

Allocation and the Hungarian Algorithm

1 (2 marks)

Row minima are 11, 12 and 15

M1

Reduced matrix: $\begin{pmatrix} 3 & 7 & 0 \\ 0 & 4 & 1 \\ 2 & 0 & 4 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 3 & 7 & 0 \\ 0 & 4 & 1 \\ 2 & 0 & 4 \end{pmatrix}$

2 marks

2 (2 marks)

Column minima are 0, 0 and 2

M1

New matrix: $\begin{pmatrix} 0 & 5 & 1 \\ 0 & 4 & 0 \\ 3 & 0 & 4 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 0 & 5 & 1 \\ 0 & 4 & 0 \\ 3 & 0 & 4 \end{pmatrix}$

2 marks

3 (2 marks)

Three lines (column 1, column 3 and row 3, for example) cover every zero

B1

Fewer than 4 lines, so no complete allocation can yet be made

E1

Answer: 3 lines; no complete allocation yet

2 marks

4 (2 marks)

Add a dummy job (a column) so that the table is 4×4

B1

Every entry in the dummy column is 0

B1

Answer: A dummy job with all costs 0

2 marks

5 (2 marks)

Subtract every entry from the largest entry in the table (or negate and add a constant)

M1

Then minimise the new table; the minimum corresponds to the maximum profit

E1

Answer: Subtract each entry from the largest entry, then minimise

2 marks

6 (3 marks)

The zeros give P-3, Q-1, R-2

B1

Cost = $8 + 7 + 8$

M1

= £23

A1

Answer: P to 3, Q to 1, R to 2; £23

3 marks

7 (4 marks)

Subtract the row minimum from each row: $\begin{pmatrix} 10 & 0 & 7 & 1 \\ 1 & 11 & 0 & 12 \\ 14 & 2 & 0 & 6 \\ 13 & 14 & 3 & 0 \end{pmatrix}$

M1

Subtract the column minimum from each column: $\begin{pmatrix} 9 & 0 & 7 & 1 \\ 0 & 11 & 0 & 12 \\ 13 & 2 & 0 & 6 \\ 12 & 14 & 3 & 0 \end{pmatrix}$

A1

The zeros now need 4 lines, the same as the number of rows, so a complete allocation exists:

M1

Ali-Breaststroke, Ben-Backstroke, Cat-Butterfly, Dev-Freestyle

Value from the original table = $28 + 25 + 28 + 24 = 105$ seconds

A1

Answer: Ali-Breaststroke, Ben-Backstroke, Cat-Butterfly, Dev-Freestyle; 105 s

4 marks

8 (4 marks)

Subtract the row minimum from each row: $\begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 0 \\ 7 & 0 & 2 \end{pmatrix}$ M1

Subtract the column minimum from each column: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 5 & 0 & 2 \end{pmatrix}$ A1

The zeros now need 3 lines, the same as the number of rows, so a complete allocation exists: M1

Mechanic X-Car 3, Mechanic Y-Car 1, Mechanic Z-Car 2

Value from the original table = $8 + 11 + 3 = 22$ hours A1

Answer: Mechanic X-Car 3, Mechanic Y-Car 1, Mechanic Z-Car 2; 22 hours 4 marks

9 (5 marks)

There are 3 students and 4 tasks, so add a dummy row of zeros to make the table square: B1

$$\begin{pmatrix} 22 & 27 & 21 & 22 \\ 30 & 27 & 23 & 15 \\ 20 & 15 & 22 & 21 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Subtract the row minimum from each row: $\begin{pmatrix} 1 & 6 & 0 & 1 \\ 15 & 12 & 8 & 0 \\ 5 & 0 & 7 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ M1

Subtract the column minimum from each column: $\begin{pmatrix} 1 & 6 & 0 & 1 \\ 15 & 12 & 8 & 0 \\ 5 & 0 & 7 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ A1

The zeros now need 4 lines, the same as the number of rows, so a complete allocation exists: M1

Amy-T3, Bao-T4, Cleo-T2, D-T1

Value from the original table = $21 + 15 + 15 = 51$ minutes A1

Answer: Amy to T3, Bao to T4, Cleo to T2; total 51 minutes 5 marks

10 (2 marks)

Subtract each entry from the largest entry, 27 M1

Matrix: $\begin{pmatrix} 9 & 2 & 6 \\ 5 & 8 & 0 \\ 3 & 7 & 4 \end{pmatrix}$

Answer: $\begin{pmatrix} 9 & 2 & 6 \\ 5 & 8 & 0 \\ 3 & 7 & 4 \end{pmatrix}$ 2 marks

11 (3 marks)

Replace the forbidden entry by M, a very large cost, so that it is never chosen B1

Row reduction: $\begin{pmatrix} 4 & 0 & 7 \\ 1 & 4 & 0 \\ 0 & M & 2 \end{pmatrix}$ M1

Column reduction: $\begin{pmatrix} 4 & 0 & 7 \\ 1 & 4 & 0 \\ 0 & M & 2 \end{pmatrix}$ A1

Answer: M in place of the dash; $\begin{pmatrix} 4 & 0 & 7 \\ 1 & 4 & 0 \\ 0 & M & 2 \end{pmatrix}$ 3 marks

12 (3 marks)

Row minima: 9, 6, 17, 6; after row reduction the column minima are 3, 0, 0, 0

M1

Total = 41

A1

Every allocation uses one entry from each row and column, so its cost is the reduced cost (at least 0) plus the amounts subtracted, which total 41

E1

Answer: 41; every allocation includes all the subtracted amounts

3 marks**13 (3 marks)**

Smallest uncovered entry is 3

B1

Subtract 3 from every uncovered entry and add 3 to the entry covered twice

M1

New matrix:
$$\begin{pmatrix} 0 & 0 & 3 & 5 \\ 4 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 4 & 3 & 0 & 0 \end{pmatrix}$$

A1

Answer:
$$\begin{pmatrix} 0 & 0 & 3 & 5 \\ 4 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 4 & 3 & 0 & 0 \end{pmatrix}$$

3 marks**14 (6 marks)**

Subtract the row minimum from each row:
$$\begin{pmatrix} 11 & 13 & 17 & 0 \\ 25 & 0 & 10 & 31 \\ 1 & 13 & 7 & 0 \\ 25 & 14 & 14 & 0 \end{pmatrix}$$

M1

Subtract the column minimum from each column:
$$\begin{pmatrix} 10 & 13 & 10 & 0 \\ 24 & 0 & 3 & 31 \\ 0 & 13 & 0 & 0 \\ 24 & 14 & 7 & 0 \end{pmatrix}$$

A1

The zeros need only 3 lines (rows Lee, Max and column Depot 4), fewer than 4. The smallest uncovered entry is 7: subtract it from the uncovered entries and add it to the entries covered twice:

M1 A1

$$\begin{pmatrix} 3 & 6 & 3 & 0 \\ 24 & 0 & 3 & 38 \\ 0 & 13 & 0 & 7 \\ 17 & 7 & 0 & 0 \end{pmatrix}$$

The zeros now need 4 lines, the same as the number of rows, so a complete allocation exists:

M1

Kay-Depot 4, Lee-Depot 2, Max-Depot 1, Noor-Depot 3

Value from the original table = $22 + 8 + 12 + 27 = £69$

A1

Answer: Kay-Depot 4, Lee-Depot 2, Max-Depot 1, Noor-Depot 3; £69

6 marks

15 (7 marks)

There are 5 drivers and 4 routes, so add a dummy column of zeros to make the table square:

B1

$$\begin{pmatrix} 23 & 23 & 33 & 38 & 0 \\ 15 & 33 & 45 & 40 & 0 \\ 18 & 30 & 31 & 22 & 0 \\ 45 & 44 & 41 & 26 & 0 \\ 20 & 18 & 33 & 44 & 0 \end{pmatrix}$$

Subtract the row minimum from each row:

$$\begin{pmatrix} 23 & 23 & 33 & 38 & 0 \\ 15 & 33 & 45 & 40 & 0 \\ 18 & 30 & 31 & 22 & 0 \\ 45 & 44 & 41 & 26 & 0 \\ 20 & 18 & 33 & 44 & 0 \end{pmatrix}$$
M1

Subtract the column minimum from each column:

$$\begin{pmatrix} 8 & 5 & 2 & 16 & 0 \\ 0 & 15 & 14 & 18 & 0 \\ 3 & 12 & 0 & 0 & 0 \\ 30 & 26 & 10 & 4 & 0 \\ 5 & 0 & 2 & 22 & 0 \end{pmatrix}$$
A1

The zeros need only 4 lines (rows D2, D3, D5 and column D), fewer than 5. The smallest uncovered entry is 2: subtract it from the uncovered entries and add it to the entries covered twice:

M1 A1

$$\begin{pmatrix} 6 & 3 & 0 & 14 & 0 \\ 0 & 15 & 14 & 18 & 2 \\ 3 & 12 & 0 & 0 & 2 \\ 28 & 24 & 8 & 2 & 0 \\ 5 & 0 & 2 & 22 & 2 \end{pmatrix}$$

The zeros now need 5 lines, the same as the number of rows, so a complete allocation exists:

M1

D1-R3, D2-R1, D3-R4, D4-D, D5-R2

Value from the original table = $33 + 15 + 22 + 18 = \text{£}88$

A1

Answer: D1 to R3, D2 to R1, D3 to R4, D5 to R2; total £88; D4 is not used

7 marks**16 (6 marks)**

Maximising, so subtract every entry from the largest entry, 88:

$$\begin{pmatrix} 15 & 35 & 13 & 38 \\ 34 & 8 & 11 & 18 \\ 46 & 0 & 4 & 2 \\ 18 & 17 & 2 & 45 \end{pmatrix}$$
M1 A1

Subtract the row minimum from each row:

$$\begin{pmatrix} 2 & 22 & 0 & 25 \\ 26 & 0 & 3 & 10 \\ 46 & 0 & 4 & 2 \\ 16 & 15 & 0 & 43 \end{pmatrix}$$
M1

Subtract the column minimum from each column:

$$\begin{pmatrix} 0 & 22 & 0 & 23 \\ 24 & 0 & 3 & 8 \\ 44 & 0 & 4 & 0 \\ 14 & 15 & 0 & 41 \end{pmatrix}$$
A1

The zeros now need 4 lines, the same as the number of rows, so a complete allocation exists: Staff 1-North, Staff 2-South, Staff 3-West, Staff 4-East

M1

Value from the original table = $73 + 80 + 86 + 86 = 325$ (£100s)

A1

Answer: Staff 1-North, Staff 2-South, Staff 3-West, Staff 4-East; 325 (£100s)

6 marks

17 (7 marks)

Replace each forbidden entry by M, a very large cost

B1

Subtract the row minimum from each row:

$$\begin{pmatrix} 17 & 8 & 0 & 19 \\ 26 & 16 & 0 & 22 \\ 23 & 0 & 27 & 14 \\ 16 & 20 & M & 0 \end{pmatrix}$$
M1

Subtract the column minimum from each column:

$$\begin{pmatrix} 1 & 8 & 0 & 19 \\ 10 & 16 & 0 & 22 \\ 7 & 0 & 27 & 14 \\ 0 & 20 & M & 0 \end{pmatrix}$$
A1

The zeros need only 3 lines (rows C, D and column Job 3), fewer than 4. The smallest uncovered entry is 1: subtract it from the uncovered entries and add it to the entries covered twice:

M1 A1

$$\begin{pmatrix} 0 & 7 & 0 & 18 \\ 9 & 15 & 0 & 21 \\ 7 & 0 & 28 & 14 \\ 0 & 20 & M & 0 \end{pmatrix}$$

The zeros now need 4 lines, the same as the number of rows, so a complete allocation exists:

M1

A-Job 1, B-Job 3, C-Job 2, D-Job 4

Value from the original table = $34 + 10 + 10 + 11 = £65$

A1

Answer: A to Job 1, B to Job 3, C to Job 2, D to Job 4; total £65

7 marks**18 (5 marks)**

Subtract the row minimum from each row:

$$\begin{pmatrix} 0 & 16 & 10 & 8 \\ 5 & 18 & 11 & 0 \\ 0 & 16 & 8 & 14 \\ 3 & 16 & 0 & 12 \end{pmatrix}$$
M1

Subtract the column minimum from each column:

$$\begin{pmatrix} 0 & 0 & 10 & 8 \\ 5 & 2 & 11 & 0 \\ 0 & 0 & 8 & 14 \\ 3 & 0 & 0 & 12 \end{pmatrix}$$
A1

The zeros now need 4 lines, the same as the number of rows, so a complete allocation exists:

M1

Tutor A-Class 1, Tutor B-Class 4, Tutor C-Class 2, Tutor D-Class 3

Value from the original table = $7 + 6 + 22 + 8 = 43$ hours

A1

The zeros allow a second allocation, A to 2, B to 4, C to 1, D to 3, which also costs 43 hours

A1

Answer: A-1, B-4, C-2, D-3, or A-2, B-4, C-1, D-3; total 43 hours

5 marks**19 (4 marks)**

Rows 3 and 4 have zeros only in column 2, so they cannot all be given a different zero; the zeros can be covered by 3 lines (rows 1, 2 and column 2), so no complete allocation exists

E1

Smallest uncovered entry is 1: subtract it from uncovered entries and add it to entries covered twice

M1

New matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 5 \\ 5 & 1 & 4 & 0 \\ 4 & 0 & 0 & 3 \\ 3 & 0 & 3 & 3 \end{pmatrix}$$
A1

Allocation from the zeros: row 1 to column 1, row 2 to column 4, row 3 to column 3, row 4 to column 2

A1

Answer: Rows 3 and 4 share one zero column; new matrix

$$\begin{pmatrix} 0 & 1 & 0 & 5 \\ 5 & 1 & 4 & 0 \\ 4 & 0 & 0 & 3 \\ 3 & 0 & 3 & 3 \end{pmatrix}; 1-1, 2-4, 3-3, 4-2$$
4 marks

20 (6 marks)

Cost matrix = hours $\times$ hourly rate for each row: $\begin{pmatrix} 84 & 42 & 84 \\ 72 & 108 & 72 \\ 45 & 105 & 75 \end{pmatrix}$ M1 A1

Subtract the row minimum from each row: $\begin{pmatrix} 42 & 0 & 42 \\ 0 & 36 & 0 \\ 0 & 60 & 30 \end{pmatrix}$ M1

Subtract the column minimum from each column: $\begin{pmatrix} 42 & 0 & 42 \\ 0 & 36 & 0 \\ 0 & 60 & 30 \end{pmatrix}$ A1

The zeros now need 3 lines, the same as the number of rows, so a complete allocation exists: M1

Pat-2, Quin-3, Ravi-1

Value from the original table = $42 + 72 + 45 = \text{£}159$ A1

Answer: Pat to Job 2, Quin to Job 3, Ravi to Job 1; total £159 6 marks

21 (9 marks)

(a) Replace each forbidden entry by M, a very large cost B1

Subtract the row minimum from each row: $\begin{pmatrix} 18 & 11 & 23 & 7 & 0 \\ 0 & 18 & 19 & 10 & 7 \\ 10 & 19 & 17 & 9 & 0 \\ 0 & 7 & 0 & 3 & 2 \\ 0 & M & 18 & 9 & 15 \end{pmatrix}$ M1

Subtract the column minimum from each column: $\begin{pmatrix} 18 & 4 & 23 & 4 & 0 \\ 0 & 11 & 19 & 7 & 7 \\ 10 & 12 & 17 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & M & 18 & 6 & 15 \end{pmatrix}$ A1

Answer: Dash replaced by M; after both reductions $\begin{pmatrix} 18 & 4 & 23 & 4 & 0 \\ 0 & 11 & 19 & 7 & 7 \\ 10 & 12 & 17 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & M & 18 & 6 & 15 \end{pmatrix}$ 3 marks

(b) The zeros need only 3 lines (row D and columns 1, 5), fewer than 5. The smallest uncovered entry is 4: subtract it from the uncovered entries and add it to the entries covered twice: M1 A1

