

Allocation and the Hungarian Algorithm

- 1 The cost matrix for an allocation problem is

$$\begin{pmatrix} 14 & 18 & 11 \\ 12 & 16 & 13 \\ 17 & 15 & 19 \end{pmatrix}$$

Subtract the smallest entry in each row from every entry in that row, and write down the resulting matrix.

[2]

- 2 A row-reduced matrix is

$$\begin{pmatrix} 0 & 5 & 3 \\ 0 & 4 & 2 \\ 3 & 0 & 6 \end{pmatrix}$$

Reduce the columns and write down the new matrix.

[2]

- 3 After reducing rows and columns, a 4×4 allocation problem gives the matrix

$$\begin{pmatrix} 0 & 3 & 5 & 2 \\ 0 & 6 & 1 & 4 \\ 2 & 0 & 0 & 7 \\ 5 & 4 & 0 & 1 \end{pmatrix}$$

Find the minimum number of lines needed to cover all of the zeros, and state whether a complete allocation can be made from the zeros.

[2]

- 4 Four workers are to be assigned to three jobs, one worker to each job, so one worker has no job.

State what must be added to the cost table before the Hungarian algorithm can be used, and give the entries that should be used.

[2]

- 5 A table gives the profit made when each of four salespeople is assigned to each of four regions. The Hungarian algorithm, which finds a minimum, is to be used to find the assignment with the greatest total profit.

Describe how the table should be changed first.

[2]

- 6 The cost table below, giving the cost in £ of assigning each of P, Q and R to each of the jobs 1, 2 and 3, was reduced by the Hungarian algorithm to the matrix on the right.

$$\begin{pmatrix} 9 & 13 & 8 \\ 7 & 12 & 10 \\ 11 & 8 & 14 \end{pmatrix} \text{ becomes } \begin{pmatrix} 1 & 5 & 0 \\ 0 & 5 & 3 \\ 3 & 0 & 6 \end{pmatrix}$$

Write down the allocation given by the zeros, and find its total cost using the original table.

[3]

- 7 A swimming coach must choose four swimmers for a medley relay, one for each stroke. The times, in seconds, that each swimmer takes for 50 m of each stroke are shown in the table. Use the Hungarian algorithm to find the assignment of swimmers to strokes that minimises the total time, and state this time.

	Backstroke	Breaststroke	Butterfly	Freestyle
Ali	38	28	35	29
Ben	25	35	24	36
Cat	42	30	28	34
Dev	37	38	27	24

[4]

- 8** Three mechanics are each to service one of three cars. The table shows the time, in hours, each mechanic would take for each car.
Use the Hungarian algorithm to find the allocation that minimises the total time taken, showing your reduced matrices.

	Car 1	Car 2	Car 3
Mechanic X	11	10	8
Mechanic Y	11	11	9
Mechanic Z	10	3	5

[4]

- 9** Three students, Amy, Bao and Cleo, are each given one of four tasks to prepare for an event. Each task is done by at most one student, so exactly one task is left undone. The table shows the number of minutes each student would need for each task.
Use the Hungarian algorithm to allocate the students to tasks so that the total preparation time is a minimum. Show how you made the table square.

	T1	T2	T3	T4
Amy	22	27	21	22
Bao	30	27	23	15
Cleo	20	15	22	21

[5]

- 10** The profit, in £100s, from assigning each of three consultants to each of three projects is given by the matrix

$$\begin{pmatrix} 18 & 25 & 21 \\ 22 & 19 & 27 \\ 24 & 20 & 23 \end{pmatrix}$$

The Hungarian algorithm is to be used to find the allocation with the maximum total profit. Write down the matrix that should be used, explaining how it is obtained.

[2]

- 11** Three delivery drivers are to be assigned to three routes. Driver C cannot take route 2. The cost matrix (£) for the other assignments is

$$\begin{pmatrix} 13 & 9 & 16 \\ 11 & 14 & 10 \\ 15 & - & 17 \end{pmatrix}$$

Explain how the entry for driver C on route 2 should be treated, and carry out the row reduction and the column reduction, writing down the matrix after each.

[3]

- 12** A 4×4 cost matrix is

$$\begin{pmatrix} 21 & 11 & 16 & 9 \\ 19 & 6 & 10 & 16 \\ 20 & 17 & 17 & 25 \\ 25 & 6 & 23 & 10 \end{pmatrix}$$

Find the total of the row minima and the column minima obtained when the matrix is reduced by rows and then by columns, and explain why the cost of any allocation must be at least this total.

[3]

- 13** After row and column reduction, a 4×4 problem gives the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 7 & 6 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 7 & 6 & 0 & 0 \end{pmatrix}$$

The zeros are covered by three lines: row 1 and columns 3 and 4. Carry out the next step of the Hungarian algorithm and write down the new matrix. (Rows and columns are numbered from 1.)

[3]

- 14** Four couriers, Kay, Lee, Max and Noor, are each to deliver to one of four depots. The table shows the cost in £ of each courier making each delivery. Use the Hungarian algorithm to find the allocation of couriers to depots which minimises the total cost, and state the minimum cost. Show the matrix at each stage.

	Depot 1	Depot 2	Depot 3	Depot 4
Kay	33	35	39	22
Lee	33	8	18	39
Max	12	24	18	11
Noor	38	27	27	13

[6]

- 15** A haulage firm has five drivers available to cover four routes. Each route is driven by one driver, and each driver drives at most one route, so one driver is not used. The table shows the cost, in £, of using each driver on each route. Use the Hungarian algorithm to find the assignment of drivers to routes that minimises the total cost, stating which driver is not used and the minimum cost.

	R1	R2	R3	R4
D1	23	23	33	38
D2	15	33	45	40
D3	18	30	31	22
D4	45	44	41	26
D5	20	18	33	44

[7]

- 16** A company will assign each of four sales staff to one of four regions. The table shows the expected monthly profit, in £100s, from each member of staff in each region. Use the Hungarian algorithm to find the assignment which maximises the total profit, and state the maximum profit.

	North	South	East	West
Staff 1	73	53	75	50
Staff 2	54	80	77	70
Staff 3	42	88	84	86
Staff 4	70	71	86	43

[6]

- 17** Four engineers are to be allocated to four repair jobs. Engineer B does not have the qualification for job 3, so cannot be given it (shown by a dash). The table shows the cost, in £, of each engineer doing each job.
- Use the Hungarian algorithm, treating the dash as a very large cost, to find the allocation with the least total cost, and state this cost.

	Job 1	Job 2	Job 3	Job 4
A	34	25	17	36
B	36	26	10	32
C	33	10	37	24
D	27	31	-	11

[7]

- 18** Four tutors are to be assigned to four classes. The table shows the preparation time, in hours, each tutor needs for each class.
- Use the Hungarian algorithm to find the minimum total preparation time, and list every allocation that achieves it.

	Class 1	Class 2	Class 3	Class 4
Tutor A	7	23	17	15
Tutor B	11	24	17	6
Tutor C	6	22	14	20
Tutor D	11	24	8	20

[5]

- 19** After row and column reduction, a 4×4 allocation problem gives the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 5 \\ 5 & 0 & 4 & 0 \\ 5 & 0 & 1 & 4 \\ 4 & 0 & 4 & 4 \end{pmatrix}$$

A student says: "Every row and every column contains a zero, so I can already allocate using the zeros."

Explain why the student is wrong. Then carry out the next step of the algorithm, using the lines through rows 1, 2 and column 2 to cover the zeros, write down the new matrix and state the allocation of rows to columns that it gives (rows and columns numbered from 1).

[4]

- 20** Three contractors, Pat, Quin and Ravi, are each to carry out one of three jobs. They charge £14, £12 and £15 per hour respectively. The table shows the number of hours each would take for each job.
- Find the cost, in £, of each contractor doing each job, and use the Hungarian algorithm to assign the contractors to the jobs at minimum total cost. State this cost.

	Job 1	Job 2	Job 3
Pat	6	3	6
Quin	6	9	6
Ravi	3	7	5

[6]

- 21** Five technicians, A to E, are to be assigned to five tasks, one each. Technician D is unable to do task 3, shown by a dash. The table gives the time, in hours, each technician would take for each task.

