

B1

Moments about C: $2.5R_D = 25g \times 1.5$

M1 A1

$R_D = 147 \text{ N}$, $R_C = 98 \text{ N}$

A1

Answer: $R_C = 98 \text{ N}$, $R_D = 147 \text{ N}$

4 marks

14 (6 marks)

Vertical: $R = 100g = 980 \text{ N}$

B1

Moments about A: $S \times 6 \sin 65^\circ = 20g \times 3 \cos 65^\circ + 80g \times 4 \cos 65^\circ$

M1 A1

$$S = \frac{g \times 380}{6 \tan 65^\circ} = 289 \text{ N}$$

A1

Horizontal: $F = S$; limiting friction $F = \mu R$

M1

$$\mu = \frac{289}{980} = 0.295$$

A1

Answer: 0.295

6 marks

15 (7 marks)

Moments about A: $T \times 2 \sin 30^\circ = 8g \times 1 \times \cos 30^\circ$

M1 A1

$T = 67.9 \text{ N}$

A1

Horizontal: $H = T = 67.9 \text{ N}$, vertical: $V = 8g = 78.4 \text{ N}$

M1

Magnitude $\sqrt{H^2 + V^2} = 104 \text{ N}$

A1

Angle with the horizontal = $\arctan \frac{78.4}{67.9} = 49.1^\circ$

M1 A1

Answer: $T = 67.9 \text{ N}$; 104 N at 49.1° above the horizontal

7 marks

16 (5 marks)

The beam tilts about C when the reaction at A becomes zero

B1

Moments about C: $60g(x - 4) = 40g(4 - 3)$

M1 A1

$$x - 4 = \frac{2}{3}$$

M1

$x = 4.67 \text{ m}$ from A

A1

Answer: 4.67 m

5 marks

17 (7 marks)

The peg is smooth, so its reaction S is perpendicular to the rod

B1

Moments about A: $S \times 2 = 8g \times 1.5 \cos 50^\circ$

M1

$S = 37.8 \text{ N}$

A1

Vertical: $R + S \cos 50^\circ = 8g$, so $R = 54.1 \text{ N}$

M1 A1

Horizontal: $F = S \sin 50^\circ = 29.0 \text{ N}$, $\mu \geq \frac{F}{R} = 0.535$

M1 A1

Answer: $S = 37.8 \text{ N}$, $\mu \geq 0.535$

7 marks

18 (7 marks)

(a) Moments about the foot: $S \times L \sin \theta = mg \times \frac{L}{2} \cos \theta$, so $S = \frac{mg}{2} \cot \theta$

M1 A1

$F = S \leq \mu R = 0.4mg$, so $\frac{1}{2} \cot \theta \leq 0.4$

M1

$$\tan \theta \geq \frac{1}{0.8} = 1.25, \theta \geq 51.3^\circ$$

A1

Answer: $\theta \geq 51.3^\circ$

4 marks

(b) Moments about the foot: $S L \sin \theta = mg \frac{L}{2} \cos \theta + 2mgL \cos \theta$, so $S = \frac{5}{2} mg \cot \theta$

M1

$R = 3mg$: $\frac{5}{2} mg \cot \theta \leq 0.4 \times 3mg$

M1

$\cot \theta \leq 0.48$, so $\theta \geq 64.4^\circ$

A1

Answer: $\theta \geq 64.4^\circ$

3 marks

19 (6 marks)

Moments about A: $T \sin 30^\circ \times 4 = 40g \times 2$

M1 A1

$T = 392 \text{ N}$

A1

Horizontal: $N = T \cos 30^\circ = 339 \text{ N}$. Vertical: $F + T \sin 30^\circ = 40g$, so $F = 196 \text{ N}$

M1 A1

$F \leq \mu N$ gives $\mu \geq \frac{196}{339} = 0.577$

A1

Answer: $T = 392 \text{ N}$, $\mu \geq 0.577$

6 marks

20 (6 marks)

Moments about B: $H_A \times 1 = 200 \times 1$ (the weight acts 1 m horizontally from B)

M1 A1

$H_A = 200 \text{ N}$

A1

Horizontal equilibrium: $H_B = 200 \text{ N}$ in the opposite direction

B1

Resultant at B = $\sqrt{200^2 + 200^2} = 283 \text{ N}$

M1 A1

Answer: $H_A = H_B = 200 \text{ N}$; 283 N

6 marks

21 (8 marks)

Vertical: $R = 90g = 882 \text{ N}$

B1

Greatest friction $F = 0.3 \times 882 = 265 \text{ N}$

M1 A1

Horizontal: $S = F$

B1

Moments about A with man at distance x : $S \times 5 \sin 60^\circ = 15g \times 2.5 \cos 60^\circ + 75g \times \cos 60^\circ$

M1 A1

$265 \times 5 \times 0.866 = 183.75 + 367.5x$, so $x = 2.62 \text{ m}$

M1 A1

Answer: 2.62 m

8 marks

22 (7 marks)

Angle $ABC = 45^\circ$ (since $AB = AC$)

B1

Moments about A: $T \sin 45^\circ \times 2 = 10g \times 1 + 4g \times 2$

M1 A1

$T = \frac{9g}{\sin 45^\circ} = 125 \text{ N}$

A1

Horizontal: $H = T \cos 45^\circ = 9g = 88.2 \text{ N}$

M1

Vertical: $V + T \sin 45^\circ = 14g$, so $V = 5g = 49 \text{ N}$

A1

Magnitude = 101 N , at 29.1° above the horizontal

A1

Answer: $T = 125 \text{ N}$; 101 N at 29.1° above the horizontal

7 marks

23 (6 marks)

$F = \mu R$ only applies in limiting equilibrium; in general $F \leq \mu R$ and F is whatever is needed for equilibrium