$\begin{pmatrix} 18 & 0 & 19 & 0 & 0 \\ 0 & 7 & 15 & 3 & 7 \\ 10 & 8 & 13 & 2 & 0 \\ 4 & 0 & 0 & 0 & 6 \\ 0 & M & 14 & 2 & 15 \end{pmatrix}$

The zeros need only 4 lines (rows A, C, D and column 1), fewer than 5. The smallest uncovered entry is 2: subtract it from the uncovered entries and add it to the entries covered twice: M1 A1

$\begin{pmatrix} 20 & 0 & 19 & 0 & 0 \\ 0 & 5 & 13 & 1 & 5 \\ 12 & 8 & 13 & 2 & 0 \\ 6 & 0 & 0 & 0 & 6 \\ 0 & M & 12 & 0 & 13 \end{pmatrix}$

The zeros now need 5 lines, the same as the number of rows, so a complete allocation exists: A-2, B-1, C-5, D-3, E-4 M1

Value from the original table = $21 + 20 + 20 + 28 + 23 = 112$ hours A1

Answer: A to 2, B to 1, C to 5, D to 3, E to 4; total 112 hours 6 marks

22 (9 marks)**(a)** Maximising, so subtract every entry from the largest entry, 86:

$$\begin{pmatrix} 41 & 29 & 51 & 25 & 39 \\ 21 & 4 & 7 & 42 & 2 \\ 8 & 0 & 34 & 21 & 28 \\ 16 & 12 & 18 & 52 & 54 \end{pmatrix}$$

M1 A1

There are 4 consultants and 5 projects, so add a dummy row of zeros to make the table square:

B1

$$\begin{pmatrix} 41 & 29 & 51 & 25 & 39 \\ 21 & 4 & 7 & 42 & 2 \\ 8 & 0 & 34 & 21 & 28 \\ 16 & 12 & 18 & 52 & 54 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer: Subtract from 86, add a dummy row:

$$\begin{pmatrix} 41 & 29 & 51 & 25 & 39 \\ 21 & 4 & 7 & 42 & 2 \\ 8 & 0 & 34 & 21 & 28 \\ 16 & 12 & 18 & 52 & 54 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3 marks**(b)** Subtract the row minimum from each row:

$$\begin{pmatrix} 16 & 4 & 26 & 0 & 14 \\ 19 & 2 & 5 & 40 & 0 \\ 8 & 0 & 34 & 21 & 28 \\ 4 & 0 & 6 & 40 & 42 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

M1

Subtract the column minimum from each column:

$$\begin{pmatrix} 16 & 4 & 26 & 0 & 14 \\ 19 & 2 & 5 & 40 & 0 \\ 8 & 0 & 34 & 21 & 28 \\ 4 & 0 & 6 & 40 & 42 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A1

The zeros need only 4 lines (rows Consultant 1, Consultant 2, D and column P2), fewer than 5.

M1 A1

The smallest uncovered entry is 4: subtract it from the uncovered entries and add it to the entries

$$\begin{pmatrix} 16 & 8 & 26 & 0 & 14 \\ 19 & 6 & 5 & 40 & 0 \\ 4 & 0 & 30 & 17 & 24 \\ 0 & 0 & 2 & 36 & 38 \\ 0 & 4 & 0 & 0 & 0 \end{pmatrix}$$

The zeros now need 5 lines, the same as the number of rows, so a complete allocation exists:

M1

Consultant 1-P4, Consultant 2-P5, Consultant 3-P2, Consultant 4-P1, D-P3

Value from the original table = 61 + 84 + 86 + 70 = £301k

A1**Answer: C1 to P4, C2 to P5, C3 to P2, C4 to P1; total £301k; P3 has no leader****6 marks****23 (8 marks)****(a)** Mechanic X is allowed two jobs, so X appears twice (as X and X') with identical costs**B1**

There are now 4 rows and 4 columns, so the table is square:

$$\begin{pmatrix} 8 & 12 & 20 & 18 \\ 8 & 12 & 20 & 18 \\ 26 & 15 & 14 & 26 \\ 17 & 21 & 22 & 18 \end{pmatrix}$$

B1**Answer: X is written twice, giving the 4 × 4 table**

$$\begin{pmatrix} 8 & 12 & 20 & 18 \\ 8 & 12 & 20 & 18 \\ 26 & 15 & 14 & 26 \\ 17 & 21 & 22 & 18 \end{pmatrix}$$

2 marks**(b)** Subtract the row minimum from each row:

$$\begin{pmatrix} 0 & 4 & 12 & 10 \\ 0 & 4 & 12 & 10 \\ 12 & 1 & 0 & 12 \\ 0 & 4 & 5 & 1 \end{pmatrix}$$

M1

Subtract the column minimum from each column: $\begin{pmatrix} 0 & 3 & 12 & 9 \\ 0 & 3 & 12 & 9 \\ 12 & 0 & 0 & 11 \\ 0 & 3 & 5 & 0 \end{pmatrix}$ **A1**

The zeros need only 3 lines (rows Y, Z and column 1), fewer than 4. The smallest uncovered entry is 3: subtract it from the uncovered entries and add it to the entries covered twice: **M1 A1**

$$\begin{pmatrix} 0 & 0 & 9 & 6 \\ 0 & 0 & 9 & 6 \\ 15 & 0 & 0 & 11 \\ 3 & 3 & 5 & 0 \end{pmatrix}$$

The zeros now need 4 lines, the same as the number of rows, so a complete allocation exists: X-1, X'-2, Y-3, Z-4 **M1**

Value from the original table = $8 + 12 + 14 + 18 = 52$ days **A1**

Answer: X: vehicles 1 and 2; Y: vehicle 3; Z: vehicle 4; total 52 days **6 marks**

24 (7 marks)

(a) C does job 2, so row C and column 2 are removed, leaving three workers and three jobs: **A1**

$$\begin{pmatrix} 15 & 11 & 29 \\ 30 & 14 & 33 \\ 31 & 20 & 17 \end{pmatrix}$$

Answer: $\begin{pmatrix} 15 & 11 & 29 \\ 30 & 14 & 33 \\ 31 & 20 & 17 \end{pmatrix}$ **1 mark**

(b) Subtract the row minimum from each row: $\begin{pmatrix} 4 & 0 & 18 \\ 16 & 0 & 19 \\ 14 & 3 & 0 \end{pmatrix}$ **M1**

Subtract the column minimum from each column: $\begin{pmatrix} 0 & 0 & 18 \\ 12 & 0 & 19 \\ 10 & 3 & 0 \end{pmatrix}$ **A1**

The zeros now need 3 lines, the same as the number of rows, so a complete allocation exists: A-1, B-3, D-4 **M1**

Value from the original table = $15 + 14 + 17 = 46$ hours **A1**

Answer: A to 1, B to 3, D to 4; 46 hours **4 marks**

(c) Total = $14 + 46 = 60$ hours **M1**

Increase = $60 - 55 = 5$ hours **A1**

Answer: 60 hours, an increase of 5 hours **2 marks**

25 (8 marks)

(a) Maximising, so subtract every entry from the largest entry, 90: $\begin{pmatrix} 17 & 41 & 36 & 8 & 22 \\ 42 & 52 & 21 & 32 & 0 \\ 56 & 5 & 14 & 49 & 45 \\ 39 & 54 & 58 & 2 & 30 \\ 47 & 30 & 60 & 21 & 44 \end{pmatrix}$ **M1 A1**

Answer: Each entry subtracted from the largest entry: $\begin{pmatrix} 17 & 41 & 36 & 8 & 22 \\ 42 & 52 & 21 & 32 & 0 \\ 56 & 5 & 14 & 49 & 45 \\ 39 & 54 & 58 & 2 & 30 \\ 47 & 30 & 60 & 21 & 44 \end{pmatrix}$ **2 marks**

(b) Subtract the row minimum from each row: $\begin{pmatrix} 9 & 33 & 28 & 0 & 14 \\ 42 & 52 & 21 & 32 & 0 \\ 51 & 0 & 9 & 44 & 40 \\ 37 & 52 & 56 & 0 & 28 \\ 26 & 9 & 39 & 0 & 23 \end{pmatrix}$ **M1**

Subtract the column minimum from each column:

0	33	19	0	14
33	52	12	32	0
42	0	0	44	40
28	52	47	0	28
17	9	30	0	23

A1

The zeros need only 4 lines (rows S1, S2, S3 and column R4), fewer than 5. The smallest uncovered entry is 9: subtract it from the uncovered entries and add it to the entries covered twice:

M1 A1

0	33	19	9	14
33	52	12	41	0
42	0	0	53	40
19	43	38	0	19
8	0	21	0	14

The zeros now need 5 lines, the same as the number of rows, so a complete allocation exists:

M1

S1-R1, S2-R5, S3-R3, S4-R4, S5-R2

Value from the original table = $73 + 90 + 76 + 88 + 60 = \text{£}387$ thousand

A1

Answer: S1 to R1, S2 to R5, S3 to R3, S4 to R4, S5 to R2; total £387 thousand

6 marks

26 (7 marks)

(a) Total cost = sum of the cost times x_{ij} over all i and j

M1

Minimise $C = 11x_{A1} + 15x_{A2} + 9x_{A3} + 14x_{B1} + 10x_{B2} + 13x_{B3} + 12x_{C1} + 16x_{C2} + 8x_{C3}$

A1

Answer: $C = 11x_{A1} + 15x_{A2} + 9x_{A3} + 14x_{B1} + 10x_{B2} + 13x_{B3} + 12x_{C1} + 16x_{C2} + 8x_{C3}$

2 marks

(b) Each worker: $x_{A1} + x_{A2} + x_{A3} = 1$, $x_{B1} + x_{B2} + x_{B3} = 1$, $x_{C1} + x_{C2} + x_{C3} = 1$

B1 B1

Each job: $x_{A1} + x_{B1} + x_{C1} = 1$, $x_{A2} + x_{B2} + x_{C2} = 1$, $x_{A3} + x_{B3} + x_{C3} = 1$

B1

Answer: Row sums = 1 and column sums = 1

3 marks

(c) $x_{A1} = 1$ for x_{A1} , x_{B2} , x_{C3} and all other $x_{ij} = 0$

B1

$C = 11 + 10 + 8 = 29$

A1

Answer: $C = 29$

2 marks

27 (12 marks)

Replace each forbidden entry by M, a very large cost

B1

Maximising, so subtract every entry from the largest entry, 85:

$$\begin{pmatrix} 51 & 46 & 41 & 28 \\ 0 & 30 & 4 & 0 \\ 10 & 42 & 42 & 11 \\ 53 & M & 15 & 8 \\ 23 & 52 & 14 & 33 \end{pmatrix}$$
M1 A1

There are 5 employees and 4 tasks, so add a dummy column of zeros to make the table square:

B1

$$\begin{pmatrix} 51 & 46 & 41 & 28 & 0 \\ 0 & 30 & 4 & 0 & 0 \\ 10 & 42 & 42 & 11 & 0 \\ 53 & M & 15 & 8 & 0 \\ 23 & 52 & 14 & 33 & 0 \end{pmatrix}$$

Subtract the row minimum from each row:

$$\begin{pmatrix} 51 & 46 & 41 & 28 & 0 \\ 0 & 30 & 4 & 0 & 0 \\ 10 & 42 & 42 & 11 & 0 \\ 53 & M & 15 & 8 & 0 \\ 23 & 52 & 14 & 33 & 0 \end{pmatrix}$$
M1

Subtract the column minimum from each column:

$$\begin{pmatrix} 51 & 16 & 37 & 28 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 10 & 12 & 38 & 11 & 0 \\ 53 & M & 11 & 8 & 0 \\ 23 & 22 & 10 & 33 & 0 \end{pmatrix}$$
A1

The zeros need only 2 lines (row E2 and column D), fewer than 5. The smallest uncovered entry is 8: subtract it from the uncovered entries and add it to the entries covered twice:

M1 A1

$$\begin{pmatrix} 43 & 8 & 29 & 20 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 2 & 4 & 30 & 3 & 0 \\ 45 & M & 3 & 0 & 0 \\ 15 & 14 & 2 & 25 & 0 \end{pmatrix}$$

The zeros need only 3 lines (rows E2, E4 and column D), fewer than 5. The smallest uncovered entry is 2: subtract it from the uncovered entries and add it to the entries covered twice:

M1 A1

$$\begin{pmatrix} 41 & 6 & 27 & 18 & 0 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 2 & 28 & 1 & 0 \\ 45 & M & 3 & 0 & 2 \\ 13 & 12 & 0 & 23 & 0 \end{pmatrix}$$

The zeros now need 5 lines, the same as the number of rows, so a complete allocation exists:

M1

E1-D, E2-T2, E3-T1, E4-T4, E5-T3

Value from the original table = $55 + 75 + 77 + 71 = \text{£}278$

A1

Answer: E2 to T2, E3 to T1, E4 to T4, E5 to T3; E1 not used; total £278 hundred

12 marks**28 (8 marks)**

(a) Every allocation contains exactly one entry from row r , so its cost is reduced by exactly k

E1

The cost of every allocation changes by the same amount k

M1

So the ordering of the allocations by cost is unchanged, and the same allocations are cheapest

E1

Answer: Every allocation uses exactly one entry of row r

3 marks

(b) Total of c over an allocation = nK – total of p over that allocation (n entries are used)

M1

nK is constant, so minimising the total of c is the same as maximising the total of p

E1

Answer: Total $c = nK$ – total p

2 marks

(c) Original cost of any allocation = reduced cost + the total R of all amounts subtracted (each row and column amount is used exactly once)

M1

Reduced entries are ≥ 0 , so reduced cost ≥ 0 and the original cost is $\geq R$

E1

The allocation on zeros has reduced cost 0, so its cost is exactly R , the least possible

E1

Answer: Cost $\geq R$ for all allocations, with equality for the zero allocation

3 marks

29 (10 marks)

Each worker can do up to two jobs, so each worker is written twice (A and A', B and B', C and C') with identical rows

B1

There are 6 rows and 5 jobs, so add a dummy column of zeros to make the table square:

B1

$$\begin{pmatrix} 12 & 17 & 16 & 15 & 35 & 0 \\ 12 & 17 & 16 & 15 & 35 & 0 \\ 18 & 38 & 37 & 10 & 29 & 0 \\ 18 & 38 & 37 & 10 & 29 & 0 \\ 40 & 27 & 32 & 26 & 37 & 0 \\ 40 & 27 & 32 & 26 & 37 & 0 \end{pmatrix}$$

Subtract the row minimum from each row:

$$\begin{pmatrix} 12 & 17 & 16 & 15 & 35 & 0 \\ 12 & 17 & 16 & 15 & 35 & 0 \\ 18 & 38 & 37 & 10 & 29 & 0 \\ 18 & 38 & 37 & 10 & 29 & 0 \\ 40 & 27 & 32 & 26 & 37 & 0 \\ 40 & 27 & 32 & 26 & 37 & 0 \end{pmatrix}$$

M1

Subtract the column minimum from each column:

$$\begin{pmatrix} 0 & 0 & 0 & 5 & 6 & 0 \\ 0 & 0 & 0 & 5 & 6 & 0 \\ 6 & 21 & 21 & 0 & 0 & 0 \\ 6 & 21 & 21 & 0 & 0 & 0 \\ 28 & 10 & 16 & 16 & 8 & 0 \\ 28 & 10 & 16 & 16 & 8 & 0 \end{pmatrix}$$

A1

The zeros need only 5 lines (rows A, A', B, B' and column D), fewer than 6. The smallest uncovered entry is 8: subtract it from the uncovered entries and add it to the entries covered twice:

M1 A1

$$\begin{pmatrix} 0 & 0 & 0 & 5 & 6 & 8 \\ 0 & 0 & 0 & 5 & 6 & 8 \\ 6 & 21 & 21 & 0 & 0 & 8 \\ 6 & 21 & 21 & 0 & 0 & 8 \\ 20 & 2 & 8 & 8 & 0 & 0 \\ 20 & 2 & 8 & 8 & 0 & 0 \end{pmatrix}$$

The zeros need only 5 lines (rows A, A' and columns 4, 5, D), fewer than 6. The smallest uncovered entry is 2: subtract it from the uncovered entries and add it to the entries covered twice:

M1 A1

$$\begin{pmatrix} 0 & 0 & 0 & 7 & 8 & 10 \\ 0 & 0 & 0 & 7 & 8 & 10 \\ 4 & 19 & 19 & 0 & 0 & 8 \\ 4 & 19 & 19 & 0 & 0 & 8 \\ 18 & 0 & 6 & 8 & 0 & 0 \\ 18 & 0 & 6 & 8 & 0 & 0 \end{pmatrix}$$

The zeros now need 6 lines, the same as the number of rows, so a complete allocation exists: A-1, A'-3, B-4, B'-5, C-2, C'-D

M1

Value from the original table = $12 + 16 + 10 + 29 + 27 = 94$ hours

A1

Answer: A does jobs 1 and 3; B does jobs 4 and 5; C does job 2; total 94 hours

10 marks**30 (11 marks)**

(a) Row minima: 14, 10, 9, 17, 14 (total 64)

M1

Reduced by rows: $\begin{pmatrix} 4 & 22 & 21 & 16 & 0 \\ 9 & 25 & 7 & 18 & 0 \\ 7 & 23 & 4 & 0 & 16 \\ 12 & 12 & 19 & 6 & 0 \\ 2 & 0 & 15 & 6 & 8 \end{pmatrix}$; column minima 2, 0, 4, 0, 0 (total 6)

A1

Answer: 64 and 6

2 marks

(b) Adjustment 1: 3 lines (rows 3, 5 and column 5), smallest uncovered entry 2; the bound rises by $2 \times (5 - 3) = 4$

M1

Adjustment 2: 4 lines (rows 1, 3, 5 and column 5), smallest uncovered entry 1; the bound rises by

M1 A1

$$1 \times (5 - 4) = 1$$

Answer: m = 2, k = 3, increase 4; m = 1, k = 4, increase 1

3 marks

(c) Minimum cost = $64 + 6 + 4 + 1 = 75$

M1

Allocation on the final zeros: 1-1, 2-3, 3-4, 4-5, 5-2 with cost $18 + 17 + 9 + 17 + 14 = 75$, which agrees

A1

Answer: Minimum cost 75

2 marks

(d) Let r rows and c columns be covered, $k = r + c < n$. The adjustment subtracts m from the $n - r$ uncovered rows, then adds m to the c covered columns

M1

Check each entry: uncovered row and column: $-m$; uncovered row and covered column: $-m + m = 0$; covered row and uncovered column: 0 ; covered row and column: $+m$

M1

Every allocation uses one entry from each row and each column, so its reduced cost changes by $-m(n - r) + mc = -m(n - k)$

M1

So the amount removed from every allocation's reduced cost, and hence added to the lower bound, is $m(n - k)$

A1

Answer: Increase = $m(n - k)$

4 marks

Transportation Problems

1 (2 marks)

Total supply = 100 and total demand = 90, so supply exceeds demand by 10

B1

Add a dummy shop with demand 10, and unit cost 0 from every depot

B1

Answer: Supply 100, demand 90; add a dummy shop of demand 10 with zero costs

2 marks

2 (2 marks)

$m + n - 1 = 4 + 5 - 1 = 8$ occupied cells

B1

If fewer than 8 are occupied the solution is degenerate (a cell must be given an allocation of 0 so the method can continue)

E1

Answer: 8 cells; fewer means degenerate

2 marks

3 (2 marks)

First: $AW = 20$

B1

Second: $AX = 10$

B1

Answer: $AW = 20$, then $AX = 10$

2 marks

4 (2 marks)

Cost = $15 \times 12 + 35 \times 8 + 5 \times 14 + 5 \times 7 + 25 \times 7 = 740$

M1 A1

Answer: Total cost 740

2 marks

5 (2 marks)

(a) $x_{A1} + x_{A2} + x_{A3} = 40$

B1

Answer: $x_{A1} + x_{A2} + x_{A3} = 40$

1 mark

(b) $x_{A2} + x_{B2} + x_{C2} = 35$

B1

Answer: $x_{A2} + x_{B2} + x_{C2} = 35$

1 mark

6 (2 marks)

No goods are actually moved to a dummy destination, so there is no cost

E1

A dummy allocation is surplus stock which stays at its source

E1

Answer: Zero cost as nothing is transported; dummy allocation is unused supply

2 marks

7 (4 marks)

North-west corner method: $AW = 20$, $BW = 10$, $CW = 15$, $CX = 5$, $CY = 5$, $CZ = 10$

M1 A1

Cost = $20 \times 8 + 10 \times 3 + 15 \times 11 + 5 \times 13 + 5 \times 15 + 10 \times 2 = 515$

M1 A1

Answer: Cost £515

4 marks

	W	X	Y	Z	Supply
A	⁸ 20	³	¹³	⁶	20
B	³ 10	⁷	⁹	¹²	10
C	¹¹ 15	¹³ 5	¹⁵ 5	² 10	35
Demand	45	5	5	10	

8 (4 marks)

Least cost method, cheapest available cell first: CX (cost 2) = 10, BY (cost 5) = 35, AY (cost 5) = 10, CY (cost 8) = 5, CW (cost 13) = 5

M1 A1

Cost = $10 \times 5 + 35 \times 5 + 5 \times 13 + 10 \times 2 + 5 \times 8 = 350$

M1 A1

Answer: Cost 350

4 marks

	W	X	Y	Supply
A	⁷	¹⁰	⁵ 10	10
B	¹⁴	¹²	⁵ 35	35
C	¹³ 5	² 10	⁸ 5	20
Demand	5	10	50	

9 (5 marks)

Total supply = 90 and total demand = 95, so add a dummy source with supply 5 and zero costs

B1

North-west corner method: AW = 35, AX = 5, BX = 15, CX = 20, CY = 15, DummyY = 5

M1 A1

Cost (dummy allocations cost 0) = $35 \times 12 + 5 \times 10 + 15 \times 9 + 20 \times 2 + 15 \times 5 + 5 \times 0 = 720$

M1 A1

Answer: Cost £720

5 marks

	W	X	Y	Supply
A	¹² 35	¹⁰ 5	¹⁴	40
B	⁹	⁹ 15	¹²	15
C	¹³	² 20	⁵ 15	35
Dummy	⁰	⁰	⁰ 5	5
Demand	35	40	20	

10 (3 marks)

For every occupied cell $R_i + K_j = c_{ij}$: $R_A + K_W = 7$; $R_A + K_X = 9$; $R_A + K_Y = 10$; $R_B + K_Y = 12$; **M1**

$R_B + K_Z = 7$; $R_C + K_Z = 3$

With $R_A = 0$, solving in turn: $R = (0, 2, -2)$ **A1**

$K = (7, 9, 10, 5)$ **A1**

Answer: $R = (0, 2, -2)$, $K = (7, 9, 10, 5)$ **3 marks**

11 (3 marks)

$I_{ij} = c_{ij} - R_i - K_j$: $I_{AX} = 14 - 0 - 13 = 1$; $I_{AY} = 14 - 0 - 7 = 7$; $I_{BX} = 15 - 3 - 13 = -1$; $I_{BY} = 13 - 3 - 7 = 3$ **M1 A1**

$I_{BX} = -1$ is negative so the allocation is not optimal (a cell can be used to reduce cost) **A1**

Answer: Not optimal; most negative index is BX **3 marks**

12 (4 marks)

Most negative index is $CW = -8$, so CW enters **B1**

Loop: $CW (+)$, $CX (-)$, $BX (+)$, $BW (-)$ **M1**

$\theta = 5$ (smallest allocation on a $(-)$ cell) **A1**

New allocation: $AW = 10$, $BX = 15$, $CW = 5$, $CX = 5$, $CY = 20$ **A1**

Answer: Enter CW , $\theta = 5$; $AW = 10$, $BX = 15$, $CW = 5$, $CX = 5$, $CY = 20$ **4 marks**

	W	X	Y	Supply
A	¹¹ 10	¹⁴	¹⁰	10
B	⁴	⁷ 15	⁶	15
C	⁴ 5	¹⁵ 5	¹² 20	30
Demand	15	20	20	

13 (5 marks)

Minimise $C = 7x_{A1} + 9x_{A2} + 6x_{A3} + 8x_{B1} + 5x_{B2} + 10x_{B3}$ **B1**

Supply: $x_{A1} + x_{A2} + x_{A3} = 45$; $x_{B1} + x_{B2} + x_{B3} = 35$ **B1 B1**

Demand: $x_{A1} + x_{B1} = 20$; $x_{A2} + x_{B2} = 30$; $x_{A3} + x_{B3} = 30$ **B1**

$x_{ij} \geq 0$ for all i and j **B1**

Answer: Minimise $C = 7x_{A1} + 9x_{A2} + 6x_{A3} + 8x_{B1} + 5x_{B2} + 10x_{B3}$ **5 marks**

14 (5 marks)

With $R_A = 0$ and $R_i + K_j = c_{ij}$ on the occupied cells: $R = (0, 8, 6)$, $K = (2, 6, 2, 2)$ **M1 A1**

Improvement indices $I_{ij} = c_{ij} - R_i - K_j$ for the empty cells: $I_{AZ} = 7 - 0 - 2 = 5$; $I_{BW} = 11 - 8$ **M1**

$- 2 = 1$; $I_{BX} = 12 - 8 - 6 = -2$; $I_{BZ} = 15 - 8 - 2 = 5$; $I_{CW} = 11 - 6 - 2 = 3$; $I_{CX} = 9 - 6 - 6$

$= -3$

Most negative index is $I_{CX} = -3$, so CX enters. The stepping-stone loop is $CX (+)$, $CY (-)$, AY **M1**

$(+)$, $AX (-)$

$\theta =$ smallest allocation on a $(-)$ cell $= 5$. New allocation: $AW = 5$, $AY = 25$, $BY = 25$, $CX = 5$, **A1**

$CY = 15$, $CZ = 10$, cost 555

Answer: Cost falls from 570 to 555 **5 marks**

Iteration 1 (cost 570)

	W	X	Y	Z	Supply	R_i
A	² 5	⁶ 5	² 20	⁷ 5	30	0
B	¹¹ 1	¹² -2	¹⁰ 25	¹⁵ 5	25	8
C	¹¹ 3	⁹ -3	⁸ 20	⁸ 10	30	6
Demand	5	5	65	10		
K_j	2	6	2	2		

Final (cost 555)

	W	X	Y	Z	Supply	R_i
A	² 5	⁶ 3	² 25	⁷ 5	30	0
B	¹¹ 1	¹² 1	¹⁰ 25	¹⁵ 5	25	8
C	¹¹ 3	⁹ 5	⁸ 15	⁸ 10	30	6
Demand	5	5	65	10		
K_j	2	3	2	2		

15 (7 marks)

Supply 80 exceeds demand 75: add a dummy destination with demand 5 and zero costs

North-west corner method: $AW = 20$, $BW = 25$, $CW = 10$, $CX = 10$, $CY = 10$, $CDummy = 5$

With $R_A = 0$: $R = (0, -2, 8)$, $K = (7, 4, 1, -8)$

Improvement indices: $I_{AX} = 15 - 0 - 4 = 11$; $I_{AY} = 9 - 0 - 1 = 8$; $I_{ADummy} = 0 - 0 - (-8) = 8$;

$I_{BX} = 10 - (-2) - 4 = 8$; $I_{BY} = 12 - (-2) - 1 = 13$; $I_{BDummy} = 0 - (-2) - (-8) = 10$

All indices are non-negative, so the north-west corner solution is optimal; minimum cost = 625

Answer: Optimal; minimum cost £625

B1**M1 A1****M1 A1****M1****A1****7 marks**

Final (cost 625)

	W	X	Y	Dummy	Supply	R_i
A	⁷ 20	¹⁵ 11	⁹ 8	⁰ 8	20	0
B	⁵ 25	¹⁰ 8	¹² 13	⁰ 10	25	-2
C	¹⁵ 10	¹² 10	⁹ 10	⁰ 5	35	8
Demand	55	10	10	5		
K_j	7	4	1	-8		

16 (7 marks)

With $R_A = 0$ and $R_i + K_j = c_{ij}$ on occupied cells: $R = (0, 5, -1)$, $K = (3, 6, 6, 3)$

M1 A1

Improvement indices: $I_{AY} = 12 - 0 - 6 = 6$; $I_{AZ} = 10 - 0 - 3 = 7$; $I_{BW} = 14 - 5 - 3 = 6$; $I_{CW} = 13 - (-1) - 3 = 11$; $I_{CX} = 6 - (-1) - 6 = 1$; $I_{CZ} = 13 - (-1) - 3 = 11$

M1 A1

All indices are positive, so no change can reduce the cost: the allocation is optimal

E1

Minimum cost = $30 \times 3 + 5 \times 6 + 5 \times 11 + 5 \times 11 + 30 \times 8 + 20 \times 5 = 570$

M1 A1

Answer: Optimal; minimum cost 570

7 marks

Final (cost 570)

	W	X	Y	Z	Supply	R_i
A	³ 30	⁶ 5	¹² 6	¹⁰ 7	35	0
B	¹⁴ 6	¹¹ 5	¹¹ 5	⁸ 30	40	5
C	¹³ 11	⁶ 1	⁵ 20	¹³ 11	20	-1
Demand	30	10	25	30		
K_j	3	6	6	3		

17 (5 marks)

Loop: CW (+), CY (-), BY (+), BX (-), AX (+), AW (-)

M1

θ = smallest allocation on the (-) cells = 5

A1

New allocation: AX = 30, BX = 15, BY = 15, CW = 5, CY = 30, CZ = 5

A1

Cost change = $\theta \times I_{CW} = 5 \times (-13) = -65$, so the cost falls from 740 to 675

M1 A1

Answer: AX = 30, BX = 15, BY = 15, CW = 5, CY = 30, CZ = 5; cost 675

5 marks

	W	X	Y	Z	Supply
A	⁹	⁴ 30	¹⁰	⁹	30
B	⁶	¹² 15	⁸ 15	⁷	30
C	² 5	¹⁵	⁶ 30	¹³ 5	40
Demand	5	45	45	5	

18 (5 marks)

North-west corner method: $AW = 35$, $AX = 0$, $BX = 10$, $BY = 15$, $CY = 20$

M1 A1

Both a supply and a demand are used up together, so only 4 cells would be occupied, fewer than $m + n - 1 = 5$. An allocation of 0 is written in AX so that the shadow costs can be found

E1

With $R_A = 0$: $R = (0, 1, -1)$, $K = (4, 12, 12)$

M1 A1

Answer: $AW = 35$, $AX = 0$, $BX = 10$, $BY = 15$, $CY = 20$; $R = (0, 1, -1)$, $K = (4, 12, 12)$

5 marks

	W	X	Y	Supply	R_i
A	⁴ 35	¹² 0	⁹	35	0
B	¹³	¹³ 10	¹³ 15	25	1
C	¹⁴	¹⁵	¹¹ 20	20	-1
Demand	35	10	35		
K_j	4	12	12		

19 (7 marks)

North-west corner method: $AW = 5$, $AX = 5$, $AY = 10$, $BY = 25$, $CY = 20$

M1 A1

NW cost = $5 \times 10 + 5 \times 12 + 10 \times 3 + 25 \times 14 + 20 \times 11 = 710$

M1

Least cost method, cheapest available cell first: AY (cost 3) = 20, CW (cost 3) = 5, CY (cost 11) = 15, BY (cost 14) = 20, BX (cost 20) = 5

M1 A1

Least cost solution cost (£) = $20 \times 3 + 5 \times 20 + 20 \times 14 + 5 \times 3 + 15 \times 11 = 620$

A1

Comparison: the least cost method gives £90 less, as it uses the cheaper routes first

E1

Answer: NW £710; least cost £620

7 marks

20 (4 marks)

The improvement index is $c_{ij} - R_i - K_j$; a negative value means the cost can be reduced, so the entering cell has the most negative index, not the largest positive one