	Task 1	Task 2	Task 3	Task 4	Task 5
A	28	21	33	17	10
B	20	38	39	30	27
C	30	39	37	29	20
D	28	35	28	31	30
E	14	-	32	23	29

- (a) Explain how the dash is dealt with, and show the matrix after the row reduction and the matrix after the column reduction. [3]
- (b) Complete the Hungarian algorithm, showing each adjustment of the matrix, to find the allocation which minimises the total time. State this time. [6]

- 22** A firm has four consultants and five projects. Each consultant is to lead exactly one project and each project has at most one leader, so one project has no consultant. The table shows the profit, in £ thousands, expected from each consultant leading each project.

	P1	P2	P3	P4	P5
Consultant 1	45	57	35	61	47
Consultant 2	65	82	79	44	84
Consultant 3	78	86	52	65	58
Consultant 4	70	74	68	34	32

- (a) Explain how the table must be altered so that the Hungarian algorithm can be used to find the maximum total profit, and write down the new table. [3]
- (b) Use the Hungarian algorithm to find the allocation which maximises the total profit. State which project is left without a consultant and the maximum profit. [6]

- 23** Three mechanics, X, Y and Z, are to service four vehicles. Mechanic X is qualified to service two vehicles, while Y and Z each service one vehicle, and every vehicle is serviced by exactly one mechanic. The table shows the time, in days, each mechanic would take to service each vehicle. (Where a mechanic services two vehicles the times simply add.)

	V1	V2	V3	V4
X	8	12	20	18
Y	26	15	14	26
Z	17	21	22	18

- (a) Explain how the table can be adapted into a 4×4 table to which the Hungarian algorithm can be applied, and write down the table. [2]
- (b) Use the Hungarian algorithm to find which vehicles each mechanic should service to minimise the total time, and state this time. [6]

- 24** Four workers, A, B, C and D, are to be given four jobs, one each. The table shows the time, in hours, each worker takes on each job.
- Without any restriction, the Hungarian algorithm gives a minimum total time of 55 hours. However, worker C has already started job 2 and must be allowed to finish it.

	1	2	3	4
A	15	35	11	29
B	30	13	14	33
C	14	14	21	15
D	31	31	20	17

- (a) Explain why the problem can now be reduced to a 3×3 allocation problem, and write down the 3×3 table. [1]
- (b) Use the Hungarian algorithm on the 3×3 table to find the best allocation of the other three workers. [4]
- (c) Find the minimum total time with C doing job 2, and the increase on the unrestricted minimum. [2]

- 25** A retailer has five salespeople, S1 to S5, and five regions, R1 to R5. One salesperson is sent to each region. The table shows the expected annual sales, in £ thousands, for each salesperson in each region.

	R1	R2	R3	R4	R5
S1	73	49	54	82	68
S2	48	38	69	58	90
S3	34	85	76	41	45
S4	51	36	32	88	60
S5	43	60	30	69	46

- (a) The Hungarian algorithm is to be used to maximise the total sales. Write down the matrix to which the algorithm should be applied. [2]
- (b) Apply the Hungarian algorithm, showing the matrix at each stage, to find the allocation that maximises the total sales, and state the maximum. [6]

- 26** Three workers A, B and C are to be assigned to three jobs 1, 2 and 3, one each. The cost, in £, of worker A doing job 1 and so on is given by

$$\begin{pmatrix} 11 & 15 & 9 \\ 14 & 10 & 13 \\ 12 & 16 & 8 \end{pmatrix}$$

Let $x_{ij} = 1$ if worker i is assigned to job j , and $x_{ij} = 0$ otherwise, for $i = A, B, C$ and $j = 1, 2, 3$.

- (a) Write down the objective function to be minimised. [2]
- (b) Write down the constraints that ensure each worker is given exactly one job and each job is given to exactly one worker. [3]
- (c) The Hungarian algorithm gives the allocation A to 1, B to 2, C to 3. State the value of each x_{ij} that is 1 in this solution, and verify the value of the objective function. [2]

- 27** A project manager has five employees, E1 to E5, and four tasks, T1 to T4. Each task is carried out by one employee and each employee carries out at most one task. The table shows the value, in £ hundreds, that each employee would add on each task. A dash means that the employee is not able to do that task.
- Use the Hungarian algorithm to find the assignment which maximises the total value added. State which employee is not used and the maximum value, showing your matrix at every stage.

	T1	T2	T3	T4
E1	34	39	44	57
E2	85	55	81	85
E3	75	43	43	74
E4	32	-	70	77
E5	62	33	71	52

[12]

- 28** An $n \times n$ allocation problem has cost matrix with entries c_{ij} . Use the notation $x_{ij} = 1$ if row i is assigned to column j , and 0 otherwise, so that every row and every column of (x_{ij}) contains exactly one 1.
- (a) The same constant k is subtracted from every entry of row r . Prove that every allocation has its total cost reduced by exactly k , and deduce that the allocations that minimise the total cost are unchanged. [3]
- (b) The profit table p_{ij} is to be maximised. Let K be the largest entry and $c_{ij} = K - p_{ij}$. Prove that an allocation minimises the total of the c_{ij} if and only if it maximises the total of the p_{ij} . [2]
- (c) After the reductions, every entry is non-negative and the zeros allow a complete allocation. Prove that this allocation minimises the total cost of the original problem. [3]

- 29** Three workers A, B and C share five jobs. Every job must be done by exactly one worker, and each worker can do at most two jobs. The table shows the time, in hours, that each worker needs for each job (times add when a worker does two jobs).
- Show how the problem can be turned into an allocation problem that can be solved by the Hungarian algorithm, and solve it to find the jobs each worker should do to minimise the total time. State the minimum total time.

	1	2	3	4	5
A	12	17	16	15	35
B	18	38	37	10	29
C	40	27	32	26	37

[10]

30 The Hungarian algorithm is applied to the 5×5 cost matrix

$$\begin{pmatrix} 18 & 36 & 35 & 30 & 14 \\ 19 & 35 & 17 & 28 & 10 \\ 16 & 32 & 13 & 9 & 25 \\ 29 & 29 & 36 & 23 & 17 \\ 16 & 14 & 29 & 20 & 22 \end{pmatrix}$$

At each stage the total of all the amounts subtracted is a lower bound for the cost of any allocation.

- Find the total of the row minima, and then the total of the column minima in the matrix after the row reduction. [2]
- The algorithm then needs 2 adjustments. For each one, state the number of lines used and the smallest uncovered entry, and use the result $m(n - k)$ from part (d) to find the amount by which the lower bound increases. [3]
- Hence find the minimum total cost, and confirm it by finding the allocation and its cost from the original matrix. [2]
- In an adjustment, k lines (r rows and c columns) are used and the smallest uncovered entry is m . Prove that the lower bound increases by $m(n - k)$ when $n \times n$ entries are adjusted in the standard way. [4]

Transportation Problems

- Three depots supply 35, 40 and 25 units of a product and four shops demand 20, 30, 25 and 15 units.
Show that the problem is not balanced, and state what should be added so that it can be solved by the transportation algorithm, including the unit costs to use. [2]
- A transportation problem has 4 sources and 5 destinations.
How many cells are occupied in a basic feasible solution that is not degenerate? What does it mean if fewer cells are occupied? [2]
- A transportation problem has three sources A, B, C and three destinations W, X, Y. The supplies and demands are shown in the table, together with the unit costs.
Use the north-west corner method to write down the first two allocations made.

	W	X	Y	Supply
A	15	6	9	30
B	12	10	10	15
C	4	12	11	35
Demand	20	50	10	

[2]

- 4 The table shows a transportation problem and a feasible allocation of units (the large figures). The small figures are the unit costs. Find the total cost of this allocation.

	W	X	Y	Supply
A	¹² 15	⁸	¹¹	15
B	⁸ 35	¹⁴ 5	¹³	40
C	¹²	⁷ 5	⁷ 25	30
Demand	50	10	25	

[2]

- 5 In a transportation problem, x_{ij} is the number of units sent from source i to destination j . There are three sources A, B, C and three destinations 1, 2, 3. Source A has a supply of 40 units, and destination 2 requires 35 units.
- (a) Write down the constraint for the supply at A. [1]
- (b) Write down the constraint for the demand at destination 2. [1]

- 6 When a transportation problem has more supply than demand, a dummy destination is added. Explain why the unit costs for the dummy destination are all zero, and say what a dummy allocation means in practice. [2]

- 7 A company has three factories A, B and C which supply four depots W, X, Y and Z. The table shows the unit transport costs (£), the supplies and the demands. Use the north-west corner method to find an initial solution, and calculate its cost.