E1

Moments about A: $S \times 4 \sin 60^\circ = 12g \times 2 \cos 60^\circ$

M1

$S = \frac{12g}{2 \tan 60^\circ} = 33.9 \text{ N}$, so $F = S = 33.9 \text{ N}$

A1

$F < 58.8 \text{ N}$, so the ladder does not slip

B1

Limiting case: $\frac{1}{2} \cot \theta = \mu = 0.5$

M1

$\tan \theta = 1$, so $\theta = 45^\circ$

A1

Answer: $F = 33.9 \text{ N}$ ($< 58.8 \text{ N}$); $\theta_{\min} = 45^\circ$

6 marks

24 (7 marks)

Moments about A: $P \times 2 = 40 \times 1 \times \cos 30^\circ$

M1 A1

$P = 17.3 \text{ N}$

A1

P acts perpendicular to the rod, so it is at 30° to the vertical: components $P \sin 30^\circ$ horizontal and $P \cos 30^\circ$ upwards

B1

Hinge components: horizontal $X = P \sin 30^\circ = 8.66 \text{ N}$ (opposite to the horizontal component of P)

M1

Vertical $Y = 40 - P \cos 30^\circ = 25 \text{ N}$

M1

Magnitude = 26.5 N

A1

Answer: $P = 17.3 \text{ N}$; hinge force 26.5 N

7 marks

25 (7 marks)

By symmetry, the normal reaction at B is $R = 10g = 98 \text{ N}$

M1 A1

Consider AB. B is $3 \cos 60^\circ = 1.5 \text{ m}$ horizontally from A; the weight acts 0.75 m horizontally from A

B1

The rope is attached at the midpoint D, at a vertical distance $1.5 \sin 60^\circ = 1.299 \text{ m}$ below A

B1

Moments about A for AB: $R \times 1.5 - 10g \times 0.75 - T \times 1.5 \sin 60^\circ = 0$

M1 A1

$$T = \frac{147 - 73.5}{1.299} = 56.6 \text{ N}$$

A1

Answer: 56.6 N

7 marks

26 (7 marks)

Vertical: floor reaction $R = 90g$

B1

Horizontal: wall reaction $S = T$

B1

Moments about A: $S \times 5 \sin 65^\circ = 20g \times 2.5 \cos 65^\circ + 70g \times x \cos 65^\circ$

M1 A1

$$S = \frac{g(50 + 70x)}{5 \tan 65^\circ}$$

M1

$$T \leq 120 \text{ gives } 50 + 70x \leq \frac{600 \tan 65^\circ}{9.8} = 131.3$$

M1

$$x \leq 1.16 \text{ m}$$

A1

Answer: 1.16 m

7 marks

27 (10 marks)

Limiting friction at both ends: $F_1 = \mu_1 R_1$ at A (towards the wall), $F_2 = \mu_2 R_2$ at B (upwards)

B1

Horizontal: $F_1 = R_2$

B1

Vertical: $R_1 + F_2 = mg$

B1

$R_1 + \mu_2 R_2 = mg$ and $R_2 = \mu_1 R_1$, so $R_1(1 + \mu_1 \mu_2) = mg$

M1

Moments about A (ladder length $2a$): $R_2 \times 2a \sin \theta + F_2 \times 2a \cos \theta = mg \times a \cos \theta$

M1 A1

$$R_2(2 \sin \theta + 2\mu_2 \cos \theta) = mg \cos \theta, \text{ with } R_2 = \frac{\mu_1 mg}{1 + \mu_1 \mu_2}$$

M1

$$\text{Hence } 2\mu_1(\sin \theta + \mu_2 \cos \theta) = (1 + \mu_1 \mu_2) \cos \theta, \text{ so } \tan \theta = \frac{1 + \mu_1 \mu_2 - 2\mu_1 \mu_2}{2\mu_1} = \frac{1 - \mu_1 \mu_2}{2\mu_1}$$

M1

Note that F_2 acts upwards, so the wall friction helps to support the weight and $R_1 < mg$

E1

$$\theta = \arctan \frac{1 - 0.1}{0.8} = \arctan 1.125 = 48.4^\circ$$

A1

$$\text{Answer: } \tan \theta = \frac{1 - \mu_1 \mu_2}{2\mu_1}; \theta = 48.4^\circ$$

10 marks

28 (11 marks)

(a) Moments about A: $S \times L \sin \theta = mg \frac{L}{2} \cos \theta + Mg x \cos \theta$

M1 A1

$$S = \frac{(mL + 2Mx)g \cos \theta}{2L \sin \theta} = \frac{(mL + 2Mx)g \cot \theta}{2L}$$

A1

Horizontal: $F = S$

B1

Answer: when the man is at a distance x from A, the frictional force at A is $\frac{(mL + 2Mx)g \cot \theta}{2L}$

4 marks

(b) $R = (m + M)g$

B1

Slipping would first occur when the man is at the top ($x = L$), where F is greatest: $F =$

M1 A1

$$\frac{(m + 2M)g \cot \theta}{2}$$

$$F \leq \mu R: \frac{(m + 2M)g \cot \theta}{2} \leq \mu(m + M)$$

M1

$$\tan \theta \geq \frac{m + 2M}{2\mu(m + M)}$$

A1

Answer: The man can reach the top of the ladder without it slipping if $\tan \theta \geq \frac{m + 2M}{2\mu(m + M)}$

5 marks

$$(c) \tan \theta \geq \frac{15 + 120}{2 \times 0.35 \times 75} = \frac{135}{52.5} = 2.571$$

