

Introduction to Algorithms

1 (2 marks)

An algorithm is a finite sequence of precise, unambiguous instructions which, when followed, solves a problem (or completes a task) and always comes to an end

E1

E1

Answer: A finite, precise sequence of steps that solves a problem and terminates

2 marks

2 (2 marks)

$$t = 7 \times 7 = 49$$

M1

$$t = 49 - 7 = 42, \text{ so } 42 \text{ is printed}$$

A1

Answer: 42

2 marks

3 (3 marks)

(a) $13 > 9$, so 13 is printed

B1

$6 > 11$ is false, so step 3 prints 11

B1

Answer: 13 and 11

2 marks

(b) It prints the larger of the two input numbers

B1

Answer: Prints the larger of a and b

1 mark

4 (3 marks)

After k additions $x = 2 + 5k$, and we need $2 + 5k > 40$, so $k > 7.6$

M1

Smallest such integer is $k = 8$

A1

$$\text{Final value } x = 2 + 40 = 42$$

A1

Answer: 8 times, $x = 42$

3 marks

5 (2 marks)

$$17 = 2 \times 8 + 1, \text{ so } r = 1 \text{ and "odd" is printed}$$

B1

$$24 = 2 \times 12 + 0, \text{ so } r = 0 \text{ and "even" is printed}$$

B1

Answer: odd; even

2 marks

6 (2 marks)

n takes the values 3, 5, 7, 9, 11, ... (always odd)

M1

so n never equals 10 and step 3 always sends us back to step 2

E1

Answer: n is always odd so it never equals 10

2 marks

7 (4 marks)

$$a = 252, b = 105: 252 = 2 \times 105 + 42, \text{ so } r = 42$$

M1

$$a = 105, b = 42: 105 = 2 \times 42 + 21, \text{ so } r = 21$$

A1

$$a = 42, b = 21: 42 = 2 \times 21 + 0, \text{ so } r = 0$$

A1

$$r = 0, \text{ so } b = 21 \text{ is printed}$$

A1

Answer: 21

4 marks

8 (4 marks)

$$n = 4729: d = 9, s = 9, n \text{ becomes } 472$$

M1

$$n = 472: d = 2, s = 11, n \text{ becomes } 47$$

A1

$$n = 47: d = 7, s = 18, n \text{ becomes } 4; \text{ then } d = 4, s = 22, n \text{ becomes } 0$$

A1

$$n = 0 \text{ so } 22 \text{ is printed: the algorithm finds the sum of the digits of } n$$

B1

Answer: 22; the digit sum of n

4 marks

9 (4 marks)

First use of step 2: $x = (3 + 10 \div 3) \div 2 = 3.16667$, and $x^2 = 10.02778$, difference 0.02778, not less than 0.001

M1 A1

Second use: $x = (3.16667 + 10 \div 3.16667) \div 2 = 3.16228$, and $x^2 = 10.00002$, difference less than 0.001

M1

The value printed is 3.16228

A1

Answer: 3.16667, then 3.16228; 3.16228 is printed

4 marks

10 (3 marks)

$45 \div 2$: quotient 22, remainder 1; $22 \div 2$: 11, r 0; $11 \div 2$: 5, r 1

M1

$5 \div 2$: 2, r 1; $2 \div 2$: 1, r 0; $1 \div 2$: 0, r 1

A1

Remainders in reverse: 101101, the binary form of N

B1

Answer: 101101 (45 in binary)

3 marks

11 (3 marks)

$m = 7$; $12 > 7$ so $m = 12$

M1

$9 > 12$ is false so m stays 12; the output is 12

A1

The algorithm prints the largest of the three numbers

B1

Answer: 12; the largest of a, b, c

3 marks

12 (3 marks)

Step 1: let $c = 0$. Step 2: take each mark m in turn

B1

Step 3: if $m \geq 40$ then increase c by 1

M1

Step 4: when all marks have been used, print c

A1

Answer: Count the marks m with $m \geq 40$ and print the count

3 marks

13 (3 marks)

$x = 50, 25, 12.5, 6.25, 3.125, 1.5625, 0.78125$

M1 A1

Halved 7 times, final value 0.78125

A1

Answer: 7 times; $x = 0.78125$

3 marks

14 (4 marks)

Since 11 is odd, $n = 3 \times 11 + 1 = 34$, then 17

M1

Continuing: 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1

A1 A1

The value 1 is reached after 14 steps, so 14 is printed

A1

Answer: 14

4 marks

15 (5 marks)

(a) $n = 91$: $d = 2, 3, 4, 5, 6$ do not divide 91 and $d^2 \leq 91$ each time

M1

$d = 7$ divides 91 ($91 = 7 \times 13$), so "not prime" is printed

A1

$n = 97$: no d from 2 to 9 divides 97; when $d = 10$, $d^2 = 100 > 97$, so "prime" is printed

A1

Answer: not prime; prime

3 marks

(b) If $n = ab$ with $a \leq b$ then $a^2 \leq ab = n$, so n has a factor not exceeding $\sqrt{n}$

M1

so if no d with $d^2 \leq n$ divides n , then n has no factor other than 1 and n

E1

Answer: Any composite n has a factor $\leq \sqrt{n}$

2 marks

16 (5 marks)

The successive values of b are 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

M1 A1

c increases by 1 each time b changes, from $c = 2$ with $b = 1$

M1

$b = 89 < 100$, so the loop continues once more to $b = 144$

A1

144 and 12 are printed (the 12th Fibonacci number is the first which is at least 100)

A1

Answer: $b = 144$, $c = 12$

5 marks

17 (5 marks)

After n years the balance is 1000×1.04^n

M1

After 10 years: $1000 \times 1.04^{10} = 1480.24$, which is not greater than 1500

A1

After 11 years: $1000 \times 1.04^{11} = 1539.45$, which is greater than 1500

A1

The loop therefore stops after 11 years

B1

Values printed: 11 and £1539.45

A1

Answer: 11 years; £1539.45

5 marks

18 (5 marks)

$$y = \frac{1}{2}(5 + 3.8) = 4.4, \text{ and } |y - x| = 0.6 \quad \text{M1}$$

$$y = \frac{1}{2}(4.4 + 4.31818) = 4.35909, \text{ and } |y - x| = 0.04091 \quad \text{A1}$$

$$y = \frac{1}{2}(4.35909 + 4.35871) = 4.35890, \text{ and } |y - x| = 0.00019 < 0.001 \quad \text{A1}$$

The algorithm stops after the third value of y , printing 4.35890 B1

It approximates $\sqrt{19}$ ($= 4.3589$ to 4 d.p.) B1

Answer: 4.35890, an approximation to $\sqrt{19}$ 5 marks

19 (4 marks)

With $M = 0$ no number in the list is greater than M , since all are negative M1

so M stays 0 and 0 is printed, which is not in the list (the largest is -2) A1

The fault is the starting value of M E1

Correction: in step 1 let M be the first number in the list B1

Answer: Prints 0 instead of -2 ; start with M equal to the first number 4 marks

20 (4 marks)

(a) For $i = 1, 2, 3, 4, 5$ the inner loop runs 1, 2, 3, 4, 5 times M1

$$S = 1 + 2 + 3 + 4 + 5 = 15 \quad \text{A1}$$

Answer: 15 2 marks

(b) The inner loop runs i times for each i , so $S = 1 + 2 + \dots + n$ M1

$$S = \frac{n(n+1)}{2} \quad \text{A1}$$

Answer: $\frac{n(n+1)}{2}$ 2 marks

21 (6 marks)

(a) Write $a = qb + r$. If d divides a and b , then d divides $a - qb = r$ M1

So d divides both b and r A1

Conversely if d divides b and r , then d divides $qb + r = a$, so d divides a and b M1

The pairs (a, b) and (b, r) have the same common divisors, so the same highest common factor; at E1

the end $r = 0$ and $\gcd(b, 0) = b$

Answer: $\gcd(a, b) = \gcd(b, r)$, so the final b is the HCF 4 marks

(b) $1071 = 2 \times 462 + 147$; $462 = 3 \times 147 + 21$ M1

$147 = 7 \times 21 + 0$, so the HCF is 21 A1

Answer: 21 2 marks

22 (6 marks)

(a) T after each digit: 1, 3, 6, 13, 27, 54, 109 M1 A1

The output is 109 A1

Answer: 109 3 marks

(b) After one digit $T = d_1$; each later digit multiplies the existing T by 2 and adds the new digit M1

So d_1 is multiplied by 2 a total of $k - 1$ times, d_2 by 2 a total of $k - 2$ times, and so on M1

T is the denary value of the binary number A1

Answer: Converts a binary number to denary 3 marks

23 (7 marks)

Let B_n be the balance after n payments: $B_n = 1.015B_{n-1} - 300$, with $B_0 = 8000$ M1

Solving: $B_n = 1.015^n(8000 - 20000) + 20000 = 20000 - 12000 \times 1.015^n$ M1 A1

$B_{34} = 20000 - 12000 \times 1.015^{34} = 92.04 > 0$, so a 35th payment is needed M1 A1

B_{35} would be negative (-206.58), so there are 35 payments in total A1

The final payment is $1.015 \times B_{34} = £93.42$ A1

Answer: 35 payments; final payment £93.42 7 marks

24 (5 marks)

- (a) 1900: step 3 gives "not leap"; 2000: step 2 gives "leap" B1
 2024: step 4 gives "leap"; 2026: not divisible by 4 so "not leap" B1
Answer: not leap, leap, leap, not leap 2 marks
 (b) Multiples of 4 in the range: from 1992 to 2096 M1
 Number of these = $(2096 - 1992) \div 4 + 1 = 27$ M1
 No year in the range is a multiple of 100 except 2000, which is a multiple of 400, so none is excluded: 27 A1
Answer: 27 3 marks

25 (7 marks)

- (a) n: 37, 36, 18, 9, 8, 4, 2, 1, 0 M1
 That is 8 steps, so 8 is printed A1
Answer: 8 2 marks
 (b) Every step reduces n by at least 1 ($n - 1$, or $n \div 2$ which is smaller for $n > 0$) M1
 n is a non-negative integer, so it must reach 0 E1
Answer: n strictly decreases through non-negative integers 2 marks
 (c) Halving removes one binary digit from the right; subtracting 1 turns a final 1 into 0 M1
 Halvings needed: 9 (to reach 1), plus one subtraction for each digit 1 (there are 6) M1
 $9 + 6 = 15$ A1
Answer: 15 3 marks