	W	X	Y	Z	Supply
A	⁸	³	¹³	⁶	20
B	³	⁷	⁹	¹²	10
C	¹¹	¹³	¹⁵	²	35
Demand	45	5	5	10	

[4]

- 8** A transportation problem has the supplies, demands and unit costs shown in the table. Use the least cost method to find an initial solution, and calculate its cost. (If two cells have the same cost, choose the one that allows the larger allocation.)

	W	X	Y	Supply
A	7	10	5	10
B	14	12	5	35
C	13	2	8	20
Demand	5	10	50	

[4]

- 9** Three factories A, B and C supply three shops W, X and Y. The table shows the supplies, the demands and the unit costs (£). Demand exceeds supply. Add a dummy source so that the problem is balanced, use the north-west corner method to find an initial solution, and find the cost of this solution.

	W	X	Y	Supply
A	12	10	14	40
B	9	9	12	15
C	13	2	5	35
Demand	35	40	20	

[5]

- 10** The table shows a feasible allocation for a transportation problem (large figures) and the unit costs (small figures).
Find the shadow costs R_i and K_j , taking $R_A = 0$.

	W	X	Y	Z	Supply
A	⁷ 10	⁹ 5	¹⁰ 25	¹⁰ 	40
B	⁹ 	¹³ 	¹² 15	⁷ 25	40
C	³ 	⁸ 	¹⁰ 	³ 30	30
Demand	10	5	40	55	

[3]

- 11** For the allocation shown, the shadow costs are $R = (0, 3, -5)$ for sources A, B, C and $K = (12, 13, 7)$ for destinations W, X, Y.
Calculate the improvement index of each empty cell, and state whether the allocation is optimal.

	W	X	Y	Supply
A	¹² 15	¹⁴ 	¹⁴ 	15
B	¹⁵ 25	¹⁵ 	¹³ 	25
C	⁷ 5	⁸ 30	² 5	40
Demand	45	30	5	

[3]

- 12** For the allocation shown, the improvement indices of the empty cells are: $AX = 0$, $AY = -1$, $BY = 2$, $CW = -8$.
Use the stepping-stone method to improve the allocation: state the entering cell, the loop, the value of θ , and the new allocation.

	W	X	Y	Supply
A	¹¹ 10	¹⁴	¹⁰	10
B	⁴ 5	⁷ 10	⁶	15
C	⁴	¹⁵ 10	¹² 20	30
Demand	15	20	20	

[4]

- 13** Two factories A and B supply 45 and 35 units respectively to three outlets 1, 2 and 3, which require 20, 30 and 30 units. The cost, in £ per unit, of sending from A to the outlets is 7, 9 and 6 and from B is 8, 5 and 10.
Let x_{ij} be the number of units sent from factory i to outlet j . Formulate the problem of minimising the total transport cost as a linear programming problem.

[5]

- 14** The table shows a transportation problem with a feasible allocation (the north-west corner solution). The cost of this allocation is 570.
Carry out one iteration of the stepping-stone method: find the shadow costs and improvement indices, choose the entering cell, find the stepping-stone loop and the new allocation, and give its cost.

	W	X	Y	Z	Supply
A	² 5	⁶ 5	² 20	⁷	30
B	¹¹	¹²	¹⁰ 25	¹⁵	25
C	¹¹	⁹	⁸ 20	⁸ 10	30
Demand	5	5	65	10	

[5]

- 15** Three warehouses A, B and C supply three shops W, X and Y. The supplies, demands and unit costs (£) are shown in the table. Supply exceeds demand. Balance the problem with a dummy destination, find the north-west corner solution and use shadow costs to show that it is optimal. State the minimum cost.

	W	X	Y	Supply
A	7	15	9	20
B	5	10	12	25
C	15	12	9	35
Demand	55	10	10	

[7]

- 16** The table shows a transportation problem and a proposed allocation. Show that the allocation is optimal, and state the minimum cost.

	W	X	Y	Z	Supply
A	³ 30	⁶ 5	12	¹⁰	35
B	¹⁴	¹¹ 5	¹¹ 5	⁸ 30	40
C	¹³	⁶	⁵ 20	¹³	20
Demand	30	10	25	30	

[7]

- 17** The table shows a feasible allocation for a transportation problem. The improvement index of the empty cell CW is -13 .
Find the stepping-stone loop for CW, the value of θ and the new allocation, and use the improvement index to find the new total cost without recalculating it from the table (the current cost is 740).

	W	X	Y	Z	Supply
A	⁹ 5	⁴ 25	¹⁰	⁹	30
B	⁶	¹² 20	⁸ 10	⁷	30
C	²	¹⁵	⁶ 35	¹³ 5	40
Demand	5	45	45	5	

[5]

- 18** The table shows a transportation problem.
Use the north-west corner method to find an initial solution. Explain why the solution is degenerate and how it should be dealt with (when both a row and a column are exhausted, write the 0 in the next cell to the right), and find the shadow costs, taking $R_A = 0$.

	W	X	Y	Supply
A	⁴	¹²	⁹	35
B	¹³	¹³	¹³	25
C	¹⁴	¹⁵	¹¹	20
Demand	35	10	35	

[5]

- 19** A firm moves stock from three warehouses A, B and C to three shops W, X and Y. The table shows the stock held, the stock needed and the cost, in £ per crate, of each route. Find the cost of the initial solution given by (i) the north-west corner method and (ii) the least cost method, and comment on the difference. (Where two cells have equal cost, choose the one allowing the larger allocation.)

	W	X	Y	Supply
A	10	12	3	20
B	19	20	14	25
C	3	12	11	20
Demand	5	5	55	

[7]

- 20** For the allocation shown, a student correctly finds the shadow costs $R = (0, -11, -15)$ and $K = (2, 15, 17, 17)$, and then the improvement index of every empty cell. The student finds that the largest index is 20 for cell CW, and chooses that cell to enter the solution. Identify the student's error, and find the correct entering cell, showing all the improvement indices.

	W	X	Y	Z	Supply
A	² 10	¹⁵ 10	12	⁵	20
B	⁷	⁴ 5	⁶ 35	¹¹	40
C	⁷	¹⁴	² 10	² 20	30
Demand	10	15	45	20	

[4]

21 A transportation problem is given in the table.

	W	X	Y	Supply
A	11	16	13	35
B	18	10	12	20
C	11	8	14	40
Demand	10	80	5	

- (a) Use the north-west corner method to find an initial solution. [2]
 (b) Carry out one iteration of the stepping-stone method, giving the shadow costs, improvement indices, loop and new allocation. [4]
 (c) Show that the new allocation is optimal and state the minimum cost. [3]

22 The table shows a transportation problem with a feasible allocation, which has cost 360.

	W	X	Y	Supply
A	⁵ 25	⁵	⁵	25
B	³ 20	⁹ 5	¹¹	25
C	¹⁵	¹² 5	¹⁴ 5	10
Demand	45	10	5	

- (a) Perform one iteration of the stepping-stone method. Show that more than one cell on the loop is reduced to zero. [5]
 (b) Explain what must be done about the zero allocation, and show that the new allocation is optimal. [4]

- 23** The table shows an allocation for a transportation problem.

	W	X	Y	Z	Supply
A	⁸ 10	³ 10	¹⁵ 15	¹⁵	35
B	⁷	¹⁰	¹⁴ 5	⁷ 30	35
C	¹²	¹³	² 20	¹⁰	20
Demand	10	10	40	30	

- (a) Show that the allocation is optimal, and state its cost. [3]
 (b) Show that the optimal solution is not unique by finding a different allocation with the same cost. [3]

- 24** A company ships from three factories A, B and C to four depots W, X, Y and Z. The table shows the supplies, demands and unit costs (£). The route from A to W is closed (shown by a dash).
 Use the least cost method (ties broken by the larger allocation) to find an initial solution, then use the stepping-stone method to find the minimum cost distribution, showing that it is optimal. State the minimum cost.

	W	X	Y	Z	Supply
A	-	¹¹	¹⁵	¹⁶	25
B	¹³	¹⁵	¹⁵	⁴	15
C	³	⁵	¹⁵	⁸	40
Demand	20	25	5	30	

[9]

- 25** Three factories A, B and C can supply at most 40, 35 and 30 units of a product. Four shops 1, 2, 3 and 4 need exactly 25, 20, 30 and 15 units. The cost, in £ per unit, of sending from A is 6, 8, 5, 9, from B is 7, 4, 8, 6 and from C is 9, 7, 6, 5 to shops 1 to 4 respectively.