M1

$$\theta \geq 68.7^\circ$$

A1

Answer: $\theta \geq 68.7^\circ$

2 marks

29 (10 marks)**(a)** Moments about C: $5R_D = 50g \times 3 + 80g(x - 1)$

M1 A1

$$R_D = \frac{g(70 + 80x)}{5} = g(14 + 16x)$$

A1

$$R_C = 130g - R_D = g(116 - 16x)$$

M1 A1

Answer: $R_D = g(14 + 16x)$, $R_C = g(116 - 16x)$

5 marks

(b) Beyond D the beam tips about D when $R_C = 0$: moments about D: $80g(x - 6) = 50g \times (6 - 4)$

M1 A1

$$x - 6 = 1.25, \text{ so } x = 7.25 \text{ m}$$

A1

Answer: 7.25 m

3 marks

(c) Man of mass M at B: moments about D: $Mg \times 2 \leq 50g \times 2$

M1

$$M \leq 50 \text{ kg}$$

A1

Answer: 50 kg

2 marks

30 (10 marks)The peg is smooth, so its reaction S is perpendicular to the rod, at angle θ to the vertical

B1

$$AC = \frac{h}{\sin\theta}; \text{ moments about A: } S \times \frac{h}{\sin\theta} = mga \cos\theta$$

M1 A1

$$S = \frac{mga \sin\theta \cos\theta}{h}$$

A1

$$\text{Vertical: } R + S \cos\theta = mg, \text{ so } R = mg(1 - \frac{a \sin\theta \cos^2\theta}{h})$$

M1 A1

$$\text{Horizontal: } F = S \sin\theta = \frac{mga \sin^2\theta \cos\theta}{h}$$

M1

$$F \leq \mu R, \text{ so } \mu \geq \frac{F}{R} = \frac{a \cos\theta \sin^2\theta}{h - a \cos^2\theta \sin\theta}$$

A1

With the data: numerator = 0.5658, denominator = 0.7252, $\mu \geq 0.780$

M1 A1

Answer: $\mu \geq 0.780$

10 marks

Moments and Couples**1 (2 marks)**Moment = force $\times$ perpendicular distance = 12×0.5

M1

$$= 6 \text{ N m}$$

A1

Answer: 6 N m

2 marks

2 (2 marks)Perpendicular distance from B to the line of the force = $0.6 \sin 30^\circ = 0.3 \text{ m}$

M1

$$\text{Moment} = 8 \times 0.3 = 2.4 \text{ N m}$$

A1

Answer: 2.4 N m

2 marks

3 (2 marks)Moment of a couple = force $\times$ perpendicular distance between the lines

M1

$$= 15 \times 0.4 = 6 \text{ N m}$$

A1

Answer: 6 N m

2 marks

4 (2 marks)

$$15d = 4.5$$

M1

$$d = 0.3 \text{ m}$$

A1

Answer: 0.3 m

2 marks

5 (2 marks)

$$\text{Moment} = x F_y - y F_x = 2 \times 4 - 1 \times 3$$

M1

$$= 5 \text{ N m anticlockwise}$$

A1

Answer: 5 N m anticlockwise

2 marks

6 (2 marks)

The perpendicular distance from C to line AB is the side length, 2 m

M1

Moment = $10 \times 2 = 20 \text{ N m}$ (anticlockwise)

A1

Answer: 20 N m anticlockwise

2 marks

7 (5 marks)

Components parallel to AB: $5 - 3 = 2 \text{ N}$; parallel to BC: 6 N

M1 A1

Resultant = $\sqrt{2^2 + 6^2} = 6.32 \text{ N}$

A1

Angle with AB = $\arctan \frac{6}{2} = 71.6^\circ$

A1

Moment about A (anticlockwise) = $6 \times 4 + 3 \times 3 = 33 \text{ N m}$

B1

Answer: R = 6.32 N at 71.6° to AB; moment 33 N m anticlockwise

5 marks

8 (4 marks)

(a) Moment = $x F_y - y F_x = 1 \times (-3) - 2 \times 4 = -11 \text{ N m}$

M1

i.e. 11 N m clockwise

A1

Answer: 11 N m clockwise

2 marks

(b) Position of A relative to B is $(-2i + 2j)$; moment = $(-2)(-3) - (2)(4) = -2 \text{ N m}$

M1

i.e. 2 N m clockwise

A1

Answer: 2 N m clockwise

2 marks

9 (5 marks)

Components parallel to AB: $6 - 6 = 0$; parallel to BC: $8 - 8 = 0$, so there is no resultant force

M1 A1

Moments about A: the force at B gives $8 \times 4 = 32$ and the force at C gives $6 \times 3 = 18$ (the forces at A and D have no moment about A), total 50 N m

M1 A1

The system is equivalent to a couple of moment 50 N m anticlockwise

A1

Answer: couple of 50 N m anticlockwise

5 marks

10 (3 marks)

A couple cannot be balanced by a single force except another couple; the weight acts at the midpoint: moment of weight about A = $3g \times 1$

M1 A1

Couple = 29.4 N m

A1

Answer: 29.4 N m

3 marks

11 (4 marks)

Moving the force by a perpendicular distance d changes its moment by $12d$, which can cancel the couple

M1

$12d = 18$

M1 A1

$d = 1.5 \text{ m}$, on the side that gives the opposite sense to the couple

A1

Answer: 12 N parallel to the original, 1.5 m away

4 marks

12 (5 marks)

The three forces are equal in magnitude and 120° apart, so their vector sum is zero

M1 A1

The moment about A is due only to the force along BC: $6 \times (\text{perpendicular distance from A to BC}) = 6 \times 2 \sin 60^\circ = 6\sqrt{3}$

M1 A1

A couple of moment $6\sqrt{3} \text{ N m}$ (anticlockwise)

A1

Answer: couple of $6\sqrt{3} \text{ N m}$

5 marks

13 (4 marks)