26 (6 marks)

- (a) Step 1: let $R = N$ and $k = 1$ B1
 Step 2: replace R by $R - k$ and k by $k + 2$; if $R > 0$ repeat this step M1
 Step 3: if $R = 0$ print "square", otherwise print "not square" A1
Answer: Subtract 1, 3, 5, ... until $R \leq 0$; zero means square 3 marks
 (b) $N = 144$: after 12 subtractions of 1, 3, ..., 23 the total subtracted is 144, so $R = 0$: "square" M1 A1
 $N = 150$: after 12 subtractions $R = 6$; subtracting 25 gives $R = -19 < 0$: "not square" A1
Answer: 144 square; 150 not square 3 marks

27 (9 marks)

- (a) Removed in order: 2, 4, 6, 8, 10, 3, 7, 1, 9 M1 A1
 The remaining person is number 5 A1
Answer: Order 2, 4, 6, 8, 10, 3, 7, 1, 9; survivor 5 3 marks
 (b) In the first round every even-numbered person is removed, leaving 1, 3, 5, ..., which is again a power of 2 in number, and the next removal starts after person 1's turn M1
 So the odd-numbered people remain, with person 1 first again, and the argument repeats M1
 Each round halves the number of people and person 1 is never removed, so 1 remains when one person is left E1
Answer: Person 1 survives every round 3 marks
 (c) After removing $100 - 64 = 36$ people, 64 remain M1
 The 36th removal is person 72, so the next person in the circle, 73, is first in a group of 64 M1
 By part (b) the first of 64 people survives: person 73 A1
Answer: 73 3 marks

28 (9 marks)

- (a) 12, 12 leaves 5, then 5: coins 12, 12, 5 M1
 Total 29 using 3 coins A1
Answer: 12, 12, 5 (3 coins) 2 marks
 (b) For 15: greedy takes 12, then 1, 1, 1 (4 coins) M1
 but $5 + 5 + 5$ uses only 3 coins A1
 Checking every amount from 1 to 14, the greedy method always uses the fewest coins M1
 So the smallest such amount is 15 A1
Answer: 15 4 marks
 (c) Greedy: $99 = 8 \times 12 + 3$, so $8 + 3 = 11$ coins M1
 Better: $99 = 7 \times 12 + 3 \times 5$, since $84 + 15 = 99$, giving 10 coins M1

A check of the possibilities shows that 9 coins cannot make 99, so the minimum is 10

Answer: Greedy 11 coins; minimum 10 coins

A1

3 marks

29 (7 marks)

(a) $(a, b) = (37, 46), (18, 92), (9, 184), (4, 368), (2, 736), (1, 1472)$

M1

Adding b when a is odd: $46 + 184 + 1472$

M1

$P = 1702$ (which is 37×46)

A1

Answer: 1702

3 marks

(b) Write $a = \sum e_i 2^i$ with each $e_i = 0$ or 1 ; then $ab = \sum e_i (2^i b)$

M1

Halving with the integer part makes the remainders 0 or 1 the binary digits $e_0, e_1, \dots$, from the right

M1

At step i the value of b is $2^i b$, and a is odd exactly when $e_i = 1$

M1

So $P = \sum e_i 2^i b = ab$

E1

Answer: $P = \sum e_i 2^i b = ab$

4 marks

30 (8 marks)

(a) $2024 = 1 \times 1394 + 630$; $1394 = 2 \times 630 + 134$; $630 = 4 \times 134 + 94$

M1

$134 = 1 \times 94 + 40$; $94 = 2 \times 40 + 14$; $40 = 2 \times 14 + 12$

A1

$14 = 1 \times 12 + 2$; $12 = 6 \times 2 + 0$, so the HCF is 2

A1

Answer: 2

3 marks

(b) $2 = 14 - 12 = 14 - (40 - 2 \times 14) = 3 \times 14 - 40$

M1

$= 3(94 - 2 \times 40) - 40 = 3 \times 94 - 7 \times 40$

A1

$= 3 \times 94 - 7(134 - 94) = 10 \times 94 - 7 \times 134$

M1

$= 10(630 - 4 \times 134) - 7 \times 134 = 10 \times 630 - 47 \times 134$

A1

$= 10 \times 630 - 47(1394 - 2 \times 630) = 104 \times 630 - 47 \times 1394 = 104 \times 2024 - 151 \times 1394$, so $x =$

A1

104 and $y = -151$

Answer: $x = 104, y = -151$

5 marks

Flowcharts and Pseudocode

1 (3 marks)

A (diamond) is a decision: a question with a Yes or No outcome

B1

B (parallelogram) is for input or output of data

B1

C (rounded box) marks the start or the end (stop) of the algorithm

B1

Answer: A decision; B input or output; C start or stop

3 marks

2 (2 marks)

$12 > 10$ is true, so "big" is printed

B1

$10 > 10$ is false, so "small" is printed

B1

Answer: big; small

2 marks

3 (2 marks)

$c = 3 \times 4 = 12$

M1

Output $= 12 + 5 = 17$

A1

Answer: 17

2 marks

4 (2 marks)

$n = 1$ is printed, then n becomes 2, and $2 > 4$ is false, so the loop repeats

M1

The values printed are 1, 2, 3, 4 (after printing 4, n becomes 5 and $5 > 4$ is true)

A1

Answer: 1, 2, 3, 4

2 marks

5 (2 marks)

total takes the values 2, 6, 12, 20 as $i = 1, 2, 3, 4$

M1

Output $= 20$

A1

Answer: 20

2 marks

6 (2 marks)

mark = 50: $50 \geq 50$ is true, so PASS is output

B1

mark = 49: $49 \geq 50$ is false, so FAIL is output

B1

Answer: PASS; FAIL

2 marks

7 (3 marks)

Start with $S = 0$, $k = 1$

B1

The table shows the values of S and k after each pass through the loop

M1

The value printed is 15

A1

Answer: 15

3 marks

Pass	S	k (after $k \leftarrow k + 1$)
1	1	2
2	3	3
3	6	4
4	10	5
5	15	6

8 (3 marks)

$x = 3$: $3 < 50$, so $x = 5$; then 9, 17, 33

M1

$x = 33$: $33 < 50$, so $x = 65$, and $65 < 50$ is false

A1

Output 65; the loop is carried out 5 times

A1

Answer: 65; 5 times

3 marks

9 (5 marks)

(a) INPUT a , b

B1

WHILE $b \neq 0$, then $r \leftarrow a \bmod b$, $a \leftarrow b$, $b \leftarrow r$ (in that order) inside the loop

M1

ENDWHILE then OUTPUT a

A1

Answer: INPUT a , b ; WHILE $b \neq 0$: $r \leftarrow a \bmod b$, $a \leftarrow b$, $b \leftarrow r$; OUTPUT a

3 marks

(b) $(48, 18) \rightarrow r = 12$; $(18, 12) \rightarrow r = 6$; $(12, 6) \rightarrow r = 0$

M1

$b = 0$ so $a = 6$ is printed

A1

Answer: 6

2 marks

10 (4 marks)

(a) (i) 2

B1

(ii) $n + 2$

B1

(iii) $n > 20$

B1

Answer: (i) 2, (ii) $n + 2$, (iii) $n > 20$

3 marks

(b) $2 + 4 + \dots + 20 = \frac{10 \times 22}{2} = 110$

B1

Answer: 110

1 mark

11 (3 marks)

$n = 1, 2$: not multiples of 3, so $n^2 = 1, 4$

M1

$n = 3$: Z; $n = 4, 5$: 16, 25; $n = 6$: Z; $n = 7$: 49

M1

Output: 1, 4, Z, 16, 25, Z, 49

A1

Answer: 1, 4, Z, 16, 25, Z, 49

3 marks

12 (4 marks)

Initially $a = 1$, $s = 0$

B1

$a = 1$: $s = 1$, $a = 3$; $a = 3$: $s = 4$, $a = 9$

M1

$a = 9$: $s = 13$, $a = 27$; $a = 27$: $s = 40$, $a = 81$

A1

$81 < 30$ is false, so the output is $s = 40$ and $a = 81$

A1

Answer: $s = 40$, $a = 81$

4 marks

Pass	s	a
start	0	1
1	1	3
2	4	9
3	13	27
4	40	81

13 (4 marks)

Start, then Input x (parallelogram)

B1

A decision $x < 0$? with the Yes branch printing "negative"

M1

On the No branch a second decision $x = 0$? giving "zero" for Yes

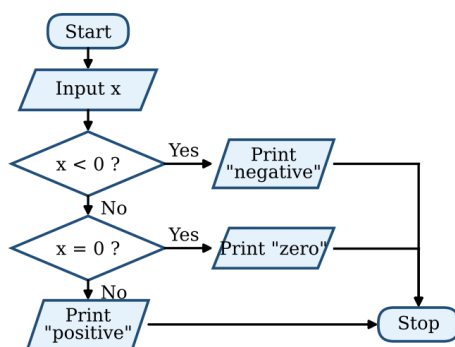
M1

"positive" for No, with all branches ending at Stop

A1

Answer: $x < 0$? Yes: print negative. No: $x = 0$? Yes: print zero, No: print positive

4 marks

**14 (5 marks)**

(a) Pass 1: $r = 0 \times 10 + 5 = 5$, $n = 127$; pass 2: $r = 57$, $n = 12$

M1

Pass 3: $r = 572$, $n = 1$; pass 4: $r = 5721$, $n = 0$

A1

Output 5721

A1

Answer: 5721

3 marks

(b) It prints the number with the digits of n in reverse order

B1

$n = 4730$: $r = 0, 3, 37, 374$, so the output is 374 (the leading zero is lost)

B1

Answer: Reverses the digits; 374

2 marks

15 (4 marks)

(a) $n = 29$: d runs 2, 3, 4, 5, 6; no value divides 29, and at $d = 6$, $36 > 29$, so "prime"