Let x_{ij} be the number of units sent from factory i to shop j .

- (a) Formulate a linear programming problem to minimise the total cost. [4]
 (b) Show how the supply constraints can be written as equations by introducing slack variables, and explain what the slack variables represent in terms of a transportation problem with a dummy destination. [2]
 (c) Show that the distribution $A_1 = 25$, $A_3 = 15$, $B_2 = 20$, $B_4 = 15$, $C_3 = 15$ satisfies the constraints, and find its cost. [2]

- 26** The table shows an optimal allocation for a transportation problem, with cost 225. The unit cost of the occupied cell BY is currently 2, but may change to t . Find the range of values of t for which this allocation remains optimal.

	W	X	Y	Supply
A	³ 15	³	⁶	15
B	⁵ 10	¹³	² 5	15
C	⁸	² 10	⁴ 25	35
Demand	25	10	30	

[7]

- 27** Four factories A, B, C and D supply four depots W, X, Y and Z. The table shows the supplies, the demands and the unit costs (£). Supply exceeds demand. The route from D to X is not available (a dash). Balance the problem, use the north-west corner method to find an initial solution, and use the stepping-stone method to find the minimum cost distribution. Show all improvement indices, and state the minimum cost and where the surplus stays.

	W	X	Y	Z	Supply
A	⁴	¹⁶	¹²	¹³	35
B	⁴	⁵	³	³	20
C	⁶	¹³	¹¹	⁴	35
D	¹³	-	⁷	⁹	25
Demand	30	45	5	25	

[12]

- 28** A firm sends goods from three depots A, B and C to three customers W, X and Y. The table shows the supplies, the demands and the profit, in £ per unit, on each route.

	W	X	Y	Supply
A	13	4	2	20
B	11	11	1	30
C	12	7	9	15
Demand	10	25	30	

- (a) Explain how the table can be changed so that the transportation algorithm, which minimises, can be used to maximise the total profit. [2]
- (b) Starting from the north-west corner solution, use the stepping-stone method to find the distribution that maximises the profit. State the maximum profit. [10]

- 29** A balanced transportation problem has m sources and n destinations, with x_{ij} units sent from source i to destination j and unit costs c_{ij} .

- (a) Prove that a basic feasible solution has at most $m + n - 1$ non-zero allocations. [4]
- (b) The shadow costs R_i and K_j satisfy $R_i + K_j = c_{ij}$ for every occupied cell. Prove that the improvement index $I_{ij} = c_{ij} - R_i - K_j$ of an empty cell equals the change in total cost when one unit is moved round its stepping-stone loop. [5]

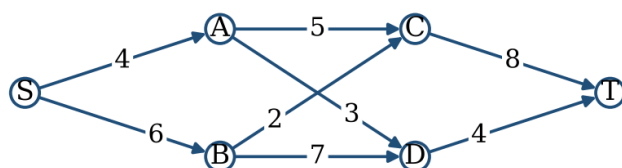
- 30** Three factories A, B and C have 25, 15 and 15 units available for three shops W, X and Y, which want 15, 20 and 45 units. The table shows the unit transport costs (£). Any unmet demand at shop W, X or Y incurs a penalty of £14, £4 and £7 per unit respectively.

	W	X	Y	Supply
A	4	3	10	25
B	14	7	9	15
C	4	13	3	15
Demand	15	20	45	

- (a) Show how the problem can be turned into a balanced transportation problem, and say what the costs in the extra row represent. [1]
- (b) Starting with the north-west corner method, use the stepping-stone method to find the plan that minimises the total of transport costs and penalties, and state this minimum. [10]
- (c) State which shops are left short, and by how many units. [1]

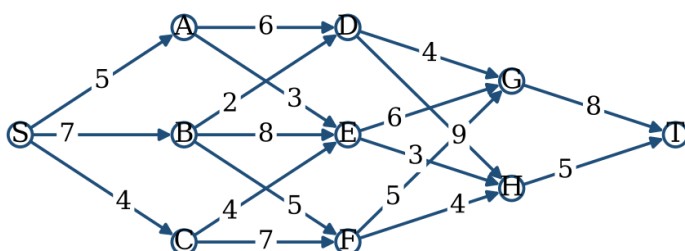
Dynamic Programming

- 1 The network shows the distances, in km, between towns. Find the length of the route S-A-D-T and of the route S-B-C-T.



[2]

- 2 The network shown is drawn in stages, with all arcs directed from left to right. Find the number of different routes from S to T.



[2]

- 3 In a dynamic programming solution to a shortest route problem, working back from the destination, $f(E) = 7$ and $f(F) = 5$. The arcs leaving C are C to E of length 4 and C to F of length 6. Find $f(C)$ and the best action at C.

[2]

- 4 Explain why dynamic programming for a shortest route problem works backwards from the destination, and what $f(X)$ stands for.

[2]

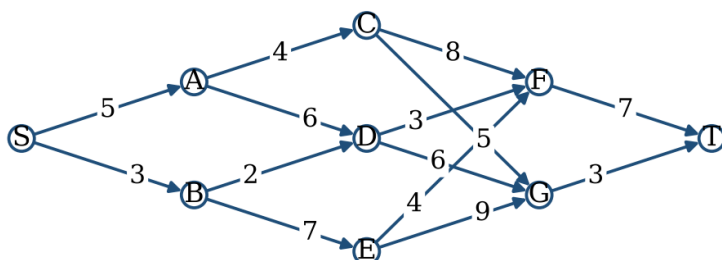
- 5 A route has arcs of capacities (maximum load in tonnes) 12, 9, 15 and 11 along it. In a maximin problem, the value of a route is the smallest capacity on it. Find the value of this route, and the value of a second route with capacities 10, 14, 13 and 10. State which route is better.

[2]

- 6 £5 million is to be shared between three projects, in whole millions of pounds, to maximise the total return. A dynamic programming method treats each project as a stage. State what the state and the action are at each stage, and what the value of the state is.

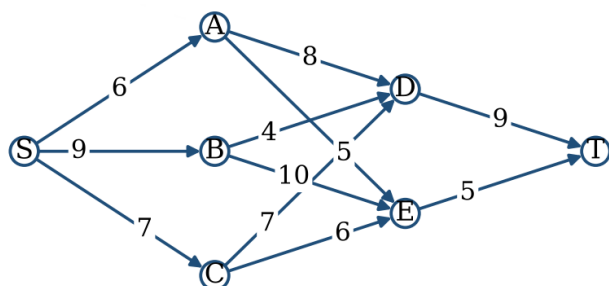
[3]

- 7 The network shows the times, in minutes, for journeys between stations. Use dynamic programming, working backwards from T, to find the shortest route from S to T. Show the stage, state, action and value at each stage.



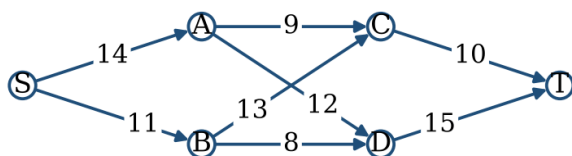
[5]

- 8** The network shows the profit, in £ hundreds, from each stage of a contract. Use dynamic programming to find the route from S to T that gives the greatest total profit, and state the profit.



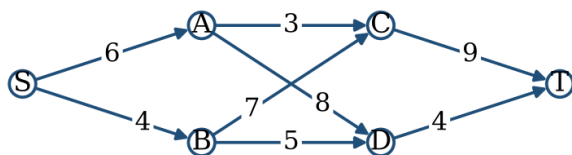
[5]

- 9** A lorry must travel from S to T. The numbers on the arcs are the maximum heights, in metres, of the lorry that can pass under bridges on that road. Use dynamic programming to find the route that allows the tallest lorry to pass, and state the maximum height.



[4]

- 10** A cyclist must travel from S to T. The numbers on the arcs are the steepest gradients, in percent, on each section of road. Use dynamic programming to find the route for which the steepest gradient is as small as possible, and state this gradient.



[4]

- 11** The dynamic programming table (working back from T) for the network in a shortest route problem is shown. Some of the values are missing; actions marked * are optimal in the completed table. Stage 1 is the stage nearest T. Calculate the missing values, and hence write down the shortest route from S to T and its length.