Resultant = $12 + 9 + 4 = 25 \text{ N}$, downwards

B1

Moments about A: $25x = 9 \times 1 + 4 \times 3 = 21$

M1 A1

$x = 0.840 \text{ m}$ from A

A1

Answer: 25 N downwards, 0.840 m from A

4 marks

14 (7 marks)

Parallel to AB: $8 - 3 = 5$ N; parallel to BC: $-5 + 2 = -3$ N

M1 A1

Resultant = $\sqrt{5^2 + 3^2} = 5.83$ N, at 31.0° below the direction AB

M1 A1

Moment about A: force at B (0, -5): $6 \times (-5) = -30$; force at C (-3, 0): $-4 \times (-3) = +12$; the others act at A: $G_A = -18$ N m

M1

Line of action: moment about A of the resultant = $x R_y$ where it crosses AB at (x, 0): $x \times (-3) = -18$

M1

$x = 6$ m from A

A1

Answer: 5.83 N at 31.0° below AB; crosses AB 6 m from A

7 marks

15 (5 marks)

Resolving parallel to BC: $P - Q = 0$, so $Q = P$

M1

Moments about A (anticlockwise): $P \times 2$ (force at B) + 8×2 (force at C) - $24 = 0$

M1 A1 A1

$2P = 8$, so $P = 4$ and $Q = 4$

A1

Answer: $P = Q = 4$

5 marks

16 (5 marks)

Moment about O = $3 \times (-2) - 4 \times 6 = -30$ N m

M1 A1

Point (x, 0) on the line: $x \times (-2) - 0 = -30$

M1 A1

$x = 15$, so the point is 15i m

A1

Answer: moment -30 N m; crosses x axis at (15, 0)

5 marks

17 (5 marks)

Resultant: $12 + P + 8 = 30$, so $P = 10$

M1 A1

Moments about A: $30 \times 2.2 = P \times AC + 8 \times 5$

M1 A1

$66 = 10 AC + 40$, so $AC = 2.60$ m

A1

Answer: $P = 10$, $AC = 2.60$ m

5 marks

18 (5 marks)

Moments about A: couple $G = 2.4g \times 0.75 \cos 40^\circ$

M1 A1

$G = 13.5$ N m

A1

A couple has no resultant force, so the hinge force balances the weight

B1

Hinge force = $2.4g = 23.5$ N, vertically upwards

A1

Answer: $G = 13.5$ N m; 23.5 N upwards

5 marks

19 (6 marks)

Take the direction of the force at A as positive: resultant = $20 - 20 + 30 = 30$ N

M1 A1

Moments about A: $30 \times 0.75 - 20 \times 1.5 = -7.5$ N m (opposite sense)

M1 A1

$30x = -7.5$ would put the line of action at $x = -0.25$ m, i.e. 0.25 m from A beyond A on the opposite side from B

M1

Single force of 30 N, 0.25 m from A, beyond A

A1

Answer: 30 N, 0.25 m from A on the side away from B

6 marks

20 (7 marks)

(a) The resultant force is zero: $x: 2 + b - 1 = 0$, $y: a - 3 + 2 = 0$

M1 A1

$b = -1$, $a = 1$

A1

Answer: $a = 1$, $b = -1$

3 marks

(b) Moment = $x F_y - y F_x$ for each force

M1

$F_1: 1 \times 1 - 0 = 1$; $F_2: 0 \times (-3) - 2 \times (-1) = 2$; $F_3: 3 \times 2 - 1 \times (-1) = 7$

M1 A1

Total = 10 N m, anticlockwise

A1

Answer: 10 N m anticlockwise

4 marks

21 (9 marks)

(a) Take A as origin, AB as the x direction and AD as the y direction. Components: x: $10 - 4 = 6$; y: 7 M1

Resultant = $\sqrt{6^2 + 7^2} = 9.22 \text{ N}$ A1

at an angle $\arctan \frac{7}{6} = 49.4^\circ$ to AB A1

Answer: 9.22 N at 49.4° to AB 3 marks

(b) Moment about A: 7×5 (force at B) + 4×3 (force at C) = 47 N m anticlockwise M1 A1

Resultant at (0, y): moment about A = $-6y$ (the x component 6 acts at height y) M1

$-6y = 47$, so $y = -7.83 \text{ m}$: the line meets AD extended beyond A, 7.83 m from A on the side away from D A1

Answer: 7.83 m from A on AD extended beyond A (away from D) 4 marks

(c) For a couple the resultant must be zero, so the fourth force is $-(6i + 7j) \text{ N}$: magnitude 9.22 N M1

The fourth force passes through A, so the moment of the system about A is unchanged: 47 N m anticlockwise A1

Answer: force of 9.22 N, opposite in direction to the resultant; couple of 47 N m anticlockwise 2 marks

22 (6 marks)

(a) The forces cancel as a resultant force, but their moments do not: they turn the body in the same sense E1

The moment of a couple is not zero; it is force $\times$ distance between the lines E1

Answer: A couple has zero resultant force but a non-zero moment ($12 \times 0.5 = 6 \text{ N m}$) 2 marks

(b) About a point on the first line: only the second force has a moment: $12 \times 0.5 = 6 \text{ N m}$ M1 A1

About the midpoint: each force has moment $12 \times 0.25 = 3 \text{ N m}$ in the same sense M1

Total = 6 N m, the same A1

Answer: 6 N m about both points 4 marks

23 (8 marks)

Components along AB: $8 + 5\cos 120^\circ + 3\cos 240^\circ = 8 - 2.5 - 1.5 = 4$ M1 A1

Components perpendicular (towards C): $5 \sin 120^\circ + 3 \sin 240^\circ = \frac{5\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = \sqrt{3}$ M1 A1

Resultant = $\sqrt{4^2 + 3} = 4.36 \text{ N}$ at $\arctan \frac{\sqrt{3}}{4} = 23.4^\circ$ above AB A1 A1