E1

Indices: $I_{AY} = 12 - 0 - 17 = -5$; $I_{AZ} = 5 - 0 - 17 = -12$; $I_{BW} = 7 - (-11) - 2 = 16$; $I_{BZ} = 11 - (-11) - 17 = 5$; $I_{CW} = 7 - (-15) - 2 = 20$; $I_{CX} = 14 - (-15) - 15 = 14$

M1 A1

The most negative is $AZ = -12$, so AZ should enter (not CW)

A1

Answer: Should choose the most negative index; correct entering cell AZ

4 marks

21 (9 marks)

(a) North-west corner method: $AW = 10$, $AX = 25$, $BX = 20$, $CX = 35$, $CY = 5$

M1 A1

Answer: $AW = 10$, $AX = 25$, $BX = 20$, $CX = 35$, $CY = 5$

2 marks

(b) With $R_A = 0$ and $R_i + K_j = c_{ij}$ on the occupied cells: $R = (0, -6, -8)$, $K = (11, 16, 22)$ **M1**

Improvement indices $I_{ij} = c_{ij} - R_i - K_j$ for the empty cells: $I_{AY} = 13 - 0 - 22 = -9$; $I_{BW} = 18 - (-6) - 11 = 13$; $I_{BY} = 12 - (-6) - 22 = -4$; $I_{CW} = 11 - (-8) - 11 = 8$ **M1**

Most negative index is $I_{AY} = -9$, so AY enters. The stepping-stone loop is AY (+), AX (-), CX (+), CY (-) **M1**

θ = smallest allocation on a (-) cell = 5. New allocation: $AW = 10$, $AX = 20$, $AY = 5$, $BX = 20$, $CX = 40$, cost 1015 **A1**

Answer: New allocation $AW = 10$, $AX = 20$, $AY = 5$, $BX = 20$, $CX = 40$ **4 marks**

(c) Shadow costs $R = (0, -6, -8)$, $K = (11, 16, 13)$; indices $I_{BW} = 18 - (-6) - 11 = 13$; $I_{BY} = 12 - (-6) - 13 = 5$; $I_{CW} = 11 - (-8) - 11 = 8$; $I_{CY} = 14 - (-8) - 13 = 9$ **M1 A1**

All improvement indices are non-negative, so this allocation is optimal; the minimum cost is 1015 **A1**

Answer: Optimal; minimum cost 1015 **3 marks**

Iteration 1 (cost 1060)

	W	X	Y	Supply	R_i
A	¹¹ 10	¹⁶ 25	¹³ + ₋₉	35	0
B	¹⁸ ₁₃	¹⁰ 20	¹² ₋₄	20	-6
C	¹¹ ₈	⁸ + 35	¹⁴ - 5	40	-8
Demand	10	80	5		
K_j	11	16	22		

Final (cost 1015)

	W	X	Y	Supply	R_i
A	¹¹ 10	¹⁶ 20	¹³ 5	35	0
B	¹⁸ ₁₃	¹⁰ 20	¹² ₅	20	-6
C	¹¹ ₈	⁸ 40	¹⁴ ₉	40	-8
Demand	10	80	5		
K_j	11	16	13		

22 (9 marks)

(a) With $R_A = 0$ and $R_i + K_j = c_{ij}$ on the occupied cells: $R = (0, -2, 1)$, $K = (5, 11, 13)$ **M1 A1**

Improvement indices $I_{ij} = c_{ij} - R_i - K_j$ for the empty cells: $I_{AX} = 5 - 0 - 11 = -6$; $I_{AY} = 5 - 0 - 13 = -8$; $I_{BY} = 11 - (-2) - 13 = 0$; $I_{CW} = 15 - 1 - 5 = 9$ **M1**

Most negative index is $I_{AY} = -8$, so AY enters. The stepping-stone loop is AY (+), AW (-), BW (+), BX (-), CX (+), CY (-) **M1**

$\theta = 5$, and 2 cells on (-) positions reach 0 together. One (BX) leaves and the other(s) stay as 0. **A1**

New allocation: $AW = 20$, $AY = 5$, $BW = 25$, $CX = 10$, $CY = 0$, cost 320

Answer: New allocation $AW = 20$, $AY = 5$, $BW = 25$, $CX = 10$, $CY = 0$ **5 marks**

(b) The solution is degenerate (fewer than $m + n - 1$ positive allocations). The 0 must be kept as an occupied cell so that there are 5 cells and the shadow costs can be found **E1**

With $R_A = 0$: $R = (0, -2, 9)$, $K = (5, 3, 5)$; indices $I_{AX} = 5 - 0 - 3 = 2$; $I_{BX} = 9 - (-2) - 3 = 8$; $I_{BY} = 11 - (-2) - 5 = 8$; $I_{CW} = 15 - 9 - 5 = 1$ **M1 A1**

All improvement indices are non-negative, so this allocation is optimal; the minimum cost is 320 **A1**

Answer: Optimal; cost 320 **4 marks**

Iteration 1 (cost 360)

	W	X	Y	Supply	R_i
A	⁵ -25	⁵ -6	⁵ +8	25	0
B	³ +20	⁹ -5	¹¹ 0	25	-2
C	¹⁵ 9	¹² +5	¹⁴ -5	10	1
Demand	45	10	5		
K_j	5	11	13		

Final (cost 320)

	W	X	Y	Supply	R_i
A	⁵ 20	⁵ 2	⁵ 5	25	0
B	³ 25	⁹ 8	¹¹ 8	25	-2
C	¹⁵ 1	¹² 10	¹⁴ 0	10	9
Demand	45	10	5		
K_j	5	3	5		

23 (6 marks)

(a) With $R_A = 0$: $R = (0, -1, -13)$, $K = (8, 3, 15, 8)$

Indices: $I_{AZ} = 15 - 0 - 8 = 7$; $I_{BW} = 7 - (-1) - 8 = 0$; $I_{BX} = 10 - (-1) - 3 = 8$; $I_{CW} = 12 - (-13) - 8 = 17$; $I_{CX} = 13 - (-13) - 3 = 23$; $I_{CZ} = 10 - (-13) - 8 = 15$; none is negative, so the allocation is optimal

Answer: Optimal; cost 655

(b) The index of BW is 0, so BW can enter without changing the cost. Loop: BW (+), BY (-), AY (+), AW (-)

$\theta = 5$: new allocation $AW = 5$, $AX = 10$, $AY = 20$, $BW = 5$, $BZ = 30$, $CY = 20$

Cost = $5 \times 8 + 10 \times 3 + 20 \times 15 + 5 \times 7 + 30 \times 7 + 20 \times 2 = 655$, the same minimum

Answer: $AW = 5$, $AX = 10$, $AY = 20$, $BW = 5$, $BZ = 30$, $CY = 20$

M1

A1 A1

3 marks

M1

A1

A1

3 marks

	W	X	Y	Z	Supply
A	⁸ 5	³ 10	¹⁵ 20	¹⁵	35
B	⁷ 5	¹⁰	¹⁴	⁷ 30	35
C	¹²	¹³	² 20	¹⁰	20
Demand	10	10	40	30	

24 (9 marks)

Give route AW a cost M, so large that it is never used. Least cost method, cheapest available cell first: CW (cost 3) = 20, BZ (cost 4) = 15, CX (cost 5) = 20, AX (cost 11) = 5, AY (cost 15) = 5, AZ (cost 16) = 15

M1 A1

With $R_A = 0$ and $R_i + K_j = c_{ij}$ on the occupied cells: $R = (0, -12, -6)$, $K = (9, 11, 15, 16)$

M1 A1

Improvement indices $I_{ij} = c_{ij} - R_i - K_j$ for the empty cells: $I_{BW} = 13 - (-12) - 9 = 16$; $I_{BX} = 15 - (-12) - 11 = 16$; $I_{BY} = 15 - (-12) - 15 = 12$; $I_{CY} = 15 - (-6) - 15 = 6$; $I_{CZ} = 8 - (-6) - 16 = -2$

M1

Most negative index is $I_{CZ} = -2$, so CZ enters. The stepping-stone loop is CZ (+), CX (-), AX (+), AZ (-)

M1

θ = smallest allocation on a (-) cell = 15. New allocation: AX = 20, AY = 5, BZ = 15, CW = 20, CX = 5, CZ = 15, cost 560

A1

Shadow costs $R = (0, -10, -6)$, $K = (9, 11, 15, 14)$; all indices non-negative, so optimal: minimum cost 560

M1

Route AW is not used, as required

A1

Answer: AX = 20, AY = 5, BZ = 15, CW = 20, CX = 5, CZ = 15; minimum cost £560

9 marks

Iteration 1 (cost 590)

	W	X	Y	Z	Supply	R_i
A	⁻ M 5	¹¹ + 5	¹⁵ 5	¹⁶ - 15	25	0
B	¹³ 16	¹⁵ 16	¹⁵ 12	⁴ 15	15	-12
C	³ 20	⁵ - 20	¹⁵ 6	⁸ + -2	40	-6
Demand	20	25	5	30		
K_j	9	11	15	16		

Final (cost 560)

	W	X	Y	Z	Supply	R_i
A	⁻ M 20	¹¹ 20	¹⁵ 5	¹⁶ 2	25	0
B	¹³ 14	¹⁵ 14	¹⁵ 10	⁴ 15	15	-10
C	³ 20	⁵ 5	¹⁵ 6	⁸ 15	40	-6
Demand	20	25	5	30		
K_j	9	11	15	14		

25 (8 marks)

(a) Minimise $C = 6x_{A1} + 8x_{A2} + 5x_{A3} + 9x_{A4} + 7x_{B1} + 4x_{B2} + 8x_{B3} + 6x_{B4} + 9x_{C1} + 7x_{C2} + 6x_{C3} + 5x_{C4}$ **B1**

Supply (at most): $x_{A1} + x_{A2} + x_{A3} + x_{A4} \leq 40$; $x_{B1} + x_{B2} + x_{B3} + x_{B4} \leq 35$; $x_{C1} + x_{C2} + x_{C3} + x_{C4} \leq 30$ **B1**

Demand (exactly): $x_{A1} + x_{B1} + x_{C1} = 25$; $x_{A2} + x_{B2} + x_{C2} = 20$; $x_{A3} + x_{B3} + x_{C3} = 30$; $x_{A4} + x_{B4} + x_{C4} = 15$ **B1**

$x_{ij} \geq 0$ **B1**

Answer: Minimise C subject to supply $\leq$ and demand = **4 marks**

(b) Supply: $x_{A1} + x_{A2} + x_{A3} + x_{A4} + s_A = 40$; $x_{B1} + x_{B2} + x_{B3} + x_{B4} + s_B = 35$; $x_{C1} + x_{C2} + x_{C3} + x_{C4} + s_C = 30$ with all $s \geq 0$ **M1**

s_i is the unused supply at factory i ; together they are the demand of a dummy destination (15 units) with zero costs **E1**

Answer: Slack = unused supply = dummy destination **2 marks**

(c) Supplies used 40, 35, 15 $\leq$ 40, 35, 30; demands 25, 20, 30, 15 are all met **M1**

Cost = $25 \times 6 + 15 \times 5 + 20 \times 4 + 15 \times 6 + 15 \times 6 = £485$ **A1**

Answer: Cost £485 **2 marks**

26 (7 marks)

Let $c_{BY} = t$. With $R_A = 0$: $R = (0, 2, 6 - t)$, $K = (3, t - 4, t - 2)$ **M1 A1**

Indices: $I_{AX} = 7 - t$; $I_{AY} = 8 - t$; $I_{BX} = 15 - t$; $I_{CW} = t - 1$ **M1 A1**

Optimal when every index ≥ 0 : $7 - t \geq 0$ gives $t \leq 7$; $8 - t \geq 0$ gives $t \leq 8$; $15 - t \geq 0$ gives $t \leq 15$; $t - 1 \geq 0$ gives $t \geq 1$ **M1 A1**

Range: $1 \leq t \leq 7$ **A1**

Answer: $1 \leq t \leq 7$ **7 marks**

27 (12 marks)

Supply 115, demand 105: add a dummy destination of demand 10 with zero costs, and give route DX the cost M **B1 B1**

North-west corner method: $AW = 30$, $AX = 5$, $BX = 20$, $CX = 20$, $CY = 5$, $CZ = 10$, $DZ = 15$, $DDummy = 10$ **M1 A1**

With $R_A = 0$ and $R_i + K_j = c_{ij}$ on the occupied cells: $R = (0, -11, -3, 2)$, $K = (4, 16, 14, 7, -2)$ **M1 A1**

Improvement indices $I_{ij} = c_{ij} - R_i - K_j$ for the empty cells: $I_{AY} = 12 - 0 - 14 = -2$; $I_{AZ} = 13 - 0 - 7 = 6$; $I_{ADummy} = 0 - 0 - (-2) = 2$; $I_{BW} = 4 - (-11) - 4 = 11$; $I_{BY} = 3 - (-11) - 14 = 0$; $I_{BZ} = 3 - (-11) - 7 = 7$; $I_{BDummy} = 0 - (-11) - (-2) = 13$; $I_{CW} = 6 - (-3) - 4 = 5$; $I_{CDummy} = 0 - (-3) - (-2) = 5$; $I_{DW} = 13 - 2 - 4 = 7$; $I_{DY} = 7 - 2 - 14 = -9$ **M1**

Most negative index is $I_{DY} = -9$, so DY enters. The stepping-stone loop is DY (+), DZ (-), CZ (+), CY (-) **M1**

θ = smallest allocation on a (-) cell = 5. New allocation: $AW = 30$, $AX = 5$, $BX = 20$, $CX = 20$, $CZ = 15$, $DY = 5$, $DZ = 10$, $DDummy = 10$, cost 745 **A1**

Shadow costs $R = (0, -11, -3, 2)$, $K = (4, 16, 5, 7, -2)$ give indices $I_{AY} = 12 - 0 - 5 = 7$; $I_{AZ} = 13 - 0 - 7 = 6$; $I_{ADummy} = 0 - 0 - (-2) = 2$; $I_{BW} = 4 - (-11) - 4 = 11$; $I_{BY} = 3 - (-11) - 5 = 9$; $I_{BZ} = 3 - (-11) - 7 = 7$; $I_{BDummy} = 0 - (-11) - (-2) = 13$; $I_{CW} = 6 - (-3) - 4 = 5$; $I_{CY} = 11 - (-3) - 5 = 9$; $I_{CDummy} = 0 - (-3) - (-2) = 5$; $I_{DW} = 13 - 2 - 4 = 7$ **M1 A1**

All improvement indices are non-negative, so this allocation is optimal; the minimum cost is 745 **A1**

Answer: £745: AW 30, AX 5, BX 20, CX 20, CZ 15, DY 5, DZ 10; surplus at D (10) **12 marks**

Iteration 1 (cost 790)

	W	X	Y	Z	Dummy	Supply	R_i
A	⁴ 30	¹⁶ 5	¹² -2	¹³ 6	⁰ 2	35	0
B	⁴ 11	⁵ 20	³ 0	³ 7	⁰ 13	20	-11
C	⁶ 5	¹³ 20	¹¹ -5	⁴ +10	⁰ 5	35	-3
D	¹³ 7	⁻ M	⁷ +9	⁹ -15	⁰ 10	25	2
Demand	30	45	5	25	10		
K_j	4	16	14	7	-2		

Final (cost 745)

	W	X	Y	Z	Dummy	Supply	R_i
A	⁴ 30	¹⁶ 5	¹² 7	¹³ 6	⁰ 2	35	0
B	⁴ 11	⁵ 20	³ 9	³ 7	⁰ 13	20	-11
C	⁶ 5	¹³ 20	¹¹ 9	⁴ 15	⁰ 5	35	-3
D	¹³ 7	⁻ M	⁷ 5	⁹ 10	⁰ 10	25	2
Demand	30	45	5	25	10		
K_j	4	16	5	7	-2		

28 (12 marks)

(a) Largest profit is 15. Subtract every profit from 15 to get unit 'costs' $\begin{pmatrix} 2 & 11 & 13 \\ 4 & 4 & 14 \\ 3 & 8 & 6 \end{pmatrix}$, and

M1 A1

minimise

Answer: Subtract each profit from 15

2 marks

(b) North-west corner method: $AW = 10$, $AX = 10$, $BX = 15$, $BY = 15$, $CY = 15$