Stage	State	Action	Value
1	F	T *	7
	G	T *	
2	C	F	$8 + 7 = 15$
		G *	$5 + 3 = 8$
	D	F	
		G *	$6 + 3 = 9$
	E	F *	
3	A	G	$9 + 3 = 12$
		C *	$4 + 8 = 12$
	B	D	
		D *	$2 + 9 = 11$
4	S	E	$7 + 11 = 18$
		A	
4	S	B *	$3 + 11 = 14$

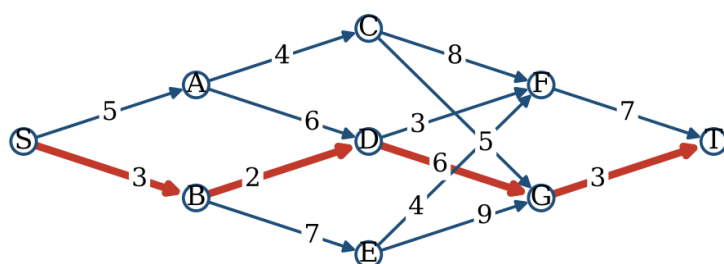
[4]

- 12** A manager has 3 units of equipment to share between two projects, and all 3 units must be used. The table shows the return, in £ thousands, from each project for each number of units. Use dynamic programming, treating each project as a stage, to find how the units should be shared to maximise the total return, and state the maximum return.

Units	0	1	2	3
Project 1	0	3	5	6
Project 2	0	2	5	7

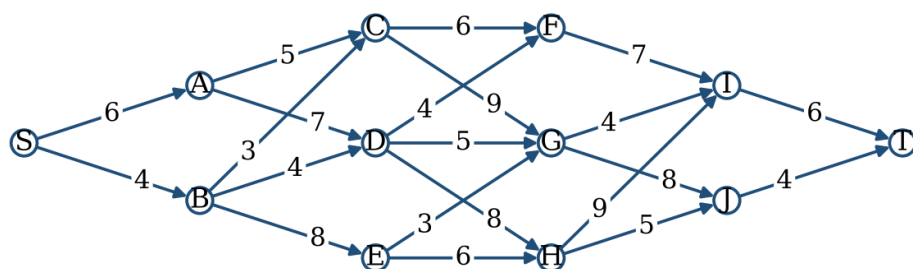
[4]

- 13** In the network shown, the shortest route from S to T is S-B-D-G-T. The length of the arc BD is increased by k. Find the greatest value of k for which S-B-D-G-T is still a shortest route.



[3]

- 14** The network shows the cost, in £ hundreds, of each stage of a production process, from S to T. Use dynamic programming to find the minimum cost route. Present your working in a table with columns for stage, state, action and value.



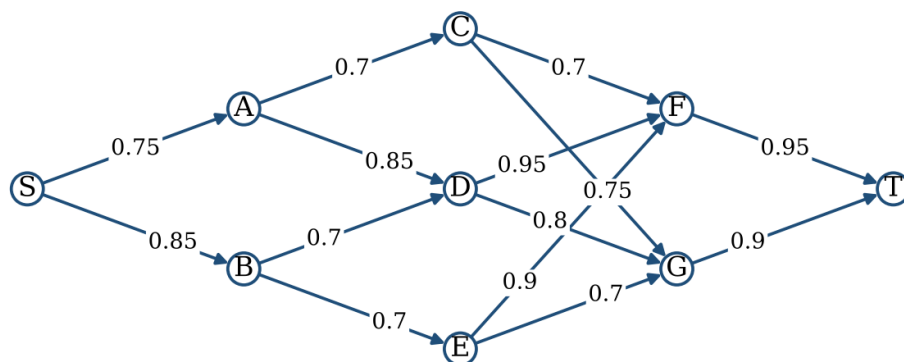
[6]

- 15** A charity has 5 units of funding to share between three projects. All 5 units must be allocated, in whole units. The table shows the benefit, in suitable units, of each project for each amount of funding. Use dynamic programming, with each project as a stage, to find the allocation that maximises the total benefit, and state this benefit.

Units	0	1	2	3	4	5
Project 1	0	4	7	9	10	10
Project 2	0	4	6	8	9	9
Project 3	0	5	8	9	9	9

[6]

- 16** A delivery van must travel from S to T. The number on each arc is the probability that the van completes that road without a breakdown, and the roads are independent. Use dynamic programming to find the route for which the probability of a journey with no breakdown is greatest. State the route and this probability to 3 significant figures.



[5]

- 17** A courier van can carry at most 11 kg. The table shows the mass and value of three types of parcel, and any number of each type may be loaded. Use dynamic programming, with the parcel types as stages and the remaining capacity as the state, to find the loading of greatest total value.

State the number of each type loaded and the total value.

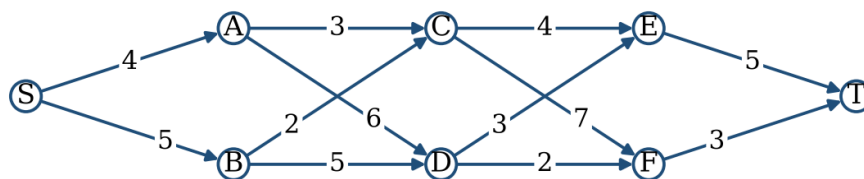
	Mass (kg)	Value (£)
Item 1	3	8
Item 2	4	11
Item 3	5	15

[6]

- 18** A factory plans production for three months. The demand is 2, 3 and 2 hundred units. Production in a month costs £4 hundred setup plus £2 hundred per hundred units made (no cost if nothing is made), at most 5 hundred units can be made in a month, and at most 3 hundred can be stored. Storage costs £1 hundred for each hundred units held at the end of a month. There is no stock at the start of month 1 and none must be left at the end of month 3. Use dynamic programming, with the months as stages and the opening stock as the state, to find the cheapest production plan. State the plan and its total cost.

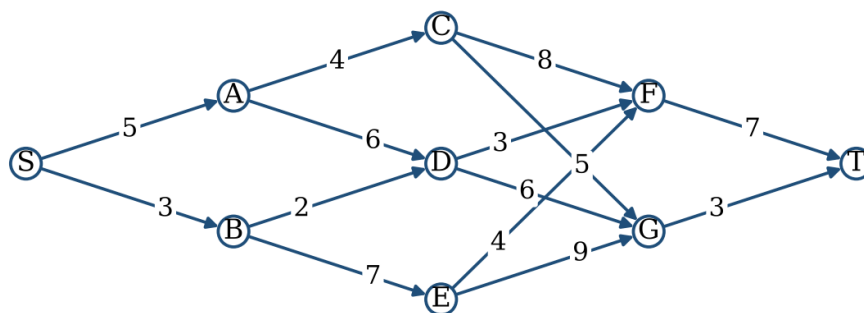
[6]

- 19** In the network, the numbers are the costs, in £, of using each arc. Use dynamic programming to find the minimum cost of a route from S to T, and list every route that achieves it.



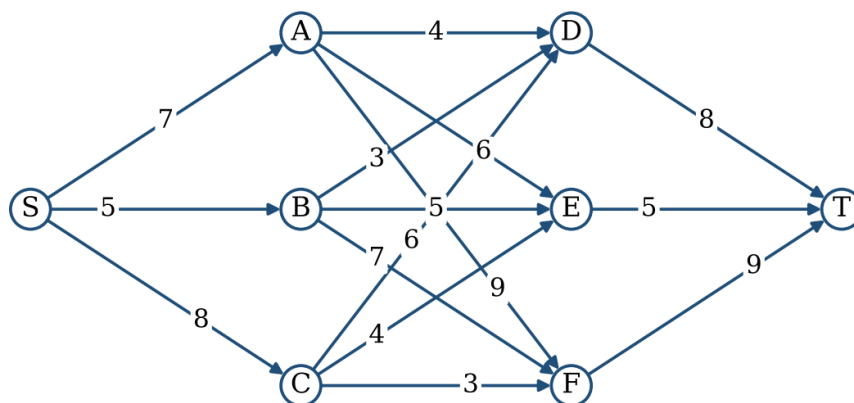
[5]

- 20** A student solves the shortest route problem below by starting at S and always moving along the shortest available arc. Find the length of the route the student finds. Explain why the method is not valid, and use dynamic programming to find the length of the true shortest route.



[4]

- 21** A project network has the activities shown, with durations in days given on the arcs of the network from S to T.



- (a) Use dynamic programming to find the shortest route from S to T, and state its length. [4]
 (b) Use dynamic programming to find the longest route from S to T, and state its length. [4]

- 22** A government department has 6 units of funding to distribute among four projects, in whole units, using all 6 units. The table shows the benefit from each project for each number of units.