Moment about A: only the force along BC contributes: $5 \times (4 \sin 60^\circ) = 10\sqrt{3} = 17.3 \text{ N m}$ M1

Perpendicular distance = $\frac{17.3}{4.36} = 3.97 \text{ m}$ A1

Answer: 4.36 N at 23.4° to AB; 3.97 m from A 8 marks

24 (8 marks)

(a) Moments about A: $6g \times 1 \times \cos \theta = 20$ M1 A1

$\cos \theta = \frac{20}{58.8} = 0.3401$, $\theta = 70.1^\circ$ A1

Answer: $\theta = 70.1^\circ$ 3 marks

(b) Couples have no resultant force E1

Hinge force = $6g = 58.8 \text{ N}$ vertically upwards B1

Answer: 58.8 N upwards 2 marks

(c) Greatest moment of weight about A is 58.8 N m, when the rod is horizontal B1

If the couple exceeds 58.8 N m, no angle gives equilibrium E1

so the weight cannot balance it and the rod rotates B1

Answer: 58.8 N m 3 marks

25 (6 marks)

Let the forces act at positions r_1, r_2, r_3 ; the total moment about P (position p) is $\sum(r_i - p) \times F_i$ **M1**
 $= \sum r_i \times F_i - p \times \sum F_i$ **M1 A1**
 $\sum F_i = 0$, so the second term vanishes and the total moment $\sum r_i \times F_i$ does not depend on p **A1**
 The system has no resultant force, so its only effect is a turning effect which is the same about every point **E1**
 This is the defining property of a couple **E1**
Answer: the moment is $\sum r_i \times F_i$, independent of P **6 marks**

26 (8 marks)

Resultant = $(3 - 1 + 2, -2 + 4 + 1) = (4, 3)$ N **M1 A1**
 Moments about O: $F_1: 1 \times (-2) - 2 \times 3 = -8$; $F_2: 4 \times 4 - 0 = 16$; $F_3: 0 - 3 \times 2 = -6$ **M1 A1**
 Total = 2 N m anticlockwise **A1**
 For a point (x, y) on the line of action: $x R_y - y R_x = 2$, so $3x - 4y = 2$ **M1 A1**
 $x = 0$: $y = -0.500$, the point $(0, -0.500)$ **M1**
Answer: R = $(4i + 3j)$ N; moment 2 N m; $3x - 4y = 2$; meets y axis at $(0, -0.500)$ **8 marks**

27 (11 marks)

(a) $x: 6 - 2 = 4$; $y: 4 - 1 = 3$ **M1**
 Resultant = $\sqrt{4^2 + 3^2} = 5$ N, at $\arctan \frac{3}{4} = 36.9^\circ$ to AB **A1 A1**
Answer: 5 N at 36.9° to AB **3 marks**
 (b) Moments about A: 4×4 (force at B) + 2×3 (force at C) = 22 N m **M1 A1**
 For a point (x, y) on the line: $3x - 4y = 22$ **A1**
Answer: $3x - 4y = 22$ **3 marks**
 (c) $y = 0$: $x = 7.33$ m from A along AB **B1**
 $x = 0$: $y = -5.50$ m (so 5.50 m below A along DA produced) **B1**
Answer: meets AB at $x = 7.33$; meets AD line at $y = -5.50$ **2 marks**
 (d) The additional force must be $(-4, -3)$, magnitude 5 N, opposite to the resultant **M1 A1**
 The resultant has moment 22 N m about A, so a clockwise couple of 22 N m is also needed **A1**
Answer: 5 N opposite the resultant; clockwise couple of 22 N m **3 marks**

28 (11 marks)

(a) Resultant magnitude = $P - Q$ in the direction of P **B1**
 Take moments about the point on the line of Q: $P \times d = (P - Q)x$ **M1 A1**
 Moments about the point on the line of P: $Q \times d = (P - Q)x$, so $x = \frac{Qd}{P - Q}$ from the line of P, on the side of Q **M1 A1**
 which lies beyond the line of Q, on the side away from P **A1**
Answer: Resultant $P - Q$, at distance $\frac{Qd}{(P - Q)}$ from the line of P, beyond Q **6 marks**
 (b) Resultant = $14 - 9 = 5$ N in the direction of the 14 N force **M1**
 $x = \frac{9 \times 0.6}{5} = 1.08$ m from the line of the 14 N force, on the side of the 9 N force **M1 A1**
Answer: 5 N; 1.08 m from the 14 N force's line, beyond the 9 N force **3 marks**
 (c) As $Q \rightarrow P$, the denominator $P - Q \rightarrow 0$ so $x \rightarrow \infty$ **E1**
 The resultant force tends to zero and its line of action moves infinitely far away: the limit is a couple **E1**
Answer: the limit is a couple **2 marks**

29 (10 marks)

The centre of mass G is the midpoint of AC: $AG = \frac{1}{2}\sqrt{0.8^2 + 0.6^2} = 0.5 \text{ m}$

M1 A1

Moment of the weight about A = $12g \times AG \sin\theta$

M1

$$12 \times 9.8 \times 0.5 \times \sin\theta = 20$$

M1 A1

$$\sin\theta = \frac{20}{58.8} = 0.3401$$

M1

$$\theta = 19.9^\circ$$

A1

A couple has no resultant force, so the hinge force is $12g = 118 \text{ N}$, vertically upwards

B1

The weight moment is greatest (58.8 N m) when AG is horizontal, so the couple cannot exceed 58.8 N m

B1 A1

Answer: $\theta = 19.9^\circ$; hinge force 118 N upwards; couple at most 58.8 N m

10 marks

30 (11 marks)