M1 A1

With $R_A = 0$ and $R_i + K_j = c_{ij}$ on the occupied cells: $R = (0, -7, -15)$, $K = (2, 11, 21)$

M1 A1

Improvement indices $I_{ij} = c_{ij} - R_i - K_j$ for the empty cells: $I_{AY} = 13 - 0 - 21 = -8$; $I_{BW} = 4 - (-7) - 2 = 9$; $I_{CW} = 3 - (-15) - 2 = 16$; $I_{CX} = 8 - (-15) - 11 = 12$

M1

Most negative index is $I_{AY} = -8$, so AY enters. The stepping-stone loop is AY (+), AX (-), BX (+), BY (-)

M1

θ = smallest allocation on a (-) cell = 10. New allocation: $AW = 10$, $AY = 10$, $BX = 25$, $BY = 5$, $CY = 15$, cost 410

A1

Shadow costs $R = (0, 1, -7)$, $K = (2, 3, 13)$; every index is non-negative: $I_{AX} = 11 - 0 - 3 = 8$;

M1 A1

$I_{BW} = 4 - 1 - 2 = 1$; $I_{CW} = 3 - (-7) - 2 = 8$; $I_{CX} = 8 - (-7) - 3 = 12$

Maximum profit = $10 \times 13 + 10 \times 2 + 25 \times 11 + 5 \times 1 + 15 \times 9 = 565$

A1

Answer: AW = 10, AY = 10, BX = 25, BY = 5, CY = 15; maximum profit £565

10 marks

Iteration 1 (cost 490)

Final (cost 410)

	W	X	Y	Supply	R_i
A	² 10	¹¹ 10	¹³ + ⁻⁸	20	0
B	⁴ ⁹	⁴ 15	¹⁴ 15	30	-7
C	³ ¹⁶	⁸ 12	⁶ 15	15	-15
Demand	10	25	30		
K_j	2	11	21		

	W	X	Y	Supply	R_i
A	² 10	¹¹ ⁸	¹³ 10	20	0
B	⁴ ¹	⁴ 25	¹⁴ 5	30	1
C	³ ⁸	⁸ 12	⁶ 15	15	-7
Demand	10	25	30		
K_j	2	3	13		

29 (9 marks)

(a) There are m supply equations and n demand equations, so $m + n$ equations in the $m \times n$ variables

B1

Adding all the supply equations gives the sum of all x_{ij} equal to the total supply, and adding all the demand equations gives the same sum equal to the total demand

M1

In a balanced problem these two totals are equal, so the equations are not independent: any one follows from the other $m + n - 1$

E1

So there are at most $m + n - 1$ independent equations, and a basic solution has at most $m + n - 1$ non-zero variables

A1

Answer: At most $m + n - 1$ independent equations

4 marks

(b) Moving one unit around the loop changes the cost by $c_{ij} - c_a + c_b - \dots$ (plus on the entering and (+) cells, minus on the (-) cells)

M1

Every cell on the loop other than the entering cell is occupied, so $c = R + K$ for each of them

M1

Replacing these costs, every R_k and every K_l on the loop appears once with a plus sign and once with a minus sign (the loop alternates along rows and columns), so they cancel

M1 E1

Only R_i and K_j for the entering cell remain, with a minus sign: change in cost = $c_{ij} - R_i - K_j = I_{ij}$

A1

Answer: I_{ij} equals the cost change per unit moved

5 marks

30 (12 marks)

(a) Demand exceeds supply by 25, so add a dummy source 'Short' of 25 units; its cost to each shop is the penalty per unit of unmet demand: £14, £4, £7

B1

Answer: Dummy source with the penalties as costs

1 mark

(b) North-west corner method: AW = 15, AX = 10, BX = 10, BY = 5, CY = 15, ShortY = 25

M1 A1

With $R_A = 0$ and $R_i + K_j = c_{ij}$ on the occupied cells: $R = (0, 4, -2, 2)$, $K = (4, 3, 5)$

M1 A1

Improvement indices $I_{ij} = c_{ij} - R_i - K_j$ for the empty cells: $I_{AY} = 10 - 0 - 5 = 5$; $I_{BW} = 14 - 4 - 4 = 6$; $I_{CW} = 4 - (-2) - 4 = 2$; $I_{CX} = 13 - (-2) - 3 = 12$; $I_{ShortW} = 14 - 2 - 4 = 8$; $I_{ShortX} = 4 - 2 - 3 = -1$

M1

Most negative index is $I_{ShortX} = -1$, so ShortX enters. The stepping-stone loop is ShortX (+),

M1

ShortY (-), BY (+), BX (-)

θ = smallest allocation on a (-) cell = 10. New allocation: AW = 15, AX = 10, BY = 15, CY = 15, ShortX = 10, ShortY = 15, cost 415

A1

Shadow costs $R = (0, 3, -3, 1)$, $K = (4, 3, 6)$ and all indices non-negative: $I_{AY} = 10 - 0 - 6 = 4$;

M1 A1

$I_{BW} = 14 - 3 - 4 = 7$; $I_{BX} = 7 - 3 - 3 = 1$; $I_{CW} = 4 - (-3) - 4 = 3$; $I_{CX} = 13 - (-3) - 3 = 13$; $I_{ShortW} = 14 - 1 - 4 = 9$

Minimum total cost (transport plus penalties) = 415

A1

Answer: AW = 15, AX = 10, BY = 15, CY = 15, ShortX = 10, ShortY = 15; minimum £415

10 marks

(c) The shortfall allocations from the dummy source are 15, 10 units, so shortages occur at Y and

A1

X

Answer: Y short by 15, X short by 10**1 mark***Iteration 1 (cost 425)*

	W	X	Y	Supply	R_i
A	15 ⁴	10 ³ ₅	10 ¹⁰	25	0
B	₆ 14	-10 ⁷	+5 ⁹	15	4
C	₂ 4	₁₂ 13	15 ³	15	-2
Short	₈ 14	+4 ⁴	-25 ⁷	25	2
Demand	15	20	45		
K_j	4	3	5		

Final (cost 415)

	W	X	Y	Supply	R_i
A	15 ⁴	10 ³ ₄	10 ¹⁰	25	0
B	₇ 14	₁ 7	15 ⁹	15	3
C	₃ 4	₁₃ 13	15 ³	15	-3
Short	₉ 14	10 ⁴	15 ⁷	25	1
Demand	15	20	45		
K_j	4	3	6		

Dynamic Programming

1 (2 marks)

S-A-D-T: $4 + 3 + 4 = 11$ km**B1**S-B-C-T: $6 + 2 + 8 = 16$ km**B1****Answer: 11 km and 16 km****2 marks**

2 (2 marks)

Number of routes from S: 3 arcs leaving S; count the routes through each vertex working back from T (G and H each have 1, D, E and F have 2 each, A has 4, B has 6, C has 4)

M1Total = $4 + 6 + 4 = 14$ **A1****Answer: 14 routes****2 marks**

3 (2 marks)

Via E: $4 + 7 = 11$. Via F: $6 + 5 = 11$ **M1**Both give 11, so $f(C) = 11$ and either E or F is a best action**A1****Answer: $f(C) = 11$ (E or F)****2 marks**

4 (2 marks)

 $f(X)$ is the length of the shortest route from X to the destination**E1**

Working backwards, the shortest routes from the next stage are already known, so only the next arc needs to be chosen at each vertex; the best route is optimal whatever the earlier choices

E1**Answer: $f(X)$ = shortest distance from X to the end; later stages are solved first****2 marks**

5 (2 marks)

Route 1: $\min(12, 9, 15, 11) = 9$; route 2: $\min(10, 14, 13, 10) = 10$ **B1**

Route 2 is better (larger minimum)

B1**Answer: 9 and 10; route 2****2 marks**

6 (3 marks)

State: the money (in £ million) still available at the start of the stage

B1

Action: the amount (£ million) given to the project at that stage

B1

Value: the greatest total return from this stage and the remaining stages starting from that state

B1**Answer: State = money left; action = amount given; value = best return from here on****3 marks**

7 (5 marks)

Stage 1: $f(F) = 7$ via T [T: 7]; $f(G) = 3$ via T [T: 3]

M1

Stage 2: $f(C) = 8$ via G [F: $8 + 7 = 15$; G: $5 + 3 = 8$]; $f(D) = 9$ via G [F: $3 + 7 = 10$; G: $6 + 3 = 9$]; $f(E) = 11$ via F [F: $4 + 7 = 11$; G: $9 + 3 = 12$]

M1

Stage 3: $f(A) = 12$ via C [C: $4 + 8 = 12$; D: $6 + 9 = 15$]; $f(B) = 11$ via D [D: $2 + 9 = 11$; E: $7 + 11 = 18$]

A1

Stage 4: $f(S) = 14$ via B [A: $5 + 12 = 17$; B: $3 + 11 = 14$]

A1

Shortest route: S-B-D-G-T with length 14

A1

Answer: S-B-D-G-T, 14 minutes

5 marks

Stage	State	Action	Value
1	F	T *	7
	G	T *	3
2	C	F	$8 + 7 = 15$
		G *	$5 + 3 = 8$
	D	F	$3 + 7 = 10$
		G *	$6 + 3 = 9$
	E	F *	$4 + 7 = 11$
3	A	G	$9 + 3 = 12$
		C *	$4 + 8 = 12$
		D	$6 + 9 = 15$
	B	D *	$2 + 9 = 11$
		E	$7 + 11 = 18$
4	S	A	$5 + 12 = 17$
		B *	$3 + 11 = 14$

8 (5 marks)

Stage 1: $f(D) = 9$ via T [T: 9]; $f(E) = 5$ via T [T: 5]

M1

Stage 2: $f(A) = 17$ via D [D: $8 + 9 = 17$; E: $5 + 5 = 10$]; $f(B) = 15$ via E [D: $4 + 9 = 13$; E: $10 + 5 = 15$]; $f(C) = 16$ via D [D: $7 + 9 = 16$; E: $6 + 5 = 11$]

M1 A1

Stage 3: $f(S) = 24$ via B [A: $6 + 17 = 23$; B: $9 + 15 = 24$; C: $7 + 16 = 23$]

A1

Longest route: S-B-E-T with length 24

A1

Answer: S-B-E-T, £24 hundred

5 marks

Stage	State	Action	Value
1	D	T *	9
	E	T *	5
2	A	D *	$8 + 9 = 17$
		E	$5 + 5 = 10$
	B	D	$4 + 9 = 13$
		E *	$10 + 5 = 15$
	C	D *	$7 + 9 = 16$
3	S	E	$6 + 5 = 11$
		A	$6 + 17 = 23$
		B *	$9 + 15 = 24$
		C	$7 + 16 = 23$

9 (4 marks)

Stage 1: $f(C) = 10$ via T [T: 10]; $f(D) = 15$ via T [T: 15]

M1

Stage 2: $f(A) = 12$ via D [C: $\min(9, 10) = 9$; D: $\min(12, 15) = 12$]; $f(B) = 10$ via C [C: $\min(13, 10) = 10$; D: $\min(8, 15) = 8$]

M1

Stage 3: $f(S) = 12$ via A [A: $\min(14, 12) = 12$; B: $\min(11, 10) = 10$]

A1

Maximin route: S-A-D-T, whose smallest value is 12

A1

Answer: S-A-D-T, 12 m

4 marks

Stage	State	Action	Value
1	C	T *	10
	D	T *	15
2	A	C	$\min(9, 10) = 9$
		D *	$\min(12, 15) = 12$
	B	C *	$\min(13, 10) = 10$
		D	$\min(8, 15) = 8$
3	S	A *	$\min(14, 12) = 12$
		B	$\min(11, 10) = 10$

10 (4 marks)

Stage 1: $f(C) = 9$ via T [T: 9]; $f(D) = 4$ via T [T: 4]

M1

Stage 2: $f(A) = 8$ via D [C: $\max(3, 9) = 9$; D: $\max(8, 4) = 8$]; $f(B) = 5$ via D [C: $\max(7, 9) = 9$; D: $\max(5, 4) = 5$]

M1

Stage 3: $f(S) = 5$ via B [A: $\max(6, 8) = 8$; B: $\max(4, 5) = 5$]

A1

Minimax route: S-B-D-T, whose largest value is 5

A1

Answer: S-B-D-T, 5 percent

4 marks

Stage	State	Action	Value
1	C	T *	9
	D	T *	4
2	A	C	$\max(3, 9) = 9$
		D *	$\max(8, 4) = 8$
	B	C	$\max(7, 9) = 9$
		D *	$\max(5, 4) = 5$
3	S	A	$\max(6, 8) = 8$
		B *	$\max(4, 5) = 5$

11 (4 marks)

Missing values: row 2 (action T) = 3; row 5 (action F) = $3 + 7 = 10$; row 7 (action F) = $4 + 7 = 11$; row 10 (action D) = $6 + 9 = 15$; row 13 (action A) = $5 + 12 = 17$

M1 A1

Stage 4: $f(S) = 14$ via B, so the shortest route is S-B-D-G-T

M1

Length 14

A1

Answer: S-B-D-G-T, length 14

4 marks

Stage	State	Action	Value
1	F	T *	7
	G	T *	3
2	C	F	$8 + 7 = 15$
		G *	$5 + 3 = 8$
	D	F	$3 + 7 = 10$
		G *	$6 + 3 = 9$
	E	F *	$4 + 7 = 11$
		G	$9 + 3 = 12$
3	A	C *	$4 + 8 = 12$
		D	$6 + 9 = 15$
	B	D *	$2 + 9 = 11$
		E	$7 + 11 = 18$
		A	$5 + 12 = 17$
4	S	A	$5 + 12 = 17$
		B *	$3 + 11 = 14$

12 (4 marks)

Last stage (project 2) with s units left: $f(s) = r_2(s)$: $f(0), f(1), f(2), f(3) = 0, 2, 5, 7$

M1

First stage with 3 units: $x = 0$: $0 + 7 = 7$; $x = 1$: $3 + 5 = 8$; $x = 2$: $5 + 2 = 7$; $x = 3$: $6 + 0 = 6$

M1 A1

Best: give 1 unit(s) to project 1 and 2 to project 2, return 8

A1

Answer: 1 unit(s) to project 1, 2 to project 2; return £8 thousand

4 marks

13 (3 marks)

Any route using arc BD increases by k , and the best route not using BD has length 17, while the original shortest has length 14

M1

S-B-D-G-T has length $14 + k$

M1

$14 + k \leq 17$, so $k \leq 3$

A1

Answer: $k = 3$

3 marks

14 (6 marks)

Stage 1: $f(I) = 6$ via T [T: 6]; $f(J) = 4$ via T [T: 4]

M1

Stage 2: $f(F) = 13$ via I [I: $7 + 6 = 13$]; $f(G) = 10$ via I [I: $4 + 6 = 10$; J: $8 + 4 = 12$]; $f(H) = 9$ via J [J: $9 + 6 = 15$; J: $5 + 4 = 9$]

M1

Stage 3: $f(C) = 19$ via F or G [F: $6 + 13 = 19$; G: $9 + 10 = 19$]; $f(D) = 15$ via G [F: $4 + 13 = 17$; G: $5 + 10 = 15$; H: $8 + 9 = 17$]; $f(E) = 13$ via G [G: $3 + 10 = 13$; H: $6 + 9 = 15$]

A1

Stage 4: $f(A) = 22$ via D [C: $5 + 19 = 24$; D: $7 + 15 = 22$]; $f(B) = 19$ via D [C: $3 + 19 = 22$; D: $4 + 15 = 19$; E: $8 + 13 = 21$]

A1

Stage 5: $f(S) = 23$ via B [A: $6 + 22 = 28$; B: $4 + 19 = 23$]