Use dynamic programming, with the projects as stages and the amount of funding left as the state, to find how the funding should be shared to maximise the total benefit. Give the maximum benefit.

Units	0	1	2	3	4	5	6
Project 1	0	5	8	11	13	15	15
Project 2	0	5	10	13	16	18	18
Project 3	0	5	10	14	18	19	20
Project 4	0	5	9	13	15	16	16

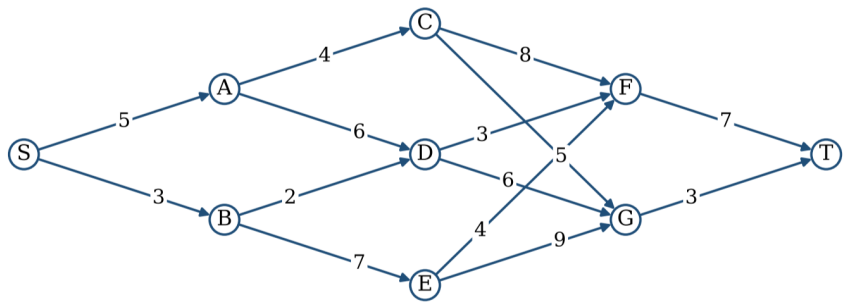
[8]

- 23** A supervisor has 6 workers to place on three machines, all 6 being used. The table shows the output of each machine, in units per hour, for each number of workers. The supervisor wants the smallest of the three outputs to be as large as possible.

Workers	0	1	2	3	4	5	6
Machine 1	0	4	7	9	10	11	11
Machine 2	0	3	6	8	9	10	10
Machine 3	0	5	7	8	9	10	11

- (a) Describe the stage, state, action and value for a dynamic programming formulation of this maximin problem. [2]
 (b) Solve the problem and state the largest possible smallest output. [6]

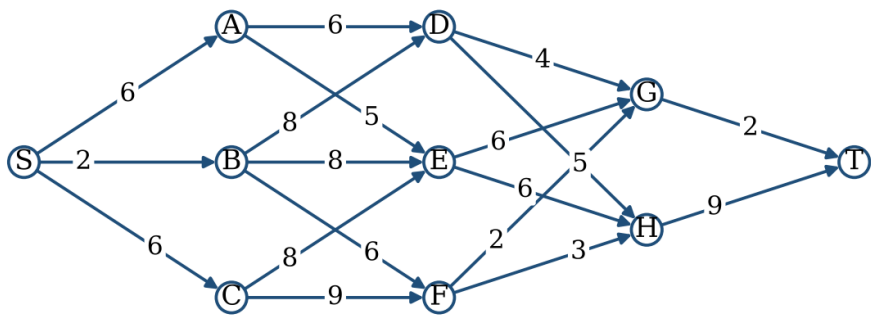
24 The table shows the results of a dynamic programming solution to a shortest route problem in the network below (f is the length of the shortest route from the vertex to T). The network is then altered. Use the table, rather than starting again, to answer each part.



Vertex	f	Best next
G	3	T
F	7	T
E	11	F
D	9	G
C	8	G
B	11	D
A	12	C
S	14	B

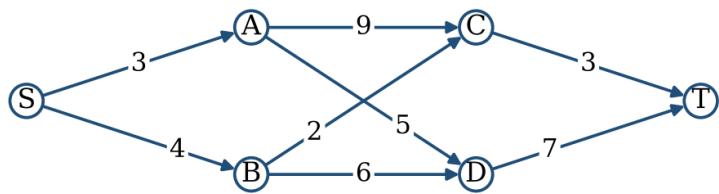
- (a) A new arc from B to G of length 1 is built, and no other change is made. Find the new shortest route from S to T and its length. [3]
- (b) Instead, the arc SB is closed and no other change is made. Find the new shortest route from S to T and its length. [3]

25 The network shows the distances, in km, between depots. A delivery vehicle must travel from S to T and must call at E on the way. Explain how dynamic programming can be adapted to find the shortest route that passes through E, and use it to find that route and its length.



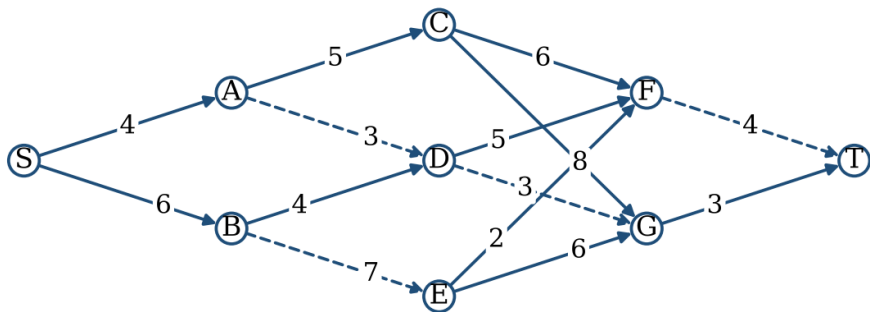
[6]

26 In the network shown, the numbers are distances.



- (a) Show that always taking the shortest arc from the current vertex does not find the shortest route from S to T, and find the true shortest route using dynamic programming. [4]
- (b) Prove that if R is a shortest route from S to T and X is a vertex on R, then the part of R from X to T is a shortest route from X to T. [3]
- (c) Explain how this result justifies working backwards in dynamic programming. [1]

27 In the network, the numbers are travel times in minutes. The arcs AD, BE, DG and FT (drawn dashed) are toll roads. A courier must travel from S to T using exactly one toll road. Use dynamic programming, with the state recording the vertex and the number of toll roads used so far, to find the quickest route and its time.



[8]

28 A company uses a machine, and at the start of each of the next 5 years it can either keep its current machine or replace it with a new one costing £12 thousand. The machine now in use is 2 years old; a machine must be replaced when it reaches age 4. The table shows the cost of running a machine for a year (according to its age at the start of the year) and its trade-in value, in £ thousand, if replaced at that age. A replacement machine is new at the start of the year, so costs the running cost of age 1 that year (the table's age-1 cost), and is age 2 the next year. Use dynamic programming to find the replacement policy that minimises the total cost over the 5 years, and state this cost.

Age (years)	Running cost	Trade-in value
1	2	8
2	3	6
3	6	4
4	9	2

[8]

- 29** A system consists of three components in series. Component k is made from m_k identical units in parallel, with $m_k \geq 1$ and $m_k \leq 3$, so it works if at least one unit works. The table shows the cost of a unit and the probability that a unit works. The units work independently, the total budget is £10, and the aim is to maximise the probability that the whole system works.

	Cost per unit (£)	Reliability of one unit
Component 1	2	0.8
Component 2	3	0.7
Component 3	1	0.9

- (a) Find the probability that component k works when it has 1, 2 and 3 units, for each k . [2]
 (b) Use dynamic programming, with the components as stages and the money left as the state, to find the best number of units for each component and the maximum probability that the system works (to 3 significant figures). [6]

- 30** A network has a source S , a sink T and k layers, each with w vertices, between them. Every vertex in a layer is joined by an arc directed from it to every vertex in the next layer, S is joined to every vertex of the first layer and every vertex of the last layer is joined to T . A route from S to T uses one vertex from each layer.

- (a) Show that the number of different routes from S to T is w^k . [2]
 (b) Dynamic programming evaluates one addition (arc length plus the value of the next vertex) for each arc other than those into T . Find the number of additions in terms of k and w . [2]
 (c) For $k = 8$ and $w = 6$ find the number of routes, the number of additions needed to find the length of every route separately, and the number of additions needed by dynamic programming. [3]
 (d) Comment on how the work needed by the two methods grows as k increases. [1]

Decision Analysis

- 1** A project has three possible outcomes, with profits (in £ thousands) of 40, 20 and -10 , which occur with probabilities 0.3, 0.5 and 0.2 respectively. Calculate the expected monetary value (EMV) of the project. [2]
2 The table shows the profit, in £ thousands, of two options in two possible states of the market. Find the EMV of each option and state which option should be chosen to maximise the EMV.

Option	Good	Poor
Option A	50	10
Option B	30	25
Probability	0.4	0.6

[3]

- 3 The table shows the profit, in £ thousands, for three options under three possible states. Use the maximin criterion to decide which option to choose.