M1 A1

$n = 91$: $d = 2, \dots, 6$ do not divide 91; $d = 7$ does, so the loop stops and 7 is output

A1

Answer: prime; 7

3 marks

(b) d is the smallest factor of n greater than 1, so n is not prime

B1

Answer: The smallest prime factor of n

1 mark

16 (4 marks)

$i = 1$: $1 \times 1 = 1$

B1

$i = 2$: $2 \times 1 + 2 \times 2 = 6$

M1

$i = 3$: $3 + 6 + 9 = 18$; $i = 4$: $4 + 8 + 12 + 16 = 40$

A1

$S = 1 + 6 + 18 + 40 = 65$

A1

Answer: 65

4 marks

17 (4 marks)

(a) After 4: $c = 1$, $T = 4$; after 7: $c = 2$, $T = 11$

M1

After -3: $c = 3$, $T = 8$; after 8: $c = 4$, $T = 16$

A1

Then 0 stops the loop, and 4 and 16 are printed

A1

Answer: $c = 4$, $T = 16$

3 marks

(b) Entering 0 is the signal to stop, so a genuine 0 would end the input early

E1

Answer: 0 is the stopping signal**1 mark****18 (3 marks)**The loop adds $1 + 2 + 3 + 4 = 10$, because i only goes up to $n - 1 = 4$ **M1**The correct sum is $1 + 2 + 3 + 4 + 5 = 15$ **A1**Change the loop to FOR i FROM 1 TO n **B1****Answer: Output 10 instead of 15; use FOR i FROM 1 TO n** **3 marks****19 (4 marks)**

(a) 7, 3 swap; 7, 8 stay; 8, 2 swap; 8, 5 swap; 8, 1 swap

M1 $L = [3, 7, 2, 5, 1, 8]$ **A1****Answer: [3, 7, 2, 5, 1, 8]****2 marks**

(b) 4 swaps (every comparison except 7 with 8)

M1

The largest item, 8, has been carried to the end of the list, its correct final position

E1**Answer: 4 swaps; 8 is now last****2 marks****20 (4 marks)**Each pass increases n by 1 and adds n^2 to T **M1** $T = 1, 5, 14, 30, 55, 91$ (all ≤ 100 , so No each time)**A1** $n = 7: T = 91 + 49 = 140 > 100$ **A1**Output $n = 7, T = 140$ **A1****Answer: $n = 7, T = 140$** **4 marks**

n	T	T > 100 ?
1	1	No
2	5	No
3	14	No
4	30	No
5	55	No
6	91	No
7	140	Yes

21 (6 marks)(a) Start $m = 12, p = 1$; 7 is not larger**B1** $i = 3: m = 19, p = 3$; 4 and the second 19 are not larger than 19**M1** $i = 6: m = 25, p = 6$; 8 is not larger; output 25, 6**A1****Answer: $m = 25, p = 6$** **3 marks**(b) Before: $m = 9, p = 2$ (the second 9 is not larger than m)**M1**After: the second 9 replaces the first, so $m = 9, p = 3$ **A1**The change makes p the position of the last occurrence of the maximum rather than the first**E1****Answer: (9, 2) before; (9, 3) after; last rather than first occurrence****3 marks****22 (7 marks)**(a) INPUT a, b **B1**WHILE $a \neq b$: IF $a > b$ THEN $a \leftarrow a - b$ ELSE $b \leftarrow b - a$ ENDIF**M1 A1****Answer: INPUT a, b ; WHILE $a \neq b$: IF $a > b$ THEN $a \leftarrow a - b$ ELSE $b \leftarrow b - a$; OUTPUT a** **3 marks**(b) $(84, 36) \rightarrow (48, 36) \rightarrow (12, 36) \rightarrow (12, 24) \rightarrow (12, 12)$ **M1** $a = b = 12$, so 12 is printed**A1****Answer: 12****2 marks**(c) Each pass reduces a or b by a positive integer while both stay positive, so $a + b$ strictly decreases; they must become equal**E1**It calculates the highest common factor of a and b , since $\gcd(a, b) = \gcd(a - b, b)$ **E1****Answer: Terminates because $a + b$ decreases; finds $\gcd(a, b)$** **2 marks**

23 (7 marks)Set $n \leftarrow 1$ and $f \leftarrow 1$

B1

WHILE $f \times (n + 1) < 1\,000\,000$: $n \leftarrow n + 1$ and $f \leftarrow f \times n$

M1 A1

OUTPUT n, f

B1

The values of f are 1, 2, 6, 24, 120, 720, 5040, 40320, 362880 for $n = 1, \dots, 9$

M1

Next value would be $362880 \times 10 = 3\,628\,800$ which is at least 1 000 000, so the loop stops

M1

Output $n = 9$ and $f = 362\,880$

A1

Answer: $n = 9, 9! = 362\,880$

7 marks

24 (6 marks)(a) n : 6, 3, 10, 5, 16, 8, 4, 2, 1

M1

 c increases by 1 each pass, reaching 8 when $n = 1$

A1

 m is updated whenever n exceeds it: 6, 10, 16

M1

Output $c = 8, m = 16$

A1

Answer: $c = 8, m = 16$

4 marks

(b) n : 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 (16 steps)

M1

Largest value reached 52: output $c = 16, m = 52$

A1

Answer: $c = 16, m = 52$

2 marks

n	c	m
6	0	6
3	1	6
10	2	10
5	3	10
16	4	16
8	5	16
4	6	16
2	7	16
1	8	16

25 (6 marks)(a) $m = 7$ and $7 < 50$ is true, so the Yes branch is taken and the algorithm stops without printing anything

M1

The Yes and No branches of the decision are the wrong way round

E1

Answer: Nothing is printed; the Yes and No branches are reversed

2 marks

(b) Change the decision to $m \geq 50$? (or swap the Yes and No branches)

M1

Then 7, 14, ..., 49 are printed and $m = 56$ ends the loop

A1

Answer: Use $m \geq 50$? with Yes leading to Stop

2 marks

(c) Numbers 7×1 to 7×7 : 7 numbers

B1

Sum = $7(1 + 2 + \dots + 7) = 7 \times 28 = 196$

B1

Answer: 7 numbers; sum 196

2 marks

26 (7 marks)(a) $n = 13$ (odd): $r = 3, a = 9, n = 6$

M1

 $n = 6$ (even): $r = 3, a = 81, n = 3; n = 3$ (odd): $r = 243, a = 6561, n = 1$

A1

 $n = 1$ (odd): $r = 243 \times 6561 = 1\,594\,323$, then $n = 0$

A1

Output 1 594 323 ($= 3^{13}$)

A1

Answer: 1 594 323

4 marks

(b) There are 10 passes, each with one multiplication $a \times a$

M1

There are 6 digits equal to 1 in 1111101000, so 6 multiplications $r \times a$

M1

Total 16 multiplications, compared with 999

A1

Answer: 16 multiplications

3 marks

27 (8 marks)**(a)** The inner loop runs 9, 8, ..., 1 times

M1

$$9 + 8 + \dots + 1 = 45$$

A1

Answer: 45

2 marks

(b) Pairs $i < j$ with $i + j$ divisible by 3 are pairs with remainders (0, 0) or (1, 2)

M1

Remainder 0: 3, 6, 9 (3 numbers); remainder 1: 1, 4, 7, 10 (4 numbers); remainder 2: 2, 5, 8 (3 numbers)

A1

Pairs: from remainder 0, $3 \times 2 \div 2 = 3$; from remainders 1 and 2, $4 \times 3 = 12$

M1

Total 15, so 15 is output

A1

Answer: 15

4 marks

(c) Remainders 0, 1, 2, 3, 4 occur for 2 numbers each; pairs (0, 0) give 1, pairs (1, 4) give 4, pairs (2, 3) give 4

M1

Total 9

A1

Answer: 9

2 marks

28 (8 marks)**(a)** $p = 2$: 4, 6, 8, ..., 30 (all even numbers from 4)

B1

 $p = 3$: 9, 15, 21, 27 are new (12, 18, 24, 30 already marked); $p = 4$ is already marked "not prime" so nothing is done

M1

 $p = 5$: 25 is new (the others, 30 only, are already marked)

A1

Answer: 2: evens from 4; 3: 9, 15, 21, 27; 4: skipped; 5: 25

3 marks

(b) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

B1

Answer: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

1 mark

(c) A composite number $n \leq 30$ has a factor p with $p^2 \leq n \leq 30$, so $p \leq 5$

M1

So every composite number up to 30 is marked by the time $p = 5$ has been dealt with

E1

A multiple of p of the form kp with $k < p$ has a smaller factor k , so it was already marked in an earlier round

M1

Hence the first multiple of p that can be new is $p \times p$

E1

Answer: Composites have a factor $\leq \sqrt{30}$; smaller multiples are already marked

4 marks

29 (8 marks)**(a)** Comparisons: 2 and 3 (take 2); 5 and 3 (3); 5 and 4 (4); 5 and 10 (5); 9 and 10 (9); 12 and 10 (10); 12 and 15 (12)

M1 A1

A is now used up after 7 comparisons, so 7 comparisons were made

A1

The remaining items 15, 18 are copied: $C = [2, 3, 4, 5, 9, 10, 12, 15, 18]$

A1

Answer: $C = [2, 3, 4, 5, 9, 10, 12, 15, 18]$; 7 comparisons

4 marks

(b) Each comparison places one item into C , and the last item is placed without a comparison (when one list is used up)

M1

So at most $m + n - 1$ comparisons are made

E1

Answer: Each comparison places one item and the last needs none

2 marks

(c) Smallest is the shorter list's length, $\min(m, n)$, when all items of the shorter list come first

M1

e.g. $A = [1, 2, 3]$, $B = [4, 5, 6, 7]$ needs 3 comparisons

A1

Answer: $\min(m, n)$

2 marks

30 (9 marks)**(a)** $k = \lceil 6 \div 5 \rceil = 2$, output $1/2$; $p = 5 \times 2 - 6 = 4$, $q = 12$

M1

 $k = \lceil 12 \div 4 \rceil = 3$, output $1/3$; $p = 4 \times 3 - 12 = 0$, so the loop stops