A1

Shortest route: S-B-D-G-I-T with length 23

A1

Answer: S-B-D-G-I-T, £23 hundred

6 marks

Stage	State	Action	Value
1	I	T *	6
	J	T *	4
2	F	I *	$7 + 6 = 13$
	G	I *	$4 + 6 = 10$
		J	$8 + 4 = 12$
	H	I	$9 + 6 = 15$
		J *	$5 + 4 = 9$
3	C	F *	$6 + 13 = 19$
		G *	$9 + 10 = 19$
	D	F	$4 + 13 = 17$
		G *	$5 + 10 = 15$
	E	H	$8 + 9 = 17$
		G *	$3 + 10 = 13$
		H	$6 + 9 = 15$
4	A	C	$5 + 19 = 24$
		D *	$7 + 15 = 22$
	B	C	$3 + 19 = 22$
		D *	$4 + 15 = 19$
		E	$8 + 13 = 21$
5	S	A	$6 + 22 = 28$
		B *	$4 + 19 = 23$

15 (6 marks)

Stage 1 (project 3): $f(s) = r_3(s) = 0, 5, 8, 9, 9, 9$ for $s = 0$ to 5

M1

Stage 2 (projects 2 and 3 with s units): $f(s) = 0, 5, 9, 12, 14, 16$ for $s = 0$ to 5, from the best split of s units

M1 A1

Stage 3 (all 5 units): $x = 0$ to 5 give 16, 18, 19, 18, 15, 10

M1 A1

Maximum return 19 with (2, 1, 2) units to projects 1, 2, 3

A1

Answer: Benefit 19 with units (2, 1, 2)

6 marks

Stage	State	Action	Value
1	0	x=0 *	0
	1	x=1 *	5
	2	x=2 *	8
	3	x=3 *	9
	4	x=4 *	9
	5	x=5 *	9
2	0	x=0 *	0 + 0 = 0
	1	x=0 *	0 + 5 = 5
		x=1	4 + 0 = 4
	2	x=0	0 + 8 = 8
		x=1 *	4 + 5 = 9
		x=2	6 + 0 = 6
	3	x=0	0 + 9 = 9
		x=1 *	4 + 8 = 12
		x=2	6 + 5 = 11
		x=3	8 + 0 = 8
	4	x=0	0 + 9 = 9
		x=1	4 + 9 = 13
		x=2 *	6 + 8 = 14
		x=3	8 + 5 = 13
		x=4	9 + 0 = 9
	5	x=0	0 + 9 = 9
		x=1	4 + 9 = 13
3	5	x=2	6 + 9 = 15
		x=3 *	8 + 8 = 16
		x=4	9 + 5 = 14
		x=5	9 + 0 = 9
		x=0	0 + 16 = 16
		x=1	4 + 14 = 18
		x=2 *	7 + 12 = 19
		x=3	9 + 9 = 18
		x=4	10 + 5 = 15
		x=5	10 + 0 = 10

16 (5 marks)

Stage 1: $f(F) = \max(0.95) = 0.95$ via T; $f(G) = \max(0.9) = 0.9$ via T

M1

Stage 2: $f(C) = \max(0.7 \times 0.95 = 0.665, 0.75 \times 0.9 = 0.675) = 0.675$ via G; $f(D) = \max(0.95 \times 0.95 = 0.9025, 0.8 \times 0.9 = 0.72) = 0.9025$ via F; $f(E) = \max(0.9 \times 0.95 = 0.855, 0.7 \times 0.9 = 0.63) = 0.855$ via F

M1

Stage 3: $f(A) = \max(0.7 \times 0.675 = 0.4725, 0.85 \times 0.9025 = 0.7671) = 0.7671$ via D; $f(B) = \max(0.7 \times 0.9025 = 0.63175, 0.7 \times 0.855 = 0.5985) = 0.63175$ via D

A1

Stage 4: $f(S) = \max(0.75 \times 0.7671 = 0.5753, 0.85 \times 0.63175 = 0.5370) = 0.5753$ via A

A1

Most reliable route: S-A-D-F-T, with probability 0.5753 (3 s.f.: 0.575)

A1

Answer: S-A-D-F-T; probability 0.575

5 marks

17 (6 marks)

Stage 1 (item 3): $f(s) = \text{value of } \lfloor s/5 \rfloor \text{ copies} = f(0) = 0, f(1) = 0, f(2) = 0, f(3) = 0, f(4) = 0, f(5) = 15, f(7) = 15, f(8) = 15, f(11) = 30$

M1

Stage 2 (items 2 and 3): for each remaining capacity s , choose the number of item 2 to maximise $11k + f(s - 4k)$: $f(2) = 0, f(5) = 15, f(8) = 22, f(11) = 30$

M1 A1

Stage 3 (all items, capacity 11): $8k + f(11 - 3k)$ for $k = 0$ to 3 gives 30, 30, 31, 24

M1 A1

Best: 2 of item 1, 0 of item 2, 1 of item 3, total value £31

A1

Answer: 2, 0 and 1 of types 1, 2 and 3; value £31

6 marks

Stage	State	Action	Value
1	0 kg	0 of item 3 *	0
	1 kg	0 of item 3 *	0
	11 kg	0 of item 3 1 of item 3	0 15
		2 of item 3 *	30
	2 kg	0 of item 3 *	0
	3 kg	0 of item 3 *	0
	4 kg	0 of item 3 *	0
	5 kg	0 of item 3	0
		1 of item 3 *	15
	7 kg	0 of item 3	0
		1 of item 3 *	15
	8 kg	0 of item 3	0
		1 of item 3 *	15
2	11 kg	0 of item 2 *	0 + 30 = 30
		1 of item 2	11 + 15 = 26
		2 of item 2	22 + 0 = 22
	2 kg	0 of item 2 *	0 + 0 = 0
	5 kg	0 of item 2 *	0 + 15 = 15
		1 of item 2	11 + 0 = 11
	8 kg	0 of item 2	0 + 15 = 15
		1 of item 2	11 + 0 = 11
		2 of item 2 *	22 + 0 = 22
3	11 kg	0 of item 1	0 + 30 = 30
		1 of item 1	8 + 22 = 30
		2 of item 1 *	16 + 15 = 31
		3 of item 1	24 + 0 = 24

18 (6 marks)

Month 3 (stage 1): closing stock must be 0, so make 2 – s: $f(0) = 8$, $f(1) = 6$, $f(2) = 0$

M1

Month 2 (stage 2): $f(s) = f(0) = 16$, $f(1) = 14$, $f(2) = 12$, $f(3) = 8$ (cost of making + holding + f of closing stock)

M1 A1

Month 1 (stage 3): opening stock 0, make x with closing stock 0 to 3: make 2: $8 + 16 = 24$; make 3: $11 + 14 = 25$; make 4: $14 + 12 = 26$; make 5: $17 + 8 = 25$

M1 A1

Best plan: make 2, 5 and 0 in months 1, 2 and 3, total cost £24 hundred

A1

Answer: Make 2, 5, 0; total cost £24 hundred

6 marks

Stage	State	Action	Value
1	stock 0	make 2 *	8
	stock 1	make 1 *	6
	stock 2	make 0 *	0
2	stock 0	make 3 make 4 make 5 *	$10 + 8 = 18$ $13 + 6 = 19$ $16 + 0 = 16$
	stock 1	make 2 make 3 make 4 *	$8 + 8 = 16$ $11 + 6 = 17$ $14 + 0 = 14$
	stock 2	make 1 make 2 make 3 *	$6 + 8 = 14$ $9 + 6 = 15$ $12 + 0 = 12$
	stock 3	make 0 * make 1 make 2	$0 + 8 = 8$ $7 + 6 = 13$ $10 + 0 = 10$
3	stock 0	make 2 * make 3 make 4 make 5	$8 + 16 = 24$ $11 + 14 = 25$ $14 + 12 = 26$ $17 + 8 = 25$

19 (5 marks)

Stage 1: $f(E) = 5$ via T [T: 5]; $f(F) = 3$ via T [T: 3]

M1

Stage 2: $f(C) = 9$ via E [E: $4 + 5 = 9$; F: $7 + 3 = 10$]; $f(D) = 5$ via F [E: $3 + 5 = 8$; F: $2 + 3 = 5$]

M1

Stage 3: $f(A) = 11$ via D [C: $3 + 9 = 12$; D: $6 + 5 = 11$]; $f(B) = 10$ via D [C: $2 + 9 = 11$; D: $5 + 5 = 10$]

A1

Stage 4: $f(S) = 15$ via A or B [A: $4 + 11 = 15$; B: $5 + 10 = 15$]

A1

Shortest length 15; the routes are S-A-D-F-T and S-B-D-F-T

A1

Answer: £15; S-A-D-F-T and S-B-D-F-T

5 marks

Stage	State	Action	Value
1	E	T *	5
	F	T *	3
2	C	E *	$4 + 5 = 9$
		F	$7 + 3 = 10$
	D	E	$3 + 5 = 8$
		F *	$2 + 3 = 5$
3	A	C	$3 + 9 = 12$
		D *	$6 + 5 = 11$
	B	C	$2 + 9 = 11$
		D *	$5 + 5 = 10$
4	S	A *	$4 + 11 = 15$
		B *	$5 + 10 = 15$

20 (4 marks)

The student's route S-B-D-F-T has length $3 + 2 + 3 + 7 = 15$

M1

The student always takes the shortest arc from the present vertex, which ignores what remains to be travelled; a short arc can lead to long arcs later

E1

Working back with dynamic programming gives $f(S) = 14$ along S-B-D-G-T

M1

So the student's route is 1 longer than the shortest route

A1

Answer: Student 15, true shortest 14 (by S-B-D-G-T)

4 marks

21 (8 marks)

(a) Stage 1: $f(D) = 8$ via T [T: 8]; $f(E) = 5$ via T [T: 5]; $f(F) = 9$ via T [T: 9]

M1

Stage 2: $f(A) = 11$ via E [D: $4 + 8 = 12$; E: $6 + 5 = 11$; F: $9 + 9 = 18$]; $f(B) = 10$ via E [D: $3 + 8 = 11$; E: $5 + 5 = 10$; F: $7 + 9 = 16$]; $f(C) = 9$ via E [D: $6 + 8 = 14$; E: $4 + 5 = 9$; F: $3 + 9 = 12$]

M1

Stage 3: $f(S) = 15$ via B [A: $7 + 11 = 18$; B: $5 + 10 = 15$; C: $8 + 9 = 17$]

A1

Shortest: S-B-E-T, length 15

A1

Answer: S-B-E-T, 15

4 marks

(b) Stage 1: $f(D) = 8$ via T [T: 8]; $f(E) = 5$ via T [T: 5]; $f(F) = 9$ via T [T: 9]

M1

Stage 2: $f(A) = 18$ via F [D: $4 + 8 = 12$; E: $6 + 5 = 11$; F: $9 + 9 = 18$]; $f(B) = 16$ via F [D: $3 + 8 = 11$; E: $5 + 5 = 10$; F: $7 + 9 = 16$]; $f(C) = 14$ via D [D: $6 + 8 = 14$; E: $4 + 5 = 9$; F: $3 + 9 = 12$]

M1

Stage 3: $f(S) = 25$ via A [A: $7 + 18 = 25$; B: $5 + 16 = 21$; C: $8 + 14 = 22$]

A1

Longest: S-A-F-T, length 25

A1

Answer: S-A-F-T, 25

4 marks

Stage	State	Action	Value
1	D	T *	8
	E	T *	5
	F	T *	9
2	A	D	$4 + 8 = 12$
		E *	$6 + 5 = 11$
		F	$9 + 9 = 18$
	B	D	$3 + 8 = 11$
		E *	$5 + 5 = 10$
		F	$7 + 9 = 16$
	C	D	$6 + 8 = 14$
		E *	$4 + 5 = 9$
		F	$3 + 9 = 12$
3	S	A	$7 + 11 = 18$
		B *	$5 + 10 = 15$
		C	$8 + 9 = 17$

22 (8 marks)

Stage 1 (project 4): $f(s) = r_4(s)$: 0, 5, 9, 13, 15, 16, 16 for $s = 0$ to 6

M1

Stage 2 (projects 3, 4): $f(s) = 0, 5, 10, 15, 19, 23, 27$ for $s = 0$ to 6

M1 A1

Stage 3 (projects 2, 3, 4): $f(s) = 0, 5, 10, 15, 20, 25, 29$ for $s = 0$ to 6

M1 A1

Stage 4 (all 6 units): $x = 0$ to 6 give 29, 30, 28, 26, 23, 20, 15

M1

Working forwards from the best first-stage choice: project 1 gets 1 (5 left), project 2 gets 2 (3 left), project 3 gets 2 (1 left), project 4 gets 1

M1

Maximum total 30 with units 1, 2, 2, 1 to projects 1 to 4

A1

Answer: Units 1, 2, 2, 1; benefit 30

8 marks

23 (8 marks)

(a) Stage: one for each machine (stage 1 is machine 3); state: the number of workers still to be placed; action: the number of workers put on the machine

B1

Value: the largest possible smallest output among the machines of this and later stages, from the state

B1

Answer: Machines are stages; state = workers left

2 marks

(b) Stage 1 (machine 3): $f(s) = q_3(s)$: 0, 5, 7, 8, 9, 10, 11 for $s = 0$ to 6

M1

Stage 2 (machines 2 and 3): $f(s) = \max \text{ over } w \text{ of } \min(q_2(w), f(s - w))$: 0, 0, 3, 5, 6, 7, 8 for $s = 0$ to 6

M1 A1

Stage 3 (6 workers): $\min(q_1(x), f(6 - x))$ for $x = 0$ to 6 gives 0, 4, 6, 5, 3, 0, 0

M1 A1

Largest possible smallest output is 6, e.g. with (2, 2, 2) workers on machines 1 to 3

A1

Answer: 6 units per hour

6 marks

Stage	State	Action	Value
1	0	w=0 *	0
	1	w=1 *	5
	2	w=2 *	7
	3	w=3 *	8
	4	w=4 *	9
	5	w=5 *	10
	6	w=6 *	11
2	0	w=0 *	min(0, 0) = 0
	1	w=0 *	min(0, 5) = 0
		w=1 *	min(3, 0) = 0
	2	w=0	min(0, 7) = 0
		w=1 *	min(3, 5) = 3
	3	w=2	min(6, 0) = 0
		w=0	min(0, 8) = 0
		w=1	min(3, 7) = 3
		w=2 *	min(6, 5) = 5
	4	w=3	min(8, 0) = 0
		w=0	min(0, 9) = 0
		w=1	min(3, 8) = 3
		w=2 *	min(6, 7) = 6
	5	w=3	min(8, 5) = 5
		w=4	min(9, 0) = 0
		w=0	min(0, 10) = 0
		w=1	min(3, 9) = 3
		w=2	min(6, 8) = 6
		w=3 *	min(8, 7) = 7
	6	w=4	min(9, 5) = 5
		w=5	min(10, 0) = 0
		w=0	min(0, 11) = 0
		w=1	min(3, 10) = 3
		w=2	min(6, 9) = 6
		w=3 *	min(8, 8) = 8
		w=4	min(9, 7) = 7
		w=5	min(10, 5) = 5
		w=6	min(10, 0) = 0
3	6	w=0	min(0, 8) = 0
		w=1	min(4, 7) = 4
		w=2 *	min(7, 6) = 6
		w=3	min(9, 5) = 5
		w=4	min(10, 3) = 3
		w=5	min(11, 0) = 0
		w=6	min(11, 0) = 0

24 (6 marks)

(a) New arc B to G gives B-G-T of length $1 + f(G) = 1 + 3 = 4$, compared with $f(B) = 11$

M1

New $f(B) = 4$; S-B route = $3 + 4 = 7$, S-A route = $5 + 12 = 17$

M1

Shortest route S-B-G-T of length 7

A1

Answer: S-B-G-T, 7

3 marks

(b) Without SB the only arc from S is SA, so the shortest route is S-A followed by the best route from A

M1

Length = $5 + f(A) = 5 + 12 = 17$

A1

Route: S-A-C-G-T

A1

Answer: S-A-C-G-T, 17

3 marks

25 (6 marks)

The route must pass through E, so remove D and F from the third layer (every route then uses E)

B1

Stage 1: $f(G) = 2$ via T [T: 2]; $f(H) = 9$ via T [T: 9]

M1

Stage 2: $f(E) = 8$ via G [G: $6 + 2 = 8$; H: $6 + 9 = 15$]

M1

Stage 3: $f(A) = 13$ via E [E: $5 + 8 = 13$]; $f(B) = 16$ via E [E: $8 + 8 = 16$]; $f(C) = 16$ via E [E: $8 + 8 = 16$]

A1

Stage 4: $f(S) = 18$ via B [A: $6 + 13 = 19$; B: $2 + 16 = 18$; C: $6 + 16 = 22$]

A1

Shortest route through E: S-B-E-G-T, length 18

A1

Answer: S-B-E-G-T, 18 km

6 marks

Stage	State	Action	Value
1	G	T *	2
	H	T *	9
2	E	G *	$6 + 2 = 8$
		H	$6 + 9 = 15$
3	A	E *	$5 + 8 = 13$
	B	E *	$8 + 8 = 16$
	C	E *	$8 + 8 = 16$
4	S	A	$6 + 13 = 19$
		B *	$2 + 16 = 18$
		C	$6 + 16 = 22$

26 (8 marks)

(a) Greedy: S to A (3), A to D (5), D to T (7): S-A-D-T has length 15

M1 A1

DP gives $f(S) = 9$ via S-B-C-T, which is shorter

M1 A1

Answer: Greedy 15; shortest 9 via S-B-C-T

4 marks

(b) Suppose the section from X to T of a shortest route R from S were not a shortest route from X to T

E1

Then replacing it with a shorter route from X to T would give a route from S to T shorter than R

M1

This contradicts R being shortest, so every section of R to T is a shortest route

E1

Answer: Proof by contradiction

3 marks

(c) So $f(X)$ can be built from the shortest routes from the next stage, and only the arcs from X need to be compared

E1

Answer: Optimal sub-routes are built from the next stage

1 mark

27 (8 marks)

State = (vertex, number of toll arcs used so far), which is 0 or 1; arcs that would make it 2 are not allowed

B1

T must be reached with exactly 1 toll arc, so only the state (T, 1) is accepted at the end

B1

Stage 1: $f(F,0) = 4$, $f(G,1) = 3$

M1

Stage 2: $f(C,0) = 10$, $f(D,0) = 6$, $f(E,1) = 9$

M1

Stage 3: $f(A,0) = 15$, $f(B,0) = 10$

A1

Stage 4: $f(S,0) = 16$

A1

Best route with exactly one toll arc: S-B-D-G-T with length 16

A1

Check: toll arcs on this route: DG

A1

Answer: S-B-D-G-T, 16 minutes

8 marks

Stage	State	Action	Value
1	F0	T *	4
	G1	T *	3
2	C0	F *	$6 + 4 = 10$
	D0	F	$5 + 4 = 9$
		G *	$3 + 3 = 6$
	E1	G *	$6 + 3 = 9$
3	A0	C *	$5 + 10 = 15$
	B0	D *	$4 + 6 = 10$
		E	$7 + 9 = 16$
4	S0	A	$4 + 15 = 19$
		B *	$6 + 10 = 16$

28 (8 marks)

Stage = year (stage 1 is year 5), state = age of the machine at the start of the year, action = keep or replace, value = least cost from then to the end

B1

Stage 1 (year 5): $f(\text{age } 2) = 3$, $f(\text{age } 3) = 6$, $f(\text{age } 4) = 12$

M1

Stage 2 (year 4): $f(\text{age } 2) = 9$, $f(\text{age } 3) = 13$, $f(\text{age } 4) = 15$

M1

Stage 3 (year 3): $f(\text{age } 2) = 16$, $f(\text{age } 3) = 19$, $f(\text{age } 4) = 21$

A1

Stage 4 (year 2): $f(\text{age } 2) = 22$, $f(\text{age } 3) = 26$

A1

Stage 5 (year 1): $f(\text{age } 2) = 29$

A1

Starting from age 2: year 1: keep, year 2: replace, year 3: keep, year 4: replace, year 5: keep

A1

Minimum total cost £29 thousand

A1

Answer: Years 1 to 5: keep, replace, keep, replace, keep; £29 thousand

8 marks

Stage	State	Action	Value
1	age 2	keep *	3
		replace	8
	age 3	keep *	6
		replace	10
	age 4	replace *	12
2	age 2	keep *	3 + 6 = 9
		replace	8 + 3 = 11
	age 3	keep	6 + 12 = 18
		replace *	10 + 3 = 13
	age 4	replace *	12 + 3 = 15
3	age 2	keep *	3 + 13 = 16
		replace	8 + 9 = 17
	age 3	keep	6 + 15 = 21
		replace *	10 + 9 = 19
	age 4	replace *	12 + 9 = 21
4	age 2	keep *	3 + 19 = 22
		replace	8 + 16 = 24
	age 3	keep	6 + 21 = 27
		replace *	10 + 16 = 26
5	age 2	keep *	3 + 26 = 29
		replace	8 + 22 = 30

29 (8 marks)

(a) Reliability with m parallel units = $1 - (1 - p)^m$

M1

Component 1: 0.8, 0.96, 0.992; component 2: 0.7, 0.91, 0.973; component 3: 0.9, 0.99, 0.999

A1

Answer: Tables of $1 - (1 - p)^m$

2 marks

(b) Stage 1 (component 3, s £ left): best m with $m \leq s$: $f(1) = 0.9$, $f(2) = 0.99$, $f(3) = 0.999$, $f(5) = 0.999$

M1

Stage 2 (components 2 and 3): maximise $r_2(m) \times f(s - 3m)$: $f(4) = 0.63000$, $f(6) = 0.69930$, $f(8) = 0.90090$

M1 A1

Stage 3 (budget 10): $m = 1, 2, 3$ give 0.72072, 0.67133, 0.62496

M1 A1

Best: 1, 2 and 2 units of components 1, 2 and 3 (cost £10), system reliability 0.72072

A1

Answer: (1, 2, 2) units; 0.721

6 marks

30 (8 marks)

(a) Each of the k layers has w vertices; a route chooses one vertex in each layer, and S and T are fixed

M1

Number of routes = w^k

A1

Answer: w^k

2 marks

(b) Arcs from S : w ; between consecutive middle layers: $(k - 1)$ arcs groups of w^2 ; arcs into T : w

M1

Dynamic programming needs one addition for each arc except those into T (for these $f(\text{vertex}) = \text{the arc length}$): $w + (k - 1)w^2$

A1

Answer: $w + (k - 1)w^2$

2 marks

(c) Routes = $6^8 = 1679616$

A1

Enumerating needs $k = 8$ additions ($k + 1$ arcs per route) for each route: 13436928 additions; DP needs $6 + 7 \times 36 = 258$ additions	M1 A1
Answer: 1679616 routes; 13436928 additions by enumeration; 258 by DP	3 marks
(d) Enumeration needs of order $k w^k$ operations, which grows exponentially with k , whereas dynamic programming needs of order $k w^2$, which grows only linearly with k	E1
Answer: Exponential against linear growth	1 mark

Decision Analysis

1 (2 marks)

EMV = $0.3 \times 40 + 0.5 \times 20 + 0.2 \times (-10)$	M1
= £20 thousand	A1
Answer: £20 thousand	2 marks

2 (3 marks)

Option A: $0.4 \times 50 + 0.6 \times 10 = 26$	M1
Option B: $0.4 \times 30 + 0.6 \times 25 = 27$	A1
Choose option B (larger EMV)	B1
Answer: A: 26, B: 27; choose B	3 marks

3 (2 marks)

Row minima: 5, 2, 7	M1
The largest of these is 7, so choose option 3	A1
Answer: Option 3 (maximin value 7)	2 marks

4 (2 marks)

Row maxima: 12, 15, 9	M1
The largest is 15, so choose option 2, value 15	A1
Answer: Option 2, value 15	2 marks

5 (2 marks)

Square: a decision made by the decision maker; circle: a chance event. At a chance node the EMV is calculated as the probability-weighted average	E1
At a decision node the branch with the best (largest) value is chosen	E1
Answer: Square = decision, circle = chance; average at circles, choose best at squares	2 marks

6 (2 marks)

Best profit in each state: 30, 25, 28	M1
Regret for option 1: 0, 7, 22	A1
Answer: 0, 7, 22	2 marks

7 (4 marks)

Project 1: $0.3 \times 60 + 0.5 \times 30 + 0.2 \times (-10) = 31$	M1 A1
Project 2: $0.3 \times 40 + 0.5 \times 35 + 0.2 \times 15 = 32.5$	A1
Project 3: 20 (certain) and project 2 has the largest EMV, so choose it	A1
Answer: EMVs 31, 32.5, 20; choose project 2	4 marks

8 (4 marks)

Row minima: 2, 7, 1; maximin value 7 for course 2	M1 A1
Row maxima: 14, 10, 20; maximax value 20 for course 3	A1
Maximin is for a cautious (pessimistic) decision maker; maximax is for an optimistic one	E1
Answer: (i) course 2 (ii) course 3	4 marks

9 (5 marks)

Best in each state: 45, 40, 50

M1

Regret table: $\begin{pmatrix} 0 & 10 & 40 \\ 15 & 0 & 25 \\ 30 & 15 & 0 \end{pmatrix}$

M1 A1

Largest regret for each option: 40, 25, 30

M1

Choose option 2 (smallest maximum regret 25)

A1

Answer: Option 2, maximum regret 25

5 marks

10 (5 marks)Large: $0.4 \times 90 + 0.35 \times 40 + 0.25 \times (-30) = 42.5$

M1 A1

Small: $0.4 \times 50 + 0.35 \times 35 + 0.25 \times 10 = 34.75$

A1

None: 0

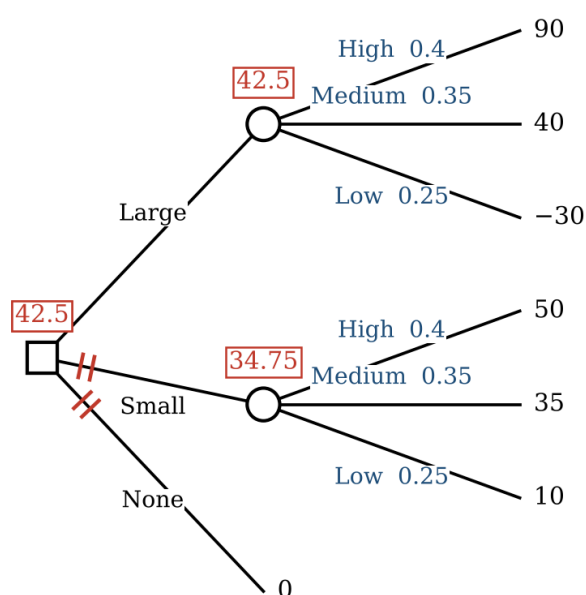
B1

Largest EMV is 42.5: build the large shop

A1

Answer: Large 42.5, small 34.75; choose large

5 marks

**11 (4 marks)**EMVs: option 1 = $0.5 \times 80 + 0.5 \times 20 = 50$; option 2 = $0.5 \times 50 + 0.5 \times 40 = 45$

M1 A1

Perfect information: best in state 1 is 80, in state 2 is 40: $0.5 \times 80 + 0.5 \times 40 = 60$

M1

EVPI = $60 - 50 = 10$

A1

Answer: EVPI = £10 thousand

4 marks

12 (3 marks)Supplier 1: $0.7 \times 12 + 0.3 \times 20 = 14.4$

M1

Supplier 2: $0.7 \times 16 + 0.3 \times 15 = 15.7$

A1

Choose supplier 1 (smaller expected cost)

B1

Answer: 14.4 and 15.7; supplier 1

3 marks

13 (4 marks) $EMV(A) = 70p - 10(1 - p) = 80p - 10$

B1

 $EMV(B) = 35p + 20(1 - p) = 20 + 15p$

B1

 $80p - 10 = 20 + 15p$, so $65p = 30$ and $p = \frac{6}{13} \approx 0.462$

M1

For larger p , A is better (its EMV rises faster)

A1

Answer: $p = \frac{6}{13}$; A better for $p > \frac{6}{13}$

4 marks

14 (7 marks)

Maximin: row minima $-5, 20, -20$, so choose option 2

M1 A1

Best in each state: 70, 40, 30, 20; regret table $\begin{pmatrix} 15 & 0 & 5 & 25 \\ 35 & 5 & 0 & 0 \\ 0 & 10 & 20 & 40 \end{pmatrix}$

M1 A1

Largest regrets 25, 35, 40, so choose option 1

A1

EMVs: 29.5, 30.5, 22, so choose option 2

M1 A1

Answer: (i) option 2 (ii) option 1 (iii) option 2

7 marks

15 (6 marks)

Full production: $0.7 \times 150 + 0.3 \times 40 = 117$, which beats selling the licence (70), so full

M1 A1

production is chosen at the second decision

Success node: $0.6 \times 117 + 0.4 \times 0 = 70.2$

M1

Prototype branch: $70.2 - 20 = 50.2$

A1

Selling the idea gives 30, less than 50.2, so build the prototype

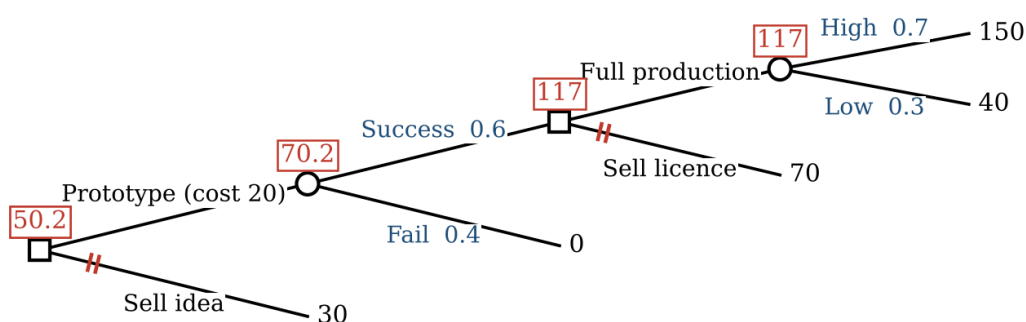
A1

Strategy: build the prototype; if it succeeds, go into full production. Expected profit £50.2 thousand

A1

Answer: Build prototype, then full production if successful; £50.2 thousand

6 marks

**16 (6 marks)**

EMVs: option 1: $0.4 \times 50 + 0.35 \times 30 + 0.25 \times 10 = 33$; option 2: $0.4 \times 35 + 0.35 \times 40 + 0.25 \times 25 = 34.25$; option 3: $0.4 \times 20 + 0.35 \times 25 + 0.25 \times 60 = 31.75$

M1 A1

Best EMV without information = 34.25

B1

With perfect information, choose the best option in each state: 50, 40, 60; expected profit $0.4 \times 50 + 0.35 \times 40 + 0.25 \times 60 = 49$

M1 A1

EVPI = $49 - 34.25 = 14.75$

A1

Answer: Best EMV 34.25; EVPI £14.75 thousand

6 marks

17 (7 marks)

Largest cost for each scheme: 90, 70, 65; the smallest of these is 65, so scheme 3

M1 A1

Smallest cost in each state: 40, 52, 58. Regret = cost – smallest cost in that state: scheme 1: 0, 3, 32; scheme 2: 10, 8, 12; scheme 3: 25, 0, 0. Largest regrets: 32, 12, 25; the smallest is 12, so scheme 2

M1 A1 A1

Expected costs: scheme 1: $0.3 \times 40 + 0.45 \times 55 + 0.25 \times 90 = 59.25$; scheme 2: $0.3 \times 50 + 0.45 \times 60 + 0.25 \times 70 = 59.5$; scheme 3: $0.3 \times 65 + 0.45 \times 52 + 0.25 \times 58 = 57.4$; the lowest is 57.4, so scheme 3

M1 A1

Answer: (i) scheme 3 (ii) scheme 2 (iii) scheme 3, expected cost 57.4

7 marks

18 (5 marks)

Campaign: expected revenue $0.55 \times 70 + 0.45 \times 20 = 47.5$

M1 A1

Net of the cost: $47.5 - 15 = 32.5$

M1

No campaign: 25, which is less than 32.5

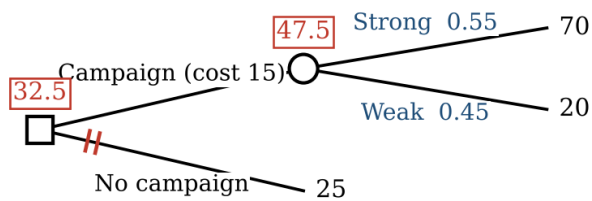
A1

Run the campaign; expected net revenue £32.5 thousand

A1

Answer: Run the campaign; £32.5 thousand

5 marks

**19 (5 marks)**

EMVs: $A = 16$, $B = 15$, $C = 28$, so EMV favours C

M1 A1

Expected utilities: $A = 0.3 \times 1 + 0.7 \times 0 = 0.3$; $B = 0.5 \times 0.7 + 0.5 \times 0.4 = 0.55$; $C = 0.6 \times 0.9 + 0.4 \times 0 = 0.54$

M1 A1

The decision maker should choose B, the option with the greatest expected utility, which differs from the EMV choice because the decision maker is risk averse

A1

Answer: EMV: C; expected utility: B

5 marks

20 (4 marks)

The student subtracted the smallest entry in each row; regret is the best payoff in the same state (column) minus the payoff, so the best of each column is used

E1

Column maxima: 40, 35, 45; regret table $\begin{pmatrix} 0 & 10 & 30 \\ 10 & 0 & 25 \\ 20 & 5 & 0 \end{pmatrix}$

M1 A1

Largest regrets 30, 25, 20: choose option 3

A1

Answer: Regret uses column maxima; option 3

4 marks

21 (10 marks)

(a) Large: $0.5 \times 100 + 0.5 \times (-40) = 30$; small: $0.5 \times 50 + 0.5 \times 20 = 35$

M1 A1

Best without a survey: small, EMV 35

A1

Answer: Small estate, EMV 35

3 marks

(b) Good result: large = $0.8 \times 100 + 0.2 \times (-40) = 72$; small = $0.8 \times 50 + 0.2 \times 20 = 44$; choose large, 72

M1 A1

Poor result: large = $0.2 \times 100 + 0.8 \times (-40) = -12$; small = $0.2 \times 50 + 0.8 \times 20 = 26$; choose small, 26

M1 A1

Survey: $0.5 \times 72 + 0.5 \times 26 = 49$, less the cost 6 gives 43

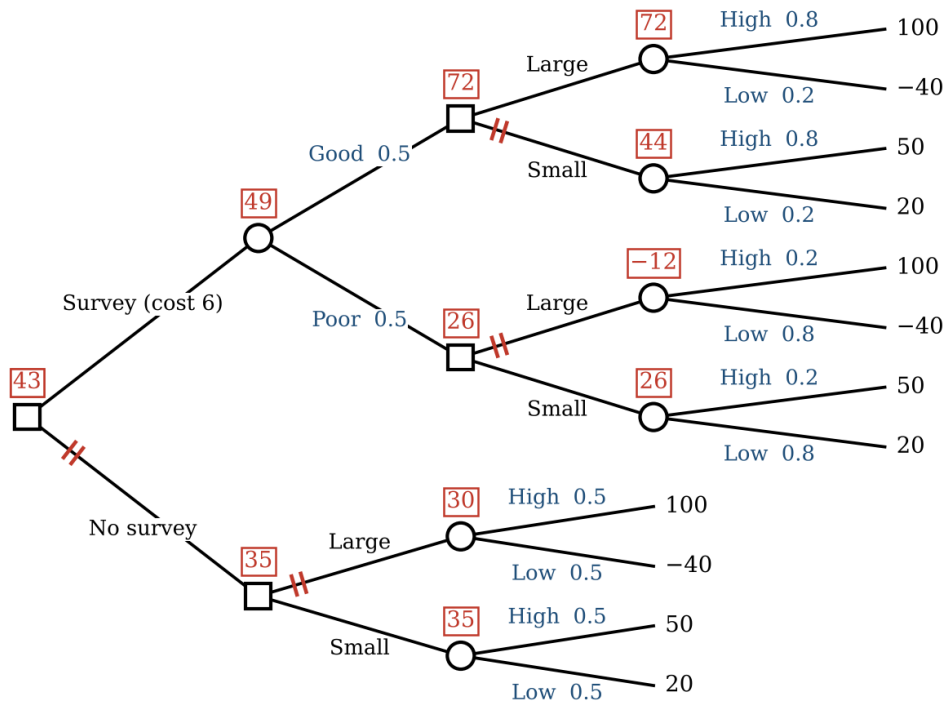
M1 A1

$43 > 35$, so commission the survey; build large if good, small if poor

A1

Answer: Commission; large if good, small if poor; EMV 43

7 marks

**22 (8 marks)****(a)** EMVs: 34.5; 40.5; 30

M1 A1

Without further information choose option 2, EMV 40.5

A1

Answer: Option 2, EMV 40.5

3 marks

(b) With perfect information: $0.25 \times 90 + 0.45 \times 50 + 0.30 \times 30 = 54$

M1 A1

EVPI = $54 - 40.5 = 13.5$

A1

Answer: EVPI £13.5 thousand

3 marks

(c) A forecast costing £12 thousand is worth buying only if its cost is less than the EVPI, £13.5 thousand; as $12 < 13.5$ it is worthwhile

E1 A1

Answer: Worth buying

2 marks

23 (7 marks)**(a)** $EMV(A) = 120p - 60(1 - p) = 180p - 60$

B1

 $EMV(B) = 60p + 10(1 - p) = 10 + 50p$; $EMV(C) = 25$

B1

Answer: $180p - 60$, $10 + 50p$, 25

2 marks

(b) B and C equal: $10 + 50p = 25$, so $p = 0.3$; B and A equal: $180p - 60 = 10 + 50p$, so $130p = 70$ and $p = \frac{7}{13} \approx 0.538$

M1 A1

A and C are equal at $p = \frac{17}{36} \approx 0.472$; at this value B is better than both, so it is not a boundary

M1

C is best for $p < 0.3$, B for $0.3 < p < \frac{7}{13}$, A for $p > \frac{7}{13}$

A1

Answer: C for $p < 0.3$; B for $0.3 < p < \frac{7}{13}$; A for $p > \frac{7}{13}$

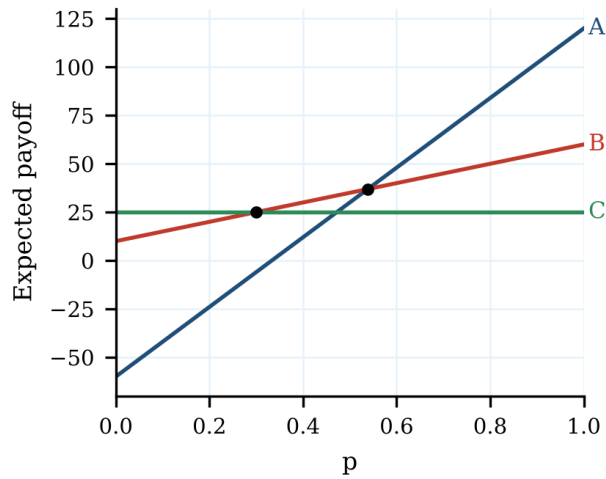
4 marks

(c) Option C is best only when the chance of a strong market is below 0.3; it is the safe choice

E1

Answer: Chance of strong market below 0.3

1 mark



24 (6 marks)

The probabilities are p , $2p$ and $1 - 3p$, so $0 < p < \frac{1}{3}$

B1

$EMV(1) = 80p + 100p + 10(1 - 3p) = 150p + 10$; $EMV(2) = 40p + 30$; $EMV(3) = 60 - 100p$

M1 A1

$EMV(1) > EMV(2)$: $110p - 20 > 0$, so $p > \frac{2}{11}$

M1

$EMV(1) > EMV(3)$: $250p - 50 > 0$, so $p > \frac{1}{5}$

A1

Option 1 is best for $\frac{1}{5} < p < \frac{1}{3}$

A1

Answer: $\frac{1}{5} < p < \frac{1}{3}$

6 marks

25 (8 marks)

(a) Best payoff in each state: 70, 45, 60; regret table $\begin{pmatrix} 0 & 15 & 40 \\ 20 & 0 & 35 \\ 40 & 5 & 0 \end{pmatrix}$

M1 A1

Answer: $\begin{pmatrix} 0 & 15 & 40 \\ 20 & 0 & 35 \\ 40 & 5 & 0 \end{pmatrix}$

2 marks

(b) Expected regrets: 18; 16.5; 14

M1 A1

Smallest expected regret 14 for option 3

A1

Answer: Option 3, 14

3 marks

(c) EMVs: 39; 40.5; 43; expected profit with perfect information 57; $EVPI = 57 - 43 = 14$

M1 A1

The smallest expected regret equals the EVPI, 14, and is for the option with the greatest EMV

E1

Answer: EVPI = 14 = least expected regret

3 marks

26 (9 marks)

(a) Launch: $0.5 \times 120 + 0.5 \times (-50) = 35$, better than abandoning (0), so launch with EMV 35

M1 A1

Answer: Launch, EMV 35

2 marks

(b) Favourable: launch = $0.7 \times 120 + 0.3 \times (-50) = 69$, so launch (69)

M1 A1

Unfavourable: launch = $0.2 \times 120 + 0.8 \times (-50) = -16$, so abandon (0)

M1 A1

Test: $0.6 \times 69 + 0.4 \times 0 = 41.4$, less the cost 5 gives 36.4

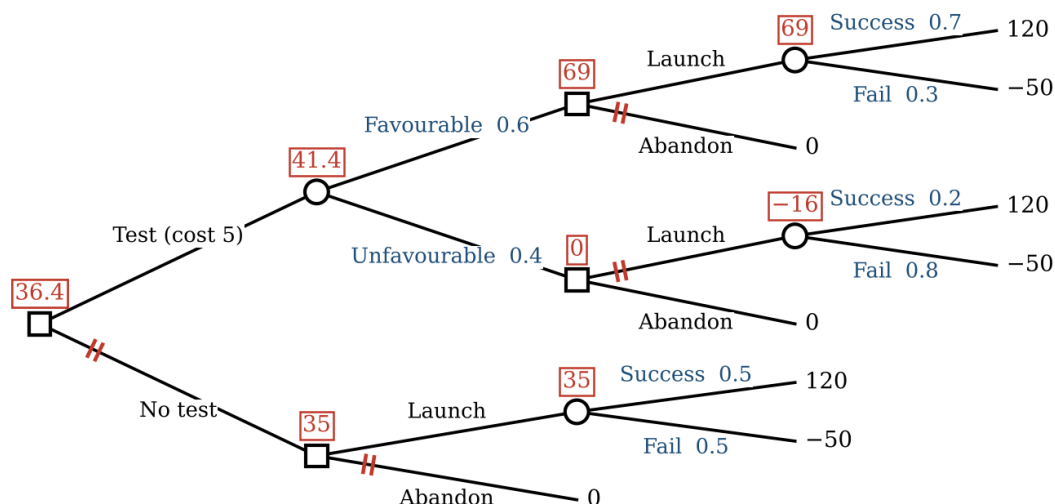
M1 A1

$36.4 > 35$: do the test, launch only if favourable

A1

Answer: Test and launch only if favourable; EMV 36.4

7 marks



27 (13 marks)

(a) $P(\text{positive}) = 0.5 \times 0.9 + 0.5 \times 0.3 = 0.6$

M1 A1

$P(\text{high} | \text{positive}) = \frac{0.5 \times 0.9}{0.6} = 0.75$

M1

$P(\text{high} | \text{negative}) = \frac{0.5 \times 0.1}{0.4} = 0.125$

A1

Answer: 0.6; 0.75; 0.125

4 marks

(b) Large: $0.5 \times 100 + 0.5 \times (-40) = 30$; small: $0.5 \times 50 + 0.5 \times 20 = 35$

M1 A1

Answer: Small, EMV 35

2 marks

(c) Positive: large = 65, small = 42.5, so large (65)

M1 A1

Negative: large = -22.5, small = 23.75, so small (23.75)

M1 A1

Answer: Large after positive; small after negative

4 marks

(d) EMV with the survey (before cost) = $0.6 \times 65 + 0.4 \times 23.75 = 48.5$; EVSI = $48.5 - 35 = 13.5$

M1 A1

The survey is worth commissioning if it costs less than £13.5 thousand (so the £6 thousand fee is acceptable); EVPI = 25, which is more than EVSI

A1

Answer: EVSI = 13.5; worthwhile

3 marks

28 (9 marks)

(a) Let $m_j = \max \text{ over } i \text{ of } u_{ij}$; the regret of option i in state j is $r_{ij} = m_j - u_{ij}$

B1

Expected regret of option $i = \sum p_j(m_j - u_{ij}) = \sum p_j m_j - \sum p_j u_{ij}$

M1

= (expected profit with perfect information) - EMV(i), since $\sum p_j m_j$ is the perfect information value

A1

Answer: Expected regret = EPPI - EMV(i)

3 marks

(b) $\sum p_j m_j$ does not depend on i , so the option with the least expected regret is the option with the greatest EMV

E1

The least expected regret = $\sum p_j m_j - (\text{greatest EMV})$

M1

This is the EVPI by definition

A1

Answer: Least expected regret = EVPI

3 marks

(c) EMVs: 34, 36.5, 38.5; perfect information value 48.5; EVPI = 10

M1 A1

Regret table $\begin{pmatrix} 0 & 20 & 15 \\ 15 & 0 & 30 \\ 25 & 10 & 0 \end{pmatrix}$; expected regrets 14.5, 12, 10, smallest 10 = EVPI

M1

Answer: EVPI = 10

3 marks

29 (8 marks)

Joint probabilities with prior p of high demand: $P(\text{high, pos}) = 0.9p$, $P(\text{low, pos}) = 0.3(1 - p)$, B1

$P(\text{high, neg}) = 0.1p$, $P(\text{low, neg}) = 0.7(1 - p)$

No survey: large $140p - 40$, small $30p + 20$; best is the larger of these M1 A1

Positive: large $102p - 12$, small $39p + 6$; negative: large $38p - 28$, small $14 - 9p$; survey value = best after positive + best after negative M1 A1

Switch points: after a positive result large is better when $102p - 12 > 39p + 6$, so $p > \frac{2}{7}$; after a M1

negative result large is better when $38p - 28 > 14 - 9p$, so $p > \frac{42}{47}$; with no survey large is better

when $140p - 40 > 30p + 20$, so $p > \frac{6}{11}$

For $\frac{2}{7} < p < \frac{6}{11}$: survey value less 6 = $(102p - 12) + (14 - 9p) - 6 = 93p - 4$, which beats $30p$ A1

+ 20 when $p > \frac{8}{21}$. For $\frac{6}{11} < p < \frac{42}{47}$: $93p - 4$ beats $140p - 40$ when $p < \frac{36}{47}$. The survey is

worthwhile when $\frac{8}{21} < p < \frac{36}{47}$ (about $0.381 < p < 0.766$)

Outside these intervals ($p < \frac{2}{7}$ or $p > \frac{42}{47}$) the survey value less 6 is $30p + 14$ or $140p - 46$, E1

always below the no-survey value, so it is never worthwhile

Answer: $\frac{8}{21} < p < \frac{36}{47}$

8 marks

30 (11 marks)

(a) Develop: $0.7 \times 200 + 0.3 \times 20 - 50 = 96$, which beats selling the findings (40) M1 A1

Promising node: $0.4 \times 96 + 0.6 \times 0 = 38.4$ M1

Research branch: $38.4 - 10 = 28.4$ A1

Licence: $0.3 \times 160 + 0.7 \times 10 - 30 = 25$ M1 A1

Do nothing: 0. Best is to do the research (EMV 28.4); if promising develop, otherwise stop A1

Answer: Research; develop if promising; EMV 28.4 7 marks

(b) Licence EMV with hit probability h : $160h + 10(1 - h) - 30 = 150h - 20$ M1 A1

The choice changes when $150h - 20 = 28.4$, so $h = \frac{48.4}{150} \approx 0.323$ M1

For h above this value, buying the licence is better A1

Answer: $h > 0.323$ 4 marks