Option	State 1	State 2	State 3
Option 1	12	8	5
Option 2	15	4	2
Option 3	9	9	7

[2]

- 4 Three options are available, and the profits (in £ thousands) in each of three possible states are: Option 1: 12, 8, 5; Option 2: 15, 4, 2; Option 3: 9, 9, 7. Find the option chosen by the maximax criterion and the value it achieves.

[2]

- 5 In a decision tree, what do a square node and a circular node represent? State what is done at each type of node when the tree is evaluated to maximise expected profit.

[2]

- 6 The table shows the profit, in £ thousands, of three options under three states of nature. Find the regret for option 1 in each of the three states.

Option	State A	State B	State C
Option 1	30	18	6
Option 2	22	25	14
Option 3	10	20	28

[2]

- 7 A company is choosing between three projects. The table shows the profit, in £ thousands, in each of three economic conditions, and the probability of each condition. Calculate the EMV of each project and state which should be chosen.

Option	Boom	Normal	Slump
Project 1	60	30	-10
Project 2	40	35	15
Project 3	20	20	20
Probability	0.3	0.5	0.2

[4]

- 8 The table shows the profit, in £ hundreds, of three courses of action under four possible scenarios. Find the course of action chosen by (i) the maximin criterion and (ii) the maximax criterion, and explain which type of decision maker would use each.

Option	S1	S2	S3	S4
Course 1	14	9	6	2
Course 2	10	10	8	7
Course 3	20	5	3	1

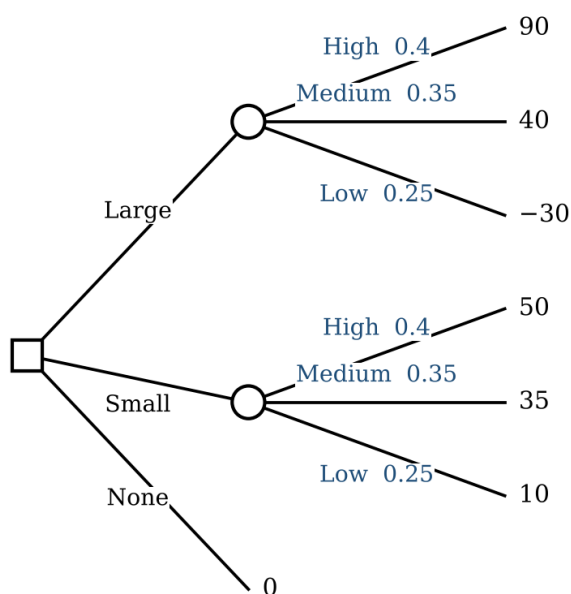
[4]

- 9** The table shows the profit, in £ thousands, for three options under three states of nature. Construct the regret (opportunity loss) table, and use the minimax regret criterion to choose an option.

Option	State A	State B	State C
Option 1	45	30	10
Option 2	30	40	25
Option 3	15	25	50

[5]

- 10** A shop owner is deciding the size of a new shop. The decision tree shows the profits, in £ thousands, of the options for each level of demand, with their probabilities (High 0.4, Medium 0.35, Low 0.25). Calculate the EMV at each chance node and state the best decision.



[5]

- 11** The table shows the profit, in £ thousands, for two options in two states of nature, which are equally likely. Find the EMV of each option, the expected profit with perfect information and the expected value of perfect information (EVPI).

Option	State 1	State 2
Option 1	80	20
Option 2	50	40
Probability	0.5	0.5

[4]

- 12** A firm must choose between two suppliers. The table shows the cost, in £ thousands, of using each supplier when demand is high or low. The probability of high demand is 0.7. Find the expected cost for each supplier and state which supplier should be chosen to minimise expected cost.

Option	High	Low
Supplier 1	12	20
Supplier 2	16	15

[3]

- 13** Option A gives a profit of £70 thousand if a product succeeds (probability p) and a loss of £10 thousand if it fails. Option B gives a profit of £35 thousand if the product succeeds and £20 thousand if it fails. Find the value of p for which the two options have the same EMV, and state which option is better when p is larger than this value.

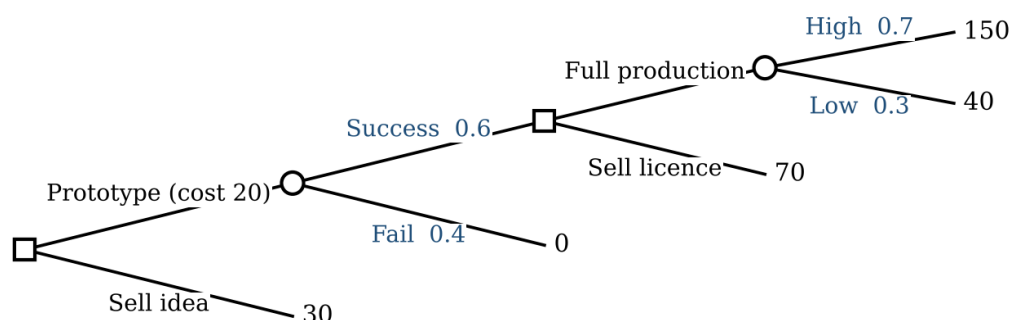
[4]

- 14** The table shows the profit, in £ thousands, of three options under four possible states of nature, together with the probabilities of the states. Determine the option that would be chosen using (i) the maximin criterion, (ii) the minimax regret criterion and (iii) the maximum EMV criterion.

Option	S1	S2	S3	S4
Option 1	55	40	25	-5
Option 2	35	35	30	20
Option 3	70	30	10	-20
Probability	0.2	0.3	0.3	0.2

[7]

- 15** A firm can sell an idea now for £30 thousand, or build a prototype at a cost of £20 thousand. The prototype succeeds with probability 0.6. If it fails the firm gets nothing. If it succeeds, the firm can either sell a licence for £70 thousand or go into full production, which brings in £150 thousand if demand is high (probability 0.7) and £40 thousand if demand is low. (All figures before the prototype cost is deducted.) Evaluate the decision tree to find the firm's best strategy and its expected profit.



[6]

- 16** The table shows the profit, in £ thousands, for three options in three states of nature, with the probability of each state.
Calculate the EMV of each option, and the expected value of perfect information (EVPI).
State the greatest amount worth paying for a perfect forecast of the state.

Option	State 1	State 2	State 3
Option 1	50	30	10
Option 2	35	40	25
Option 3	20	25	60
Probability	0.4	0.35	0.25

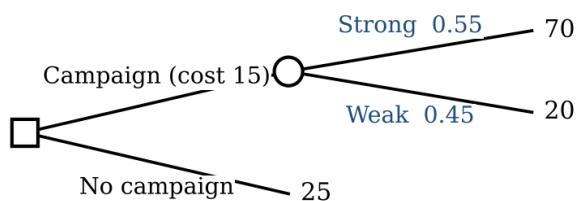
[6]

- 17** A council must choose one of three flood defence schemes. The table shows the cost, in £ thousands, of each scheme in three possible weather conditions, with the probability of each condition.
Find the scheme chosen by (i) the minimax cost criterion, (ii) the minimax regret criterion, and (iii) the lowest expected cost.

Scheme	Dry	Normal	Wet
Scheme 1	40	55	90
Scheme 2	50	60	70
Scheme 3	65	52	58
Probability	0.3	0.45	0.25

[7]

- 18** A company can run an advertising campaign costing £15 thousand. Revenue (before the campaign cost) is £70 thousand if the response is strong (probability 0.55) and £20 thousand if it is weak. Without the campaign the revenue is £25 thousand for certain.
Use the decision tree to decide whether the campaign should be run, and state the expected revenue after all costs.



[5]

- 19** A decision maker's utilities for profits (in £ thousands) are shown in the first table. The second table shows the profits and probabilities for three options. Find the option with the greatest EMV and the option with the greatest expected utility, and explain why they differ.

Profit (£k)	Utility
−20	0
0	0.4
30	0.7
60	0.9
100	1

Option	Outcome 1 (prob.)	Outcome 2 (prob.)
A	100 (0.3)	−20 (0.7)
B	30 (0.5)	0 (0.5)
C	60 (0.6)	−20 (0.4)

[5]

- 20** The table shows the profit, in £ thousands, of three options in three states. A student forms a regret table by subtracting the smallest entry in each row from every entry in that row, and uses it to apply the minimax regret criterion. Explain the student's error, and find the correct option by the minimax regret criterion, showing the correct regret table.