A1

Output $1/2$ and $1/3$ (which sum to $5/6$)

A1

Answer: $1/2, 1/3$

3 marks

(b) $k = 4$: output $1/4$; $p = 4 \times 4 - 13 = 3$, $q = 52$; $k = \lceil 52 \div 3 \rceil = 18$: output $1/18$

M1 A1

 $p = 3 \times 18 - 52 = 2$, $q = 936$; $k = 468$: output $1/468$; $p = 2 \times 468 - 936 = 0$

A1

$$\text{Check: } \frac{1}{4} + \frac{1}{18} + \frac{1}{468} = \frac{(117 + 26 + 1)}{468} = \frac{144}{468} = \frac{4}{13}$$

Answer: $1/4, 1/18, 1/468$

3 marks

(c) Since $k = \lceil q \div p \rceil$, we have $\frac{q}{p} \leq k < \frac{q}{p} + 1$

M1

Multiplying by p gives $q \leq pk < q + p$, so $0 \leq pk - q < p$

A1

So p is a strictly decreasing sequence of non-negative integers, which must reach 0 in a finite number of steps

E1

Answer: $0 \leq p_k - q < p$, so p strictly decreases to 0

3 marks

Sorting Algorithms

1 (2 marks)

Comparisons: 4, 9 stay; 9, 2 swap; 9, 7 swap; 9, 5 swap

M1

List after the first pass: 4, 2, 7, 5, 9

A1

Answer: 4, 2, 7, 5, 9

2 marks

2 (2 marks)

The first pass compares neighbouring pairs: $8 - 1 = 7$ comparisons

B1

At most 7 passes involve swaps ($n - 1$ passes)

B1

Answer: 7 comparisons; 7 passes

2 marks

3 (3 marks)

Compare the first two items and swap them if the left one is larger; then compare the second and third items, and so on to the end of the list

M1

Each comparison moves the larger item of the pair one place to the right, so the largest item is carried along to the end

A1

The largest item is in its final position at the right-hand end

B1

Answer: Compare neighbouring items in turn, swapping if out of order; the largest item ends in its final position

4 (2 marks)

The middle item is 15, so 15 is the pivot

B1

Items smaller than 15 stay in order to its left: 12, 7, 3, 9, 15

B1

Answer: Pivot 15; list 12, 7, 3, 9, 15

2 marks

5 (2 marks)

Second pass: 4, 3 swap; 4, 6 stay; 6, 2 swap; 6, 8 stay: the list is 3, 4, 2, 6, 8, 9

M1

2 is out of place, so a third pass is needed

A1

Answer: 3, 4, 2, 6, 8, 9; yes

2 marks

6 (2 marks)

One pass of 5 comparisons is made, to check the order

B1

No swaps are made, so the algorithm stops after this pass: 5 comparisons, 0 swaps

B1

Answer: 5 comparisons; 0 swaps

2 marks

7 (4 marks)

Pass 1: 3, 7, 4, 1, 6, 9

M1

Pass 2: 3, 4, 1, 6, 7, 9

A1

Pass 3: 3, 1, 4, 6, 7, 9

Pass 4: 1, 3, 4, 6, 7, 9

A1

Pass 5: 1, 3, 4, 6, 7, 9

A1

The final pass has no swaps, so the list is sorted.

Answer: 1, 3, 4, 6, 7, 9

4 marks

8 (5 marks)

Pass 1 (pivot 21): 14, 6, 9, 3, 17, 10, 21

M1

Pass 2 (pivot 3): 3, 14, 6, 9, 17, 10, 21

A1

Pass 3 (pivot 9): 3, 6, 9, 14, 17, 10, 21

A1

Pass 4 (pivot 17): 3, 6, 9, 14, 10, 17, 21

A1

Pass 5 (pivot 10): 3, 6, 9, 10, 14, 17, 21

A1

Answer: 3, 6, 9, 10, 14, 17, 21

5 marks

9 (4 marks)

After the 2nd item: 2, 5, 8, 1, 6

M1

After the 3rd item: 2, 5, 8, 1, 6

A1

After the 4th item: 1, 2, 5, 8, 6

A1

After the 5th item: 1, 2, 5, 6, 8

A1

Answer: 1, 2, 5, 6, 8

4 marks

10 (4 marks)

Pass 1: 9, 12, 7, 15, 4, 10, 6, 3

M1 A1

Pass 2: 12, 9, 15, 7, 10, 6, 4, 3

A1 A1

Answer: After pass 2: 12, 9, 15, 7, 10, 6, 4, 3

4 marks

11 (3 marks)Comparisons: $7 - 1 = 6$

B1

Swaps: 8, 3; 8, 6; 8, 1; 9, 4; 9, 7 (8, 9 does not swap): 5 swaps

M1

List: 3, 6, 1, 8, 4, 7, 9

A1

Answer: List 3, 6, 1, 8, 4, 7, 9; 6 comparisons; 5 swaps

3 marks

12 (4 marks)

Pass 1: 2, 6, 4, 7, 9, 11

M1

Pass 2: 2, 4, 6, 7, 9, 11

A1

Pass 3: 2, 4, 6, 7, 9, 11

A1

The third pass has no swaps, so 3 passes in total

B1

Answer: 3 passes

4 marks

13 (4 marks)Pass 1: Ben, Mia, Alf, Eve, **Zoe**

M1

Pass 2: Ben, Alf, Eve, **Mia, Zoe**

A1

Pass 3: Alf, Ben, **Eve, Mia, Zoe**

A1

Pass 4: **Alf, Ben, Eve, Mia, Zoe**

A1

Answer: Alf, Ben, Eve, Mia, Zoe

4 marks

14 (6 marks)Pass 1 (pivot 6): 23, 9, 31, 14, 27, 19, 35, 12, 21, **6**

M1 A1

Pass 2 (pivot 27): 31, 35, **27**, 23, 9, 14, 19, 12, 21, **6**

A1

Pass 3 (pivots 35, 19): **35**, 31, **27**, 23, 21, **19**, 9, 14, 12, **6**

A1

Pass 4 (pivots 21, 14): **35**, **31**, **27**, 23, **21**, **19**, **14**, 9, 12, **6**

A1

Pass 5 (pivot 12): **35**, **31**, **27**, 23, **21**, **19**, **14**, **12**, 9, **6**

A1

Answer: 35, 31, 27, 23, 21, 19, 14, 12, 9, 6

6 marks

15 (6 marks)Pass 1: 12, 8, 5, 15, 9, 3, 11, **1** (7 comparisons, 5 swaps)

M1

Pass 2: 12, 8, 15, 9, 5, 11, **3**, **1** (6 comparisons, 3 swaps)Pass 3: 12, 15, 9, 8, 11, **5**, **3**, **1** (5 comparisons, 3 swaps)

A1

Pass 4: 15, 12, 9, 11, **8**, **5**, **3**, **1** (4 comparisons, 2 swaps)

A1

Pass 5: 15, 12, 11, **9**, **8**, **5**, **3**, **1** (3 comparisons, 1 swaps)Pass 6: **15**, **12**, **11**, **9**, **8**, **5**, **3**, **1** (2 comparisons, 0 swaps)

A1

Totals: 27 comparisons and 14 swaps

B1 B1

Answer: 15, 12, 11, 9, 8, 5, 3, 1; 27 comparisons, 14 swaps

6 marks

16 (6 marks)Pass 1 (pivot 40): 31, 17, 25, 8, 12, 29, 6, 21, **40**

M1

Pass 2 (pivot 12): 8, 6, **12**, 31, 17, 25, 29, 21, **40**

A1

Pass 3 (pivots 6, 25): **6**, 8, **12**, 17, 21, **25**, 31, 29, **40**

A1

Pass 4 (pivots 21, 29): **6**, **8**, **12**, **17**, **21**, **25**, **29**, **31**, **40**

A1

4 passes were used

B1

Pivots: 40, 12, 6, 25, 21, 29

B1

Answer: 6, 8, 12, 17, 21, 25, 29, 31, 40

6 marks

17 (4 marks)

Comparisons in total $\leq (n - 1) + (n - 2) + \dots + 2 + 1$

M1

This is an arithmetic series with $n - 1$ terms: $\text{sum} = \frac{(n - 1)}{2}(\text{first} + \text{last}) = \frac{(n - 1)}{2} \times n = \frac{n(n - 1)}{2}$

M1 A1

$$n = 12: \frac{12 \times 11}{2} = 66$$

A1

Answer: $\frac{n(n - 1)}{2}$; 66

4 marks

18 (5 marks)

After swap 1: 8, 14, 36, 29, 21, 17

M1

After swap 2: 8, 14, 36, 29, 21, 17

After swap 3: 8, 14, 17, 29, 21, 36

A1

After swap 4: 8, 14, 17, 21, 29, 36

After swap 5: 8, 14, 17, 21, 29, 36

A1

Finding the smallest of 6, 5, 4, 3, 2 items takes $5 + 4 + 3 + 2 + 1 = 15$ comparisons

M1 A1

Answer: 8, 14, 17, 21, 29, 36; 15 comparisons

5 marks

19 (5 marks)

Split: [13, 5, 22, 9] and [17, 3, 11, 8], then [13, 5], [22, 9], [17, 3], [11, 8]

B1

Merge pairs: [5, 13], [9, 22], [3, 17], [8, 11]

M1 A1

Merge again: [5, 9, 13, 22] and [3, 8, 11, 17]

A1

Final merge: [3, 5, 8, 9, 11, 13, 17, 22]

A1

Answer: 3, 5, 8, 9, 11, 13, 17, 22

5 marks

20 (5 marks)

List A, pass 1: 2, 3, 4, 5, 1, 6

M1

List A, pass 2: 2, 3, 4, 1, 5, 6

List A, pass 3: 2, 3, 1, 4, 5, 6

List A, pass 4: 2, 1, 3, 4, 5, 6

List A, pass 5: 1, 2, 3, 4, 5, 6

A1

List B, pass 1: 1, 2, 3, 4, 5, 6; pass 2: no swaps

B1

A needs 5 passes (the maximum for 6 items) and B only 2

B1

In each pass an item can move only one place to the left, so the 1 in A needs five passes to reach the front, whereas the large 6 in B is carried to the end in a single pass

E1

Answer: A: 5 passes; B: 2 passes

5 marks

21 (6 marks)

(a) Take 2, 1, 3, 4, 5, 6: pass 1 swaps 2 and 1, giving 1, 2, 3, 4, 5, 6

M1

Pass 2 makes no swaps, so there are 2 passes

A1

Answer: e.g. 2, 1, 3, 4, 5, 6

2 marks

(b) Take 2, 3, 4, 5, 6, 1: the 1 moves only one place left per pass, so five passes are needed

M1 A1

After k passes the k largest items are in their final places, so after 5 passes all 6 items are in place

E1

Answer: e.g. 2, 3, 4, 5, 6, 1; at most 5 passes

3 marks

(c) Each of the 5 passes moves the 1 one place to the left: 5 swaps

B1

Answer: 5

1 mark

22 (6 marks)

(a) A swap exchanges two neighbouring items which are the wrong way round

M1

This corrects that pair and does not change the order of any other pair, so the number of inversions falls by exactly 1

E1

The sorted list has none, so the total number of swaps equals the initial number of inversions

E1

Answer: Total swaps = number of inversions

3 marks

(b) Inversions with 5 on the left: (5, 3), (5, 1), (5, 2)

M1

With 3 on the left: (3, 1), (3, 2); with 8: (8, 1), (8, 2); with 9: (9, 2)

M1

Total $3 + 2 + 2 + 1 = 8$ inversions, so 8 swaps

A1

Answer: 8

3 marks

23 (7 marks)

- (a) Pass 1: pivot 1; nothing is smaller, so the sublist 2, ..., 7 remains M1
 Pass 2: pivot 2, sublist 3, ..., 7; pass 3: pivot 3, sublist 4, ..., 7 A1
 Each pass fixes only one item and leaves a sublist just one item shorter M1
 6 passes are needed in total (pivots 1, 2, 3, 4, 5, 6) A1
Answer: 6 passes 4 marks
 (b) Pass 1: pivot 4, giving sublists 1, 2, 3 and 5, 6, 7 M1
 Pass 2: pivots 2 and 6, leaving only single items A1
 So 2 passes are needed A1
Answer: 2 passes 3 marks

24 (7 marks)

- Split: [17, 4, 29, 11, 36] and [8, 23, 14, 2, 31], and so on down to single items M1
 merge [17] and [4]: 1 comparison M1
 merge [4, 17] and [29]: 2 comparisons
 merge [11] and [36]: 1 comparison
 merge [4, 17, 29] and [11, 36]: 4 comparisons A1
 merge [8] and [23]: 1 comparison
 merge [8, 23] and [14]: 2 comparisons A1
 merge [2] and [31]: 1 comparison
 merge [8, 14, 23] and [2, 31]: 4 comparisons
 merge [4, 11, 17, 29, 36] and [2, 8, 14, 23, 31]: 9 comparisons A1
 Total number of comparisons = $1 + 2 + 1 + 4 + 1 + 2 + 1 + 4 + 9 = 25$ A1
 Sorted list: 2, 4, 8, 11, 14, 17, 23, 29, 31, 36 A1
Answer: 2, 4, 8, 11, 14, 17, 23, 29, 31, 36; 25 comparisons 7 marks

25 (7 marks)

- (a) Pass 1: 22 is larger than the pivot 13 but has been placed to its left; the items smaller than 13 are 7 and 9 only M1 A1
 Pass 2: for the right-hand sublist 18, 22, 25, 16 (four items) the pivot is the right-hand middle item, 25, not 18 M1 A1
Answer: 22 on wrong side of pivot; pivot of right-hand sublist should be 25 4 marks
 (b) Pass 1 (pivot 13): 7, 9, **13**, 18, 22, 25, 16 M1
 Pass 2 (pivots 9, 25): 7, **9**, **13**, 18, 22, 16, 25
 Pass 3 (pivot 22): 7, **9**, **13**, 18, 16, **22**, **25** A1
 Pass 4 (pivot 16): 7, **9**, **13**, **16**, **18**, **22**, **25** A1
Answer: 7, 9, 13, 16, 18, 22, 25 3 marks

26 (7 marks)

- Pass 1: 3, 7, 1, 8, 2, **9** (5 comparisons, 5 swaps) M1
 Pass 2: 3, 1, 7, 2, **8**, **9** (4 comparisons, 2 swaps)
 Pass 3: 1, 3, 2, 7, **8**, **9** (3 comparisons, 2 swaps) A1
 Pass 4: 1, 2, 3, 7, **8**, **9** (2 comparisons, 1 swaps)
 Pass 5: **1**, **2**, **3**, 7, **8**, **9** (1 comparisons, 0 swaps) A1
 Method 1 total: 15 comparisons, 10 swaps B1
 After item 2: 3, 9, 7, 1, 8, 2 M1
 After item 3: 3, 7, 9, 1, 8, 2
 After item 4: 1, 3, 7, 9, 8, 2
 After item 5: 1, 3, 7, 8, 9, 2
 After item 6: 1, 2, 3, 7, 8, 9 A1
 Method 2 total: 13 comparisons (each item is compared until the item to its left is smaller) A1
Answer: Bubble: 15 comparisons; shuttle: 13 comparisons 7 marks

27 (8 marks)

(a) Let M be the largest item. Once M is compared with its right-hand neighbour it is swapped (it is larger), so M moves one place right M1
 It continues to be the larger item in each later comparison, so it is carried to the end of the list M1
 If M was already to the right of earlier items, it is not moved until it is reached, and the same argument applies E1

Answer: The largest item is carried to the end 3 marks

(b) For $k = 1$ this is part (a) B1

Assume it is true after k passes. Pass $k + 1$ deals with the remaining items, and by the argument of M1

(a) it carries the largest of them to the right-hand end of those items

That is the $(k + 1)$ th largest item, in its final place, so the result follows by induction E1

Answer: After k passes the k largest items are in their final positions 3 marks

(c) After $n - 1$ passes the $n - 1$ largest items are correctly placed, so the last item must be too: at most $n - 1$ passes involve swaps M1

If the list becomes sorted earlier, one more pass with no swaps is the only way to detect this; after $n - 1$ passes the sort is known to be complete E1

Answer: $n - 1$ passes 2 marks

28 (9 marks)

(a) $M(1) = 0$ (nothing to compare), $M(2) = 1$ B1

$M(3) = M(2) + M(1) + 2 = 3$, $M(4) = M(2) + M(2) + 3 = 5$ B1

Answer: 0, 1, 3, 5 2 marks

(b) The two parts need at most $M(\lceil n/2 \rceil)$ and $M(\lfloor n/2 \rfloor)$ comparisons, and the final merge at most $n - 1$ M1

$M(8) = 2M(4) + 7 = 17$ A1

$M(5) = M(3) + M(2) + 4 = 8$, so $M(10) = 2M(5) + 9 = 25$ A1

Answer: $M(8) = 17$, $M(10) = 25$ 3 marks

(c) $k = 1$: $M(2) = 1 = 0 \times 2 + 1$, true B1

Assume $M(2^k) = (k - 1)2^k + 1$. Then $M(2^{k+1}) = 2M(2^k) + 2^{k+1} - 1$ M1

$= 2(k - 1)2^k + 2 + 2^{k+1} - 1 = (k - 1)2^{k+1} + 2^{k+1} + 1$ M1

$= k \times 2^{k+1} + 1$, which is the formula for $k + 1$; true by induction A1

Answer: $M(2^k) = (k - 1)2^k + 1$ for all $k \geq 1$ 4 marks

29 (8 marks)

(a) The list can be in any of $n!$ different orders, and the algorithm must be able to identify which m comparisons give at most 2^m different sequences of outcomes, so $2^m \geq n!$, that is $m \geq \log_2(n!)$ M1

Answer: $m \geq \log_2(n!)$ M1 E1

(b) $n = 4$: $4! = 24$, $2^4 = 16 < 24 \leq 32 = 2^5$, so at least 5 3 marks

$n = 5$: $120 \leq 128 = 2^7$ and $2^6 = 64 < 120$, so at least 7 B1

$n = 6$: $720 \leq 1024 = 2^{10}$ and $2^9 = 512 < 720$, so at least 10 B1

Answer: 5, 7, 10 B1

(c) Bubble sort: $\frac{5 \times 4}{2} = 10$ comparisons, compared with a lower bound of 7 3 marks

Bubble sort is not the most efficient possible: no method can use fewer than 7, but 10 is more than needed M1

Answer: 10 compared with at least 7 E1

30 (9 marks)

(a) 123: 0; 132: 1; 213: 1; 231: 2; 312: 2; 321: 3 swaps 2 marks

Mean = $\frac{0 + 1 + 1 + 2 + 2 + 3}{6} = \frac{9}{6} = 1.5$ M1 A1

Answer: Mean 1.5 A1

(b) Take any two of the n numbers. By symmetry the larger is to the left of the smaller in exactly 3 marks

half of the orderings, so the pair is an inversion with probability $\frac{1}{2}$ M1 A1

There are $\frac{n(n-1)}{2}$ pairs, and the mean number of inversions is the sum of the probabilities for the pairs M1

$$\text{Mean} = \frac{1}{2} \times \frac{n(n-1)}{2} = \frac{n(n-1)}{4} \quad \text{A1}$$

Answer: $\frac{n(n-1)}{4}$ 4 marks

(c) $n = 3$: $\frac{3 \times 2}{4} = 1.5$, agreeing with part (a); $n = 8$: $\frac{8 \times 7}{4} = 14$ M1

The greatest possible number of swaps is the number of pairs, $\frac{8 \times 7}{2} = 28$, which is twice the mean A1

Answer: 14; the maximum is 28 2 marks

Searching Algorithms

1 (2 marks)

Compare 17 with 8, then 3, then 17: a match M1

17 is in position 3, after 3 comparisons A1

Answer: Position 3; 3 comparisons 2 marks

2 (1 mark)

Every item must be checked: 6 comparisons B1

Answer: 6 1 mark

3 (2 marks)

Smallest: 1, if the item is first B1

Greatest: 20, if the item is last B1

Answer: 1 and 20 2 marks

4 (2 marks)

Start with 4; $11 > 4$, so the maximum becomes 11; 7 is not larger M1

$15 > 11$, so the maximum becomes 15; 9 is not larger. The values are 4, 11, 15 A1

Answer: 4, 11, 15 2 marks

5 (2 marks)

Check Leo, Priya, Sam, Amira M1

Amira is in position 4, found after 4 names are checked A1

Answer: Position 4; 4 names 2 marks

6 (2 marks)

Advantage: it works on a list which is not in order (or is simple to apply) B1

Disadvantage: it may have to check every item, so it is slow for long lists B1

Answer: Works on unsorted lists; slow for long lists 2 marks

7 (3 marks)

Compare 14 with 3, 6, 9, 12: each is smaller, so continue M1

15 is larger than 14, so stop: since the list is in order 14 cannot appear later A1

5 comparisons are made B1

Answer: Not in the list; 5 comparisons 3 marks

8 (3 marks)

All 8 items must be checked, since a 7 could be anywhere M1

7 occurs in positions 1, 3 and 6 A1

8 comparisons are needed B1

Answer: Positions 1, 3, 6; 8 comparisons 3 marks

9 (4 marks)

After 6, 15: (a, b) = (15, 6)

B1

After 9: (15, 9); after 21: (21, 15)

M1

After 12: no change, (21, 15); after 18: $18 > 15$ so (21, 18)

A1

The two largest numbers are 21 and 18

A1

Answer: 21 and 18

4 marks

10 (3 marks)

t = 7: i takes the values 1, 2, 3, 4; L[4] = 7, so the loop stops with i = 4

M1

Output 4

A1

t = 20: no item equals 20, so i reaches $6 > n = 5$, and "not found" is output

B1

Answer: 4; not found

3 marks

11 (3 marks)A search for the item in position r takes r comparisons, with probability $\frac{1}{n}$

M1

$$\text{Mean} = \frac{1}{n}(1 + 2 + \dots + n) = \frac{1}{n} \times \frac{n(n+1)}{2} = \frac{(n+1)}{2}$$

A1

$$n = 25: \frac{26}{2} = 13$$

B1

Answer: $\frac{(n+1)}{2}$; 13

3 marks

12 (3 marks)

Items 12, 7, 31, 5, then 19, 23, 8, 14, then 2, 27, 16

M1

16 is in row 3, column 3

A1

11 items are checked

A1

Answer: Row 3, column 3; 11 items

3 marks

13 (3 marks)

Compare in turn with 50: 12, 25, 31, 44 are not greater than 50

M1

52 is the first which is greater than 50

A1

It is in position 5, found after 5 comparisons

A1

Answer: 52 in position 5; 5 comparisons

3 marks

14 (5 marks)

31: compare with 5, 12, 18, 23, 31: found after 5 comparisons

M1

47: found after 7 comparisons

A1

20: compare with 5, 12, 18, 23; $23 > 20$ so stop, not found, after 4 comparisons

A1

70: every item is smaller than 70, so not found after all 9 comparisons

A1

Summary: 31 found (5), 47 found (7), 20 not found (4), 70 not found (9)

B1

Answer: 5, 7, 4, 9 comparisons

5 marks

15 (5 marks)21: the extended list is 8, 13, 5, 21, 3, 21; the first match is in position 4 after 4 comparisons; $4 \leq 5$, so 21 is in the list

M1 A1

9: the extended list ends with the sentinel 9; the first match is in position 6 after 6 comparisons; $6 > 5$, so 9 is not in the list

M1 A1

Advantage: each step needs only one comparison (with the target), since there is no need to test whether the end of the list has been reached

E1

Answer: 21 at position 4 (4 comparisons); 9 not found (6 comparisons)

5 marks

16 (5 marks)

(a) A search for the item in position r takes r comparisons

M1

$$\text{Mean} = 1(0.10) + 2(0.35) + 3(0.05) + 4(0.30) + 5(0.20) = 0.1 + 0.7 + 0.15 + 1.2 + 1.0 = 3.15$$

A1

Answer: 3.15

2 marks

(b) Put the items in descending order of probability: B, D, E, A, C

M1

$$\text{Mean} = 1(0.35) + 2(0.30) + 3(0.20) + 4(0.10) + 5(0.05)$$

M1

$$= 0.35 + 0.6 + 0.6 + 0.4 + 0.25 = 2.2$$

A1

Answer: B, D, E, A, C; 2.2**3 marks****17 (5 marks)****(a)** $i = 1$: $L[1] = 6$, no change; $i = 2$: $L[2] = 3$, so first = 2 and last = 2

M1

 $i = 4$: last = 4; $i = 6$: last = 6; other items are not 3

A1

Output 2, 6

A1

Answer: 2, 6**3 marks****(b)** No item is 7, so first and last stay 0: the output is 0, 0

B1

It finds the first and last positions of t in the list (0 meaning t is not present)

B1

Answer: 0, 0; finds first and last positions of t **2 marks****18 (4 marks)** $T = 16$: pairs (9, 4) = 13, (9, 11) = 20, (9, 7) = 16: found at the third pair

M1 A1

 $T = 25$: no pair has sum 25, so all of the pairs are examined: $\frac{6 \times 5}{2} = 15$

A1

For n items the greatest number of pairs is $\frac{n(n-1)}{2}$

B1

Answer: 3 pairs; 15 pairs; $\frac{n(n-1)}{2}$ **4 marks****19 (4 marks)**Time per item = $0.04 \div 2000 = 0.00002$ seconds

M1

10 000 items: $10\,000 \times 0.00002 = 0.2$ seconds

A1

1.5 seconds: $1.5 \div 0.00002 = 75\,000$ items

M1 A1

Answer: 0.2 seconds; 75 000 items**4 marks****20 (5 marks)**

Start 1: a, b, a match, then a against b: mismatch (4 comparisons)

M1

Start 2: b against a (1); start 3: a matches, then a against b (2)

A1

Start 4: a, b match, then b against a (3); start 5: b against a (1); start 6: b against a (1)

A1

Start 7: a, b, a, b all match (4 comparisons): the pattern starts at position 7

A1

Total $4 + 1 + 2 + 3 + 1 + 1 + 4 = 16$ comparisons

A1

Answer: Position 7; 16 comparisons**5 marks****21 (6 marks)****(a)** If present: mean $\frac{10+1}{2} = 5.5$ comparisons

M1

If absent: 10 comparisons

B1

Mean = $0.8 \times 5.5 + 0.2 \times 10 = 4.4 + 2 = 6.4$

A1

Answer: 6.4**3 marks****(b)** Mean = $5.5p + 10(1-p) = 10 - 4.5p$

M1

 $10 - 4.5p = 7$

M1

 $p = \frac{3}{4.5} = \frac{2}{3}$

A1

Answer: $\frac{2}{3}$ **3 marks****22 (7 marks)****(i)** Positions 3, 3, 5, 3, 2: total 16

M1 A1

(ii) C: 3 (list becomes C, A, B, D, E); C: 1; E: 5 (list E, C, A, B, D); C: 2 (list C, E, A, B, D); B: 4

M1 A1

Total $3 + 1 + 5 + 2 + 4 = 15$

A1

(iii) C: 3 (A, C, B, D, E); C: 2 (C, A, B, D, E); E: 5 (C, A, B, E, D); C: 1; B: 3. Total 14

M1 A1

Answer: (i) 16, (ii) 15, (iii) 14**7 marks**

23 (7 marks)**(a)** 27, then 20 ($19 < 27$ so left; $19 < 20$ so left)

M1

12 ($19 > 12$, so down), then 15 ($19 > 15$, so down)

A1

19, found: the numbers compared are 27, 20, 12, 15, 19

A1

Answer: 27, 20, 12, 15, 19

3 marks

(b) 27, 20, 24, 15, 19, 26, 22

M1

7 comparisons

A1

Answer: 7

2 marks

(c) Each comparison (other than the last) moves one place left or down, and there are at most $4 + 5 - 1 = 8$ numbers on any such path

M1

If the target is smaller than a number, it is smaller than everything below it in that column, so the column can be dropped (similarly the row if larger)

E1

Answer: At most $4 + 5 - 1 = 8$

2 marks

24 (6 marks)**(a)** After 3, 3: $c = 3$, $k = 2$

B1

After 4: $k = 1$; after 2: $k = 0$

M1

After 3: $c = 3$, $k = 1$; 3: $k = 2$; 5: $k = 1$; 3: $k = 2$; 1: $k = 1$

A1

Output $c = 3$

A1

Answer: 3

4 marks

(b) 3 appears 5 times out of 9, and $5 > 4.5$, so it is a majority item

B1

If no item occurs more than half the time the algorithm still outputs some candidate, so it must be checked by counting

E1

Answer: 3 appears 5 of 9 times; the candidate must be checked

2 marks

25 (6 marks)**(a)** $2 + 23 = 25 > 22$, so move right pointer left; $2 + 19 = 21 < 22$, so move left pointer right

M1

 $5 + 19 = 24 > 22$; $5 + 14 = 19 < 22$

A1

 $8 + 14 = 22$: found, using 8 and 14

A1

Answer: 8 and 14

3 marks

(b) Sums: $2 + 23 = 25$, $5 + 23 = 28$, $8 + 23 = 31$, $11 + 23 = 34$, $14 + 23 = 37$, $19 + 23 = 42$

M1

Then $19 + 19$ would be needed, but the pointers have met: 6 additions, no pair found

A1

Answer: 6 additions; none found

2 marks

(c) Every addition moves one pointer one place nearer the other, and the pointers start $n - 1$ places apart

E1

Answer: Pointers are $n - 1$ places apart

1 mark

26 (6 marks)**(a)** The pattern can start at positions 1 to 9: $12 - 4 + 1 = 9$ starts

M1

Each start compares aaa and then b with a , which is 4 comparisons: $9 \times 4 = 36$

A1

Answer: 36

2 marks

(b) There are $n - m + 1$ starting positions

M1

Each uses at most m comparisons, so at most $m(n - m + 1)$

A1

Answer: $m(n - m + 1)$

2 marks

(c) $m(13 - m)$ is greatest when m is nearest to 6.5, i.e. at $m = 6$ or 7

M1

Greatest value $6 \times 7 = 42$

A1

Answer: $m = 6$ or 7; 42

2 marks

27 (8 marks)**(a)** $H = \frac{60 + 30 + 20 + 15 + 12}{60} = \frac{137}{60}$

M1

Mean $= \sum r \times \frac{1}{r} \div H = \frac{5}{H} = \frac{5 \times 60}{137} = \frac{300}{137}$

M1

 $= 2.19$ to 2 d.p.

A1

Answer: $\frac{300}{137} = 2.19$

3 marks

(b) $\frac{1 + 2 + 3 + 4 + 5}{5} = 3$

B1

Answer: 3**1 mark**

(c) Only the two swapped items change position, so the change in the mean is $r p_{r+1} + (r + 1)p_r - r p_r - (r + 1)p_{r+1}$
 $= p_r - p_{r+1}$

M1**A1**

This is negative when $p_r < p_{r+1}$, so the mean is reduced

E1

Repeated swaps show the mean is least when the probabilities are in descending order, as in this list

E1**Answer: Descending order of probability****4 marks****28 (9 marks)**

(a) lo = 1, hi = 7, mid = 4: $L[4] = 3$ is not greater than $L[7] = 11$, so hi = 4

M1

lo = 1, hi = 4, mid = 2: $18 > 3$, so lo = 3; then lo = 3, hi = 4, mid = 3: $22 > 3$, so lo = 4

A1

lo = hi = 4: output $L[4] = 3$

A1**Answer: 3****3 marks**

(b) If $L[mid] > L[hi]$, then the rotation point (the smallest number) must lie after mid, since $L[mid]$ is in the first ascending run and $L[hi]$ in the second

M1 E1

Otherwise $L[mid] \leq L[hi]$, so $L[mid]$ to $L[hi]$ is ascending and the smallest number lies at or before mid

E1

Each pass makes the range of positions shorter by at least one, so $lo < hi$ eventually fails

E1**Answer: The invariant keeps the minimum between lo and hi****4 marks**

(c) Each pass leaves at most $\lceil \frac{\text{size}}{2} \rceil$ positions, so the range of size n is reduced to 1 after at most $\lceil \log_2 n \rceil$ passes

M1

$n = 1000$: $2^9 = 512 < 1000 \leq 1024 = 2^{10}$, so at most 10 repetitions

A1**Answer: 10****2 marks****29 (8 marks)**

(a) Every number other than the largest must lose at least one comparison (be shown to be smaller than something), otherwise it could be the largest

M1

Each comparison has exactly one loser, so at least $n - 1$ comparisons are needed

M1 E1**Answer: At least $n - 1$ comparisons****3 marks**

(b) Pairs: (14, 3), (9, 21), (6, 18), (11, 2), (25, 8): 5 comparisons, larger 14, 21, 18, 11, 25 and smaller 3, 9, 6, 2, 8

M1

Largest of the larger items: 4 comparisons, giving 25; smallest of the smaller items: 4 comparisons, giving 2

A1

Total $5 + 4 + 4 = 13$

A1**Answer: Largest 25, smallest 2; 13 comparisons****3 marks**

(c) $\frac{n}{2} + (\frac{n}{2} - 1) + (\frac{n}{2} - 1) = \frac{3n}{2} - 2$

M1

Separately: $(n - 1) + (n - 1) = 2n - 2$, which is greater for $n > 0$ (for $n = 10$: 18 compared with 13)

A1**Answer: $\frac{3n}{2} - 2$** **2 marks****30 (9 marks)**

(a) $25 \rightarrow 3$; $14 \rightarrow 3$ is full, so 4; $8 \rightarrow 8$; $36 \rightarrow 3$ and 4 full, so 5

M1

$19 \rightarrow 8$ full, so 9; $52 \rightarrow 8$ and 9 full, so 10

A1

$33 \rightarrow 0$. Positions: 0: 33, 3: 25, 4: 14, 5: 36, 8: 8, 9: 19, 10: 52

A1**Answer: 0: 33, 3: 25, 4: 14, 5: 36, 8: 8, 9: 19, 10: 52****3 marks**

(b) 25: 1; 14: 2; 8: 1; 36: 3; 19: 2; 52: 3; 33: 1

M1 A1

Total 13, so mean = $\frac{13}{7} \approx 1.86$ positions

A1**Answer: $\frac{13}{7} \approx 1.86$** **3 marks**

(c) $47 \bmod 11 = 3$: positions 3, 4, 5 are full and 6 is empty: 4 positions

M1 A1

$41 \bmod 11 = 8$: positions 8, 9, 10, 0 are full and 1 is empty: 5 positions

A1**Answer: 4 and 5 positions****3 marks**

Algorithm Complexity

1 (3 marks)

(a) Linear: order n

B1

Answer: n

1 mark

(b) Quadratic: order n^2

B1

Answer: n^2

1 mark

(c) Order $n \log n$

B1

Answer: $n \log n$

1 mark

2 (2 marks)

The list is 3 times as long, so the time is multiplied by $3^2 = 9$

M1

$0.5 \times 9 = 4.5$ seconds

A1

Answer: 4.5 seconds

2 marks

3 (2 marks)

The size is 4 times larger, so the time is 4 times longer

M1

$3 \times 4 = 12$ seconds

A1

Answer: 12 seconds

2 marks

4 (2 marks)

(a) $2^2 = 4$

B1

Answer: $\times 4$

1 mark

(b) $3^2 = 9$

B1

Answer: $\times 9$

1 mark

5 (2 marks)

$n = 10$: $100 + 50 = 150$; $n = 100$: $10\,000 + 500 = 10\,500$

B1

The n^2 term is more important (it is 95% of the total at $n = 100$)

B1

Answer: 150; 10 500; the n^2 term

2 marks

6 (1 mark)

$\log n$, n , n^2 , n^3 , 2^n

B1

Answer: $\log n$, n , n^2 , n^3 , 2^n

1 mark

7 (3 marks)

In the worst case the item r has to be compared with all $r - 1$ bins in use before it is placed in a new bin, so about $1 + 2 + \dots + (n - 1)$ comparisons are needed

M1

This is $\frac{n(n-1)}{2}$, which is of order n^2

A1

For 10 items: $\frac{10 \times 9}{2} = 45$ comparisons

A1

Answer: Quadratic order; 45 comparisons

3 marks

8 (3 marks)

Time $\propto n \log n$, so time $= 1.8 \times \frac{18\,000 \log 18\,000}{600 \log 600}$

M1

$= 1.8 \times \frac{18\,000 \times 9.798}{600 \times 6.397}$

A1

$= 82.7$ seconds

A1

Answer: 82.7 seconds

3 marks

9 (4 marks)

$n = 24$: the time is multiplied by $2^4 = 16$: $0.8 \times 16 = 12.8$ seconds

M1 A1

1 hour = 3600 s; the time is $0.8 \times 2^{n-20}$; $3600 \div 0.8 = 4500$, and $2^{12} = 4096 < 4500 < 8192 = 2^{13}$

M1

So $n - 20 = 12$ and the largest n is 32

A1

Answer: 12.8 seconds; $n = 32$

4 marks

10 (3 marks)

$n = 15$: $\frac{15 \times 14}{2} = 105$; $n = 30$: $\frac{30 \times 29}{2} = 435$

B1 B1

$\frac{n(n-1)}{2} = \frac{n^2 - n}{2}$: for large n the n^2 term dominates, so the time is of order n^2

E1

Answer: 105; 435

3 marks

11 (3 marks)

$n = 12$: $\frac{11!}{2} = \frac{39\,916\,800}{2}$

M1

= 19 958 400 tours

A1

Time = $19\,958\,400 \div 10^6 = 20.0$ seconds (3 s.f.)

A1

Answer: 19 958 400 tours; about 20 seconds

3 marks

12 (2 marks)

For large n the term $3n^2$ is much larger than $2n + 7$, and constants multiply the time only by a fixed factor

M1

So the algorithm is of order n^2

A1

Answer: Order n^2

2 marks

13 (4 marks)

B is faster when $50n < 2n^2$

M1

Dividing by $2n$ ($n > 0$): $25 < n$

M1 A1

So B is faster for $n > 25$ (that is $n \geq 26$ for whole numbers)

A1

Answer: $n > 25$

4 marks

14 (5 marks)

Comparisons for $n = 100$: $\frac{100 \times 99}{2} = 4950$; for $n = 1000$: $\frac{1000 \times 999}{2} = 499\,500$

M1 A1

(a) Time = $0.2 \times \frac{499\,500}{4950} = 20.18$ seconds

A1

(b) Time = $0.2 \times 10^2 = 20$ seconds

A1

The values are close; n^2 slightly overestimates since it ignores the $-n$ term, and the relative difference shrinks as n increases

E1

Answer: 20.18 s; 20 s

5 marks

15 (5 marks)

Doubling n from 1000 to 2000 multiplies the time by $1.6 \div 0.4 = 4$

M1

Doubling from 2000 to 4000 also multiplies it by $6.4 \div 1.6 = 4$

A1

A factor of 4 for doubling indicates a quadratic order

E1

$n = 10\,000$ is 10 times 1000, so the time is 0.4×10^2

M1

= 40 seconds

A1

Answer: Quadratic; 40 seconds

5 marks

16 (4 marks)

One million = 1000^2 , so $\log_2(10^6) = 2 \log_2(10^3)$

M1

The time is doubled: 4 microseconds

A1

Multiplying the size of the list by 1000 only doubles the time: the time grows very slowly

E1

(Check: $\log_2 10^6 \div \log_2 10^3 = 19.93 \div 9.97 = 2$)

B1

Answer: 4 microseconds

4 marks

17 (5 marks)

For each value of i the middle loop runs i times and the inner loop 5 times, adding $5i$

M1

$$\text{count} = 5(1 + 2 + \dots + n)$$

M1

$$= 5 \times \frac{n(n+1)}{2} = \frac{5n(n+1)}{2}$$

A1

$$n = 100: \frac{5 \times 100 \times 101}{2} = 25\,250$$

A1

The order is n^2

B1

Answer: 25 250; order n^2

5 marks

18 (4 marks)

(a) In 1 minute the computer performs 6×10^9 operations

B1

$$n^2 \leq 6 \times 10^9 \text{ gives } n \leq 77\,459.7, \text{ so } n = 77\,459$$

A1

Answer: 77 459

2 marks

(b) $2^{32} = 4.29 \times 10^9 \leq 6 \times 10^9$, but $2^{33} = 8.59 \times 10^9 > 6 \times 10^9$

M1

So $n = 32$

A1

Answer: 32

2 marks

19 (5 marks)

(a) Order n : size multiplies by 100

B1

Answer: $\times 100$

1 mark

(b) $(\text{new size})^2 = 100 n^2$

M1

So the size multiplies by 10

A1

Answer: $\times 10$

2 marks

(c) $2^m = 100 \times 2^n$, so $m = n + \log_2 100 = n + 6.64$

M1

The size increases by only 6 (whole numbers)

A1

Answer: +6

2 marks

20 (4 marks)

$$n^2 = 10^{12}$$

B1

$$n \log_2 n = 10^6 \times 19.93 = 1.993 \times 10^7$$

M1 A1

$$\text{Ratio} = 10^{12} \div (1.993 \times 10^7) = 5.0 \times 10^4, \text{ so about } 50\,000 \text{ times larger}$$

A1

Answer: About 50 000 times larger

4 marks

21 (6 marks)

(a) Fix the starting city; the other $n - 1$ cities can be visited in $(n - 1)!$ orders

M1

Each tour, and the same tour in the reverse direction, correspond to two of these orders

M1

Hence the number of different tours is $\frac{(n-1)!}{2}$

A1

Answer: $\frac{(n-1)!}{2}$

3 marks

(b) $n = 13: \frac{12!}{2} = 239\,500\,800 < 10^9$

M1

$n = 14: \frac{13!}{2} = 3\,113\,510\,400 > 10^9$

A1

So the smallest n is 14

A1

Answer: 14

3 marks

22 (5 marks)

$$\text{Time} \propto n^3: \text{time} = 3.2 \times \left(\frac{100}{40}\right)^3$$

M1

$$= 3.2 \times 15.625 = 50 \text{ seconds}$$

A1

$$3600 = 3.2 \times \left(\frac{n}{40}\right)^3, \text{ so } \left(\frac{n}{40}\right)^3 = 1125$$

M1

$$n = 40 \times 1125^{1/3} = 40 \times 10.40 = 416.1$$

M1

So the largest size is 416

A1

Answer: 50 seconds; $n = 416$

5 marks

23 (6 marks)

Q is faster when $200n \log_2 n < \frac{n^2}{10}$, that is $2000 \log_2 n < n$

M1 A1

Try $n = 25\,000$: $2000 \log_2 25\,000 = 29219 > 25\,000$, so P is still faster

M1

Try $n = 30\,000$: $2000 \log_2 30\,000 = 29745 < 30\,000$, so Q is faster

A1

The change occurs where $n = 2000 \log_2 n$: by trial, $n = 29719$ gives $2000 \log_2 n = 29718 < n$, while $n = 29718$ does not

M1

The smallest whole number is $n = 29719$

A1

Answer: $n = 29719$

6 marks

24 (6 marks)

Sorting: $5 \times 10^{-7} \times (10^5)^2 = 5000$ seconds

M1 A1

Searching: $10^5 \times 2 \times 10^{-6} \times \log_2 10^5 = 0.2 \times 16.61 = 3.32$ seconds

M1 A1

Total ≈ 5003 seconds; the sorting stage is more than 99.9% of the time

A1

The program is of order n^2 , the order of the larger term

E1

Answer: Total 5003 s; order n^2

6 marks

25 (6 marks)

$n = 10$: $2^{10} = 1024 > 1000 = 10^3$, so true for $n = 10$

B1

Assume $2^k > k^3$ for some $k \geq 10$. Then $2^{k+1} = 2 \times 2^k > 2k^3$

M1

Now $2k^3 \geq (k+1)^3$ is equivalent to $(1 + \frac{1}{k})^3 \leq 2$

M1

For $k \geq 10$, $(1 + \frac{1}{k})^3 \leq 1.1^3 = 1.331 < 2$

A1

So $2^{k+1} > (k+1)^3$

A1

True for $n = 10$, and true for $k+1$ whenever true for $k \geq 10$, so true for all $n \geq 10$ by induction

E1

Answer: $2^n > n^3$ for all integers $n \geq 10$

6 marks

26 (6 marks)

$10\,000a + 100b = 0.5$ and $40\,000a + 200b = 1.6$

M1 A1

Subtract $2 \times$ (first) from the second: $20\,000a = 0.6$, so $a = 3 \times 10^{-5}$

M1 A1

$100b = 0.5 - 0.3$, so $b = 0.002$

A1

$n = 1000$: $T = 3 \times 10^{-5} \times 10^6 + 0.002 \times 1000 = 30 + 2 = 32$ seconds

A1

Answer: $a = 3 \times 10^{-5}$, $b = 0.002$; 32 seconds

6 marks

27 (9 marks)

(a) The first pass compares about n items with the pivot, and leaves two sublists of $\frac{n}{2}$ items

M1

Each sublist is sorted in the same way, needing $C(\frac{n}{2})$ comparisons each

E1

Answer: $C(n) = 2C(n/2) + n$

2 marks

(b) $k = 0$: $C(1) = 0 = 0 \times 1$, true

B1

Assume $C(2^k) = k \times 2^k$. Then $C(2^{k+1}) = 2C(2^k) + 2^{k+1}$

M1

$= 2k \times 2^k + 2^{k+1} = (k+1)2^{k+1}$

A1

So the formula holds for $k+1$ and by induction for all $k \geq 0$

E1

Answer: $C(2^k) = k \times 2^k$ for all $k \geq 0$

4 marks

(c) $n = 2^{10}$, so $C(1024) = 10 \times 1024 = 10\,240$

M1

Bubble sort: $\frac{1024 \times 1023}{2} = 523\,776$

A1

The bubble sort needs about 51 times as many comparisons

A1

Answer: 10 240; 523 776

3 marks

28 (8 marks)

(a) 1 hour: 3.6×10^{12} operations; $15! = 1.31 \times 10^{12} \leq 3.6 \times 10^{12} < 16! = 2.09 \times 10^{13}$, so $n = 15$

M1 A1

1000 times faster: 3.6×10^{15} ; $17! = 3.56 \times 10^{14} \leq 3.6 \times 10^{15} < 18! = 6.40 \times 10^{15}$, so $n = 17$

A1

Answer: 15, then 17

3 marks

(b) 3.6×10^{12} ; $2^{41} = 2.20 \times 10^{12} \leq 3.6 \times 10^{12} < 2^{42} = 4.40 \times 10^{12}$, so $n = 41$

M1 A1

3.6×10^{15} ; $2^{51} = 2.25 \times 10^{15} \leq 3.6 \times 10^{15} < 2^{52} = 4.50 \times 10^{15}$, so $n = 51$

A1

Answer: 41, then 51

3 marks

(c) A speed-up by 1000 increases the size by only 2 for $n!$ and by about 10 ($\log_2 1000$) for 2^n

M1

Faster computers do not make large problems of exponential or factorial order feasible; better algorithms or heuristics are needed

E1

Answer: +2 and +10: faster hardware helps very little

2 marks

29 (8 marks)

(a) $T(n) = 3T(\frac{n}{2})$

B1

$k = 0$: $T(1) = 1 = 3^0$

B1

Assume $T(2^k) = 3^k$. Then $T(2^{k+1}) = 3T(2^k) = 3^{k+1}$

M1

So true for $k + 1$, and by induction for all $k \geq 0$

E1

Answer: $T(2^k) = 3^k$

4 marks

(b) $n = 2^{10}$: $T = 3^{10} = 59\,049$

M1

Standard: $1024^2 = 1\,048\,576$, about 17.8 times as many

A1

Answer: 59 049 compared with 1 048 576

2 marks

(c) $n = 2^k$ gives $3^k = (2^{\log_2 3})^k = (2^k)^{\log_2 3} = n^{\log_2 3}$, and $\log_2 3 = 1.58$

M1

Order $n^{1.58}$, which grows more slowly than n^2 for large n

A1

Answer: $n^{1.58}$

2 marks

30 (8 marks)

(a) Copying happens when the array holds 1, 2, 4, ..., 512 items

M1

Total = $1 + 2 + 4 + \dots + 512 = 2^{10} - 1$

M1

= 1023

A1

Answer: 1023

3 marks

(b) Copying occurs at sizes 1, 2, ..., 2^m : geometric series with sum $2^{m+1} - 1$

M1 A1

Since $2^m < n$, $2^{m+1} < 2n$, so the number of copies is less than $2n$

E1

Answer: $2^{m+1} - 1 < 2n$

3 marks

(c) Mean number of copies per item $< \frac{2n}{n} = 2$

M1

The total time is of order n (linear), although individual additions can be expensive

E1

Answer: Less than 2 per item; order n

2 marks

Binary Search

1 (1 mark)

The items must be in order (sorted), for example ascending numerical or alphabetical order

B1

Answer: The list must be sorted

1 mark

2 (2 marks)

The list has 7 items, so the middle item is the 4th, which is 17

B1

$25 > 17$, so the search continues with the items after 17

B1

Answer: 17; the items after it

2 marks

3 (3 marks)

First middle item 17; $12 < 17$, so use 3, 8, 12

M1

Middle item 8; $12 > 8$, so use 12

A1

Middle item 12: found

A1

Answer: 17, 8, 12

3 marks

4 (2 marks)

Each comparison leaves at most half the items: $15 \rightarrow 7 \rightarrow 3 \rightarrow 1$

M1

At most 4 comparisons

A1

Answer: 4

2 marks

5 (2 marks)

(a) The two middle items are 4th and 5th, so the 5th

B1

Answer: 5th

1 mark

(b) The 3rd and 4th are in the middle, so the 4th

B1

Answer: 4th

1 mark

6 (2 marks)

(a) 1000

B1

Answer: 1000

1 mark

(b) 10, since $2^{10} = 1024 \geq 1000$ (the list is halved at most 10 times)

B1

Answer: 10

1 mark