(a) The forces along AB and DA pass through A, so have no moment about A

B1

Force at B: $0.5 \times 6 = 3 \text{ N m}$; force at C: $0.5 \times 10 = 5 \text{ N m}$, both anticlockwise

M1 A1

Total = 8 N m anticlockwise, independent of X

A1

Answer: The moment about A is 8 N m anticlockwise for every X

4 marks

(b) Components: along AB $10 - 10 = 0$, along BC $6 - X$

M1

The resultant force is zero when $X = 6$

A1

The system is then a couple of moment 8 N m (anticlockwise)

A1

Answer: $X = 6$, couple of 8 N m anticlockwise

3 marks

(c) The resultant is $6 - 2 = 4 \text{ N}$ parallel to BC

M1 A1

Moment about A of the resultant = $4d = 8$, so $d = 2 \text{ m}$

M1 A1

Answer: 4 N; 2 m from A

4 marks

Toppling and Sliding**1 (2 marks)**

The block is about to topple when the vertical through the centre of mass passes through the pivot edge

M1

The centre of mass is 0.2 m horizontally and 0.6 m vertically from the edge: $\tan\alpha = \frac{0.2}{0.6}$, $\alpha = 18.4^\circ$

A1

Answer: 18.4°

2 marks

2 (2 marks)

Limiting friction = $\mu R = 0.4 \times 5g$

M1

$$P = 19.6 \text{ N}$$

A1

Answer: 19.6 N

2 marks

3 (2 marks)

The block is on the point of sliding when $\tan\theta = \mu$

M1

$$\tan 25^\circ = 0.466 < 0.5, \text{ so the block does not slide}$$

A1

Answer: it does not slide

2 marks

4 (2 marks)

Maximum friction = $0.3 \times 12g = 35.28 \text{ N}$, which exceeds 30 N, so the block does not move

M1

$$\text{Friction} = 30 \text{ N}$$

A1

Answer: 30 N

2 marks

5 (2 marks)

Moments about the pivot edge: $P \times 0.4 = 20g \times 0.2$

M1

$$P = 98 \text{ N}$$

A1

Answer: 98 N

2 marks

6 (2 marks)

It topples when the vertical through G passes through the lower edge of the base: $\tan\theta = \frac{0.1}{0.25}$ M1

$$\theta = 21.8^\circ$$
A1

Answer: 21.8° 2 marks

7 (5 marks)

Toppling about the far bottom edge: $P \times 0.8 = 15g \times 0.15$ M1 A1

$$P_{\text{topple}} = 27.6 \text{ N}$$
A1

Sliding needs $P = \mu R = 0.4 \times 15g = 58.8 \text{ N}$ M1

$27.6 < 58.8$, so it topples first, at $P = 27.6 \text{ N}$ A1

Answer: P = 27.6 N; it topples 5 marks

8 (5 marks)

Toppling when $\tan\theta = \frac{0.1}{0.25} = 0.4$, $\theta = 21.8^\circ$ M1 A1

Sliding when $\tan\theta = \mu = 0.6$, $\theta = 31.0^\circ$ M1 A1

$21.8^\circ < 31.0^\circ$, so the block topples first A1

Answer: topples at 21.8°; would slide at 31.0°; topples first 5 marks

9 (5 marks)

Toppling: $P \times 0.4 = mg \times 0.3$, so $P = 0.75mg$ M1 A1

Resolving: friction $F = P$ and $R = mg$ B1

For no sliding $F \leq \mu R$: $\mu \geq 0.75$ M1 A1

Answer: $\mu \geq 0.75$ 5 marks

10 (4 marks)

Toppling: G is above the lower edge when $\tan\theta = \frac{1}{2}a \div \frac{1}{2}a = 1$, $\theta = 45^\circ$ M1 A1

Sliding when $\tan\theta = 0.8$, $\theta = 38.7^\circ$ M1

It slides first, at 38.7° A1

Answer: slides first at 38.7° (would topple at 45°) 4 marks

11 (5 marks)

Vertical: $R + T \sin 30^\circ = 10g$ M1

Horizontal: $T \cos 30^\circ = 0.35R$ M1 A1

$$T(\cos 30^\circ + 0.35 \sin 30^\circ) = 0.35 \times 98$$
M1

$$T = 32.9 \text{ N}$$
A1

Answer: 32.9 N 5 marks

12 (4 marks)

G is 0.2 m along AB from the lower end and 0.45 m above AB B1

Topples when $\tan\theta = \frac{0.2}{0.45}$ M1 A1

$$\theta = 24.0^\circ$$
A1

Answer: 24.0° 4 marks

13 (4 marks)

Toppling when $\tan\theta = \frac{5}{10} = 0.5$ M1 A1

For no sliding up to that angle $\mu \geq \tan\theta$ M1

$$\mu \geq 0.5$$
A1

Answer: $\mu \geq 0.5$ 4 marks

14 (5 marks)

The centre of mass is on the axis, $\frac{h}{4} = 3$ cm above the base

B1

Topples when the vertical through G passes through the lower rim: $\tan\theta = \frac{4}{3}$

M1 A1

$$\theta = 53.1^\circ$$

A1

No sliding needs $\mu \geq \tan\theta = \frac{4}{3}$, so $\mu \geq 1.33$

M1

Answer: $\theta = 53.1^\circ$; $\mu \geq 1.33$

5 marks

15 (6 marks)

Moments about the pivot: $P \cos 20^\circ \times 1.2 - P \sin 20^\circ \times 0.4 = 20g \times 0.2$

M1 A1 A1

$$P = 39.6 \text{ N}$$

A1

Friction needed $F = P \cos 20^\circ = 37.2 \text{ N}$; $R = 20g + P \sin 20^\circ = 210 \text{ N}$

M1

$\mu R = 62.9 \text{ N} > 37.2 \text{ N}$, so it does not slide: it tips

A1

Answer: $P = 39.6 \text{ N}$; it tips

6 marks

16 (5 marks)

Sliding when $P = \mu mg = 0.4mg$

B1

Toppling when $P h = mg \times 0.25$

M1

Sliding first requires $P_{\text{slide}} \leq P_{\text{topple}}$: $0.4mg \leq \frac{0.25mg}{h}$

M1 A1

$$h \leq 0.625 \text{ m}$$

A1

Answer: $h \leq 0.625 \text{ m}$

5 marks

17 (5 marks)

Take the lower bottom edge as origin, x up the slope and y perpendicular to it

B1

$$\bar{x} = \frac{6 \times 0.2 + 2 \times 0}{8} = 0.1500, \bar{y} = \frac{6 \times 0.3 + 2 \times 0.6}{8} = 0.3750$$

M1 A1

Topples when $\tan\theta = \frac{\bar{x}}{\bar{y}} = 0.4000$

M1

$$\theta = 21.8^\circ$$

A1

Answer: 21.8°

5 marks

18 (5 marks)

G is $\frac{4r}{3\pi}$ above the diameter and r along it from each end

B1

Topples when the vertical through G passes through the lower end of the diameter

M1

$$\tan\theta = \frac{\frac{r}{4r}}{\frac{3\pi}{3\pi}} = \frac{3\pi}{4}$$

M1 A1

$$\theta = 67.0^\circ$$

A1

Answer: 67.0°

5 marks

19 (5 marks)

Toppling: $0.8P = 20g \times 0.3$

M1 A1

$$P = 73.5 \text{ N}$$

A1

Sliding needs $P = 0.5 \times 20g = 98 \text{ N}$

M1

$73.5 \text{ N} < 98 \text{ N}$, so the block topples about its far bottom edge before sliding

A1

Answer: $P = 73.5 \text{ N}$; topples

5 marks

20 (4 marks)

The condition for toppling is that the line of action of the weight through G passes through the pivot edge

E1

This depends only on the position of G, i.e. on the dimensions, not on the mass or the weight

E1

$$\tan\theta = \frac{0.2}{0.25} = 0.8$$

M1

$$\theta = 38.7^\circ$$

A1

Answer: 38.7°

4 marks

21 (9 marks)

(a) G must be vertically above/below the point of contact C

M1

OC = r is perpendicular to the plane, at angle θ to the vertical, so G is at horizontal distance $r \sin\theta$ from O

M1 A1

G is also at horizontal distance $\frac{3r}{8} \sin\phi$ from O, so $\frac{3}{8} \sin\phi = \sin\theta$

A1

Answer: $\sin\phi = \frac{8}{3} \sin\theta$

4 marks

(b) $\sin\phi \leq 1$ gives $\sin\theta \leq \frac{3}{8}$, $\theta \leq 22.0^\circ$

M1 A1

$\phi = \arcsin(\frac{8}{3} \sin 15^\circ) = 43.6^\circ$

A1

Answer: $\theta \leq 22.0^\circ$; $\phi = 43.6^\circ$

3 marks

(c) Friction $F = mg \sin\theta$, $R = mg \cos\theta$

M1

$\mu \geq \tan 15^\circ = 0.268$

A1

Answer: $\mu \geq 0.268$

2 marks

22 (6 marks)

(a) The centre of mass is at half the height, so G is 0.45 m above the plane, not 0.9 m

E1

$\tan\theta = \frac{0.15}{0.45} = \frac{1}{3}$, $\theta = 18.4^\circ$

M1 A1

Answer: G is 0.45 m above the plane, so toppling occurs at $\theta = 18.4^\circ$

3 marks

(b) Sliding occurs when $\tan\theta = 0.25$, $\theta = 14.0^\circ$

M1 A1

This is less than 18.4° , so the block slides before it topples

A1

Answer: The block slides at 14.0° , before it would topple

3 marks

23 (7 marks)

Origin at the lower bottom edge of A, x up the slope and y perpendicular to it

B1

$\bar{x} = \frac{12 \times 0.2 + 3 \times 0.1}{15} = 0.1800$, $\bar{y} = \frac{12 \times 0.2 + 3 \times 0.5}{15} = 0.2600$

M1 A1

Topples when $\tan\theta = \frac{\bar{x}}{\bar{y}} = 0.6923$, $\theta = 34.7^\circ$

M1 A1

Slides when $\tan\theta = 0.6$, $\theta = 31.0^\circ$

M1

It slides first ($31.0^\circ < 34.7^\circ$)

A1

Answer: topple at 34.7° , slide at 31.0° ; slides first

7 marks

24 (7 marks)

Toppling is about the upper bottom edge: moments about it: $0.6P = mg \cos 20^\circ \times 0.2 + mg \sin 20^\circ \times 0.3$

M1 A1 A1

$P_{\text{topple}} = 47.5 \text{ N}$

A1

Sliding up the slope: $P = mg \sin 20^\circ + \mu mg \cos 20^\circ = 98.0 \text{ N}$

M1 A1

$47.5 \text{ N} < 98.0 \text{ N}$, so the block topples at $P = 47.5 \text{ N}$

A1

Answer: $P = 47.5 \text{ N}$; topples

7 marks

25 (8 marks)

(a) Volumes proportional to $0.01 \times 0.3 = 0.003$ and $\frac{1}{3} \times 0.01 \times 0.2 = \frac{0.002}{3}$

B1

Heights of the centres: 0.15 and $0.3 + 0.05 = 0.35$

B1

$\bar{y} = \frac{0.00045 + 0.0002333}{0.0036667} = 0.1864$

M1 A1

Answer: The centre of mass is 0.186 m above the base of the cylinder

4 marks

(b) $\tan\theta = \frac{0.1}{\bar{y}} = 0.5366$

M1

$\theta = 28.2^\circ$

A1

Answer: 28.2°

2 marks

(c) $\mu \geq \tan\theta$

M1

$\mu \geq 0.537$

A1

Answer: $\mu \geq 0.537$

2 marks

26 (8 marks)**(a)** Sliding at $P = 0.4 \times 30g = 117.6 \text{ N}$ **B1**Toppling at $P \times x = 30g \times 0.25$, i.e. $P = \frac{73.5}{x}$ **M1**Sliding first needs $117.6 < \frac{73.5}{x}$, so $x < 0.625$ **M1 A1****Answer: The block slides before it topples if $x < 0.625$** **4 marks****(b)** $P = 73.5 \text{ N}$ (topples, since $x > 0.625$)**M1 A1****Answer: $P = 73.5 \text{ N}$; topples****2 marks****(c)** $P = 117.6 \text{ N}$ (slides, since $x < 0.625$)**M1 A1****Answer: $P = 117.6 \text{ N}$; slides****2 marks****27 (11 marks)****(a)** Toppling when $P \times 2b = mg a$, so $P = \frac{mga}{2b}$ **M1 A1**Sliding when $P = \mu mg$, so toppling comes first when $\frac{mga}{2b} < \mu mg$, i.e. $\mu > \frac{a}{2b}$ **M1 A1****Answer: The block topples before it slides if $\mu > \frac{a}{2b}$** **4 marks****(b)** $\mu > \frac{0.3}{1.0} = 0.3$ **M1 A1** $P = \frac{40 \times 9.8 \times 0.3}{1.0} = 118 \text{ N}$ **M1 A1****Answer: $\mu > 0.3$; $P = 118 \text{ N}$** **4 marks****(c)** Toppling $P kb = mga$, $P = \frac{mga}{kb}$; sliding $P = \mu mg$, so sliding is first if $\mu < \frac{a}{kb}$ **M1 A1**For $a = 0.3$, $b = 0.5$, $\mu = 0.5$: toppling first needs $0.5 > \frac{0.3}{0.5k}$, $k > 1.20$ **A1****Answer: $k > 1.2$** **3 marks****28 (11 marks)****(a)** G is on the median, $\frac{1}{3}$ of the height above BC: height $= \frac{1}{3} \times 0.6 \times \frac{\sqrt{3}}{2} = 0.1732 \text{ m}$ **B1**

Horizontal distance of G from B along BC = 0.3 m

B1 $\tan \theta = \frac{0.3}{0.1732} = \sqrt{3}$, $\theta = 60^\circ$ **M1 A1****Answer: The lamina topples when $\theta = 60^\circ$ if no other mass is attached****4 marks****(b)** Combined centre of mass: $\bar{x} = 0.3$ (by symmetry)**B1** $\bar{y} = \frac{3m \times 0.1732 + m \times 0.5196}{4m} = 0.2598$ **M1 A1** $\tan \theta = \frac{0.3}{0.2598} = 1.155$, $\theta = 49.1^\circ$ **M1 A1****Answer: $\theta = 49.1^\circ$** **5 marks****(c)** Sliding when $\tan \theta = \mu$ **M1**Toppling first if $\mu > \tan \theta_{\text{topple}} = 1.155$ **A1****Answer: $\mu > 1.15$** **2 marks****29 (12 marks)****(a)** Moments about the point of contact C**M1**The rim point is at height $r + r \sin \phi$ above C**B1**G is at horizontal distance $\frac{3r}{8} \sin \phi$ from C**B1** $P r (1 + \sin \phi) = mg \times \frac{3r}{8} \sin \phi$ **M1 A1**so $P = \frac{3mg \sin \phi}{8(1 + \sin \phi)}$ **A1****Answer: $P = \frac{3mg \sin \phi}{8(1 + \sin \phi)}$** **6 marks****(b)** $P = \frac{3 \times 2 \times 9.8 \times 0.5}{8 \times 1.5}$ **M1**

$$P = 2.45 \text{ N}$$

A1

Answer: 2.45 N

2 marks

(c) Friction $F = P$ and normal reaction $R = mg$

B1

$$\frac{F}{R} = \frac{3 \sin \phi}{8(1 + \sin \phi)}, \text{ which increases with } \phi$$

M1

$$\text{Maximum at } \phi = 90^\circ: \frac{3}{16}$$

M1

$$\text{So } \mu \geq \frac{3}{16} = 0.1875 \text{ suffices}$$

A1

Answer: $\mu \geq 3/16$

4 marks

30 (12 marks)

(a) G is on the axis, $\frac{h}{4}$ from the base

B1

Topples when the vertical through G passes through the lower rim

M1

$$\tan \theta = \frac{r}{\frac{h}{4}} = \frac{4r}{h}$$

A1

Answer: The cone topples when $\tan \theta = \frac{4r}{h}$

3 marks

$$(b) \tan \theta = \frac{20}{12} = 1.667, \theta = 59.0^\circ$$

M1 A1

$$\mu \geq 1.667$$

A1

Answer: 59.0° ; $\mu \geq 1.667$

3 marks

(c) Origin at the lowest rim point, x up the slope: cone G at $(r, h/4)$, particle at $(2r, 0)$

B1

$$\bar{x} = \frac{M r + \frac{M}{4}(2r)}{\frac{5M}{4}} = \frac{6r}{5}$$

M1 A1

$$\bar{y} = \frac{M \frac{h}{4}}{\frac{5M}{4}} = \frac{h}{5}$$

M1 A1

$$\tan \theta = \frac{\frac{6r}{5}}{\frac{h}{5}} = \frac{6r}{h} = 2.500, \theta = 68.2^\circ$$

A1

Answer: $\theta = 68.2^\circ$

6 marks