Option	State 1	State 2	State 3
Option 1	40	25	15
Option 2	30	35	20
Option 3	20	30	45

[4]

- 21** A developer can build a large or a small estate. Profits (£ thousands) are: large, 100 if demand is high and −40 if low; small, 50 if high and 20 if low. The probability of high demand is 0.5. The developer can first commission a survey costing £6 thousand. The survey result is 'good' with probability 0.5, and then the probability of high demand is 0.8; otherwise the result is 'poor', and the probability of high demand is 0.2.

- (a) Find the best decision and its EMV if no survey is commissioned. [3]
 (b) Evaluate the strategy of commissioning the survey, including its cost, and state whether it should be commissioned and how the developer should respond to each result. [7]

- 22** The table shows the profit, in £ thousands, of three options in three states of nature, together with the probabilities of the states.

Option	S1	S2	S3
Option 1	90	40	−20
Option 2	60	50	10
Option 3	30	30	30
Probability	0.25	0.45	0.3

- (a) Find the EMV of each option and the best option if no further information is available. [3]
 (b) Find the expected value of perfect information. [3]
 (c) A consultant offers a perfectly reliable forecast of the state for £12 thousand. Decide whether it is worth buying. [2]

- 23** Three options A, B and C are available. If the market is strong (probability p) their profits, in £ thousands, are 120, 60 and 25 respectively; if the market is weak (probability $1 - p$) their profits are −60, 10 and 25.
- (a) Write down the EMV of each option as a function of p . [2]
 (b) Find the range of values of p for which each option has the greatest EMV. [4]
 (c) Describe the circumstances in which option C is the best choice. [1]

- 24 The table shows the profit, in £ thousands, of three options in three states S1, S2 and S3. The probabilities of the states are p , $2p$ and $1 - 3p$ respectively. Find the range of values of p for which option 1 has the greatest EMV.

Option	S1 (p)	S2 ($2p$)	S3 ($1-3p$)
Option 1	80	50	10
Option 2	40	45	30
Option 3	20	30	60

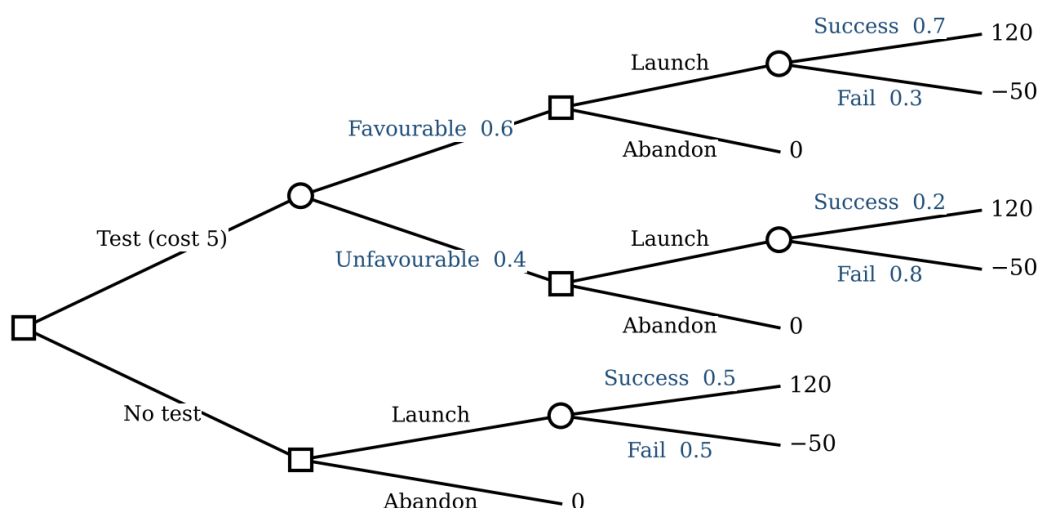
[6]

- 25 The table shows the profit, in £ thousands, for three options in three states, with probabilities 0.3, 0.4 and 0.3.

Option	S1	S2	S3
Option 1	70	30	20
Option 2	50	45	25
Option 3	30	40	60
Probability	0.3	0.4	0.3

- (a) Construct the regret table. [2]
(b) Find the expected regret of each option, and the option with the smallest expected regret. [3]
(c) Calculate the EVPI and comment on its relationship with your answer to part (b). [3]

- 26 A company is deciding whether to launch a new product. A launch makes a profit of £120 thousand if it succeeds and a loss of £50 thousand if it fails; abandoning the product gives nothing. The prior probability of success is 0.5. The company can first pay £5 thousand for a test market. The test is 'favourable' with probability 0.6, in which case the probability of success is 0.7, and 'unfavourable' otherwise, in which case the probability of success is 0.2.



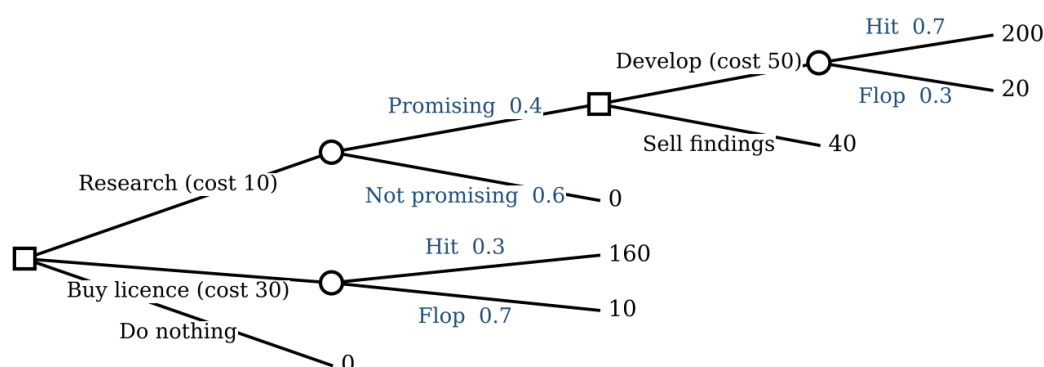
- (a) Find the EMV of the best decision if the company does not do the test. [2]
(b) Evaluate the decision tree and decide whether the test should be done, and what the company should do after each test result. [7]

- 27 A developer chooses between a large estate (profit £100 thousand if demand is high, –£40 thousand if low) and a small estate (£50 thousand if high, £20 thousand if low). The prior probability of high demand is 0.5. A survey, costing £6 thousand, gives a positive result with probability 0.9 when demand is high and with probability 0.3 when demand is low.
- Find the probability of a positive result, and the probability of high demand after a positive result and after a negative result. [4]
 - Find the best decision without the survey and its EMV. [2]
 - Find the best decision after each survey result. [4]
 - Find the expected value of the survey information (EVSI) and decide whether the £6 thousand survey is worthwhile. [3]

- 28 A decision problem has options $i = 1, \dots, n$ and states $j = 1, \dots, m$, with profit u_{ij} and probability p_j for state j .
- Show that the expected regret of option i equals the expected profit with perfect information minus the EMV of option i . [3]
 - Deduce that the least expected regret equals the EVPI and occurs for the option with the greatest EMV. [3]
 - Verify this for the profits (in £ thousands) 55, 25, 35 / 40, 45, 20 / 30, 35, 50 (options 1 to 3, states 1 to 3) with probabilities 0.2, 0.5, 0.3. [3]

- 29 A developer chooses between a large estate (profit £100 thousand if demand is high, –£40 thousand if low) and a small estate (£50 thousand if high, £20 thousand if low). The prior probability of high demand is p . A survey costing £6 thousand gives a positive result with probability 0.9 if demand is high and 0.3 if demand is low. Find the range of values of p for which commissioning the survey gives a greater expected profit than not commissioning it. Give your answer as exact fractions where possible. [8]

- 30 A firm has three options for a new product, all figures being in £ thousands. It can (i) carry out research at a cost of £10 thousand; the result is ‘promising’ with probability 0.4, and then the firm can either develop the product at a further cost of £50 thousand (a hit, probability 0.7, brings in £200 thousand, a flop brings in £20 thousand) or sell the findings for £40 thousand; if the result is not promising the project stops with no further income; (ii) buy a licence at a cost of £30 thousand, which brings in £160 thousand if it is a hit (probability 0.3) and £10 thousand otherwise; (iii) do nothing.



- Evaluate the decision tree, and state the best strategy and its expected value. [7]
- The probability h of a hit for the licence is uncertain. Find the value of h above which buying the licence is better than doing the research. [4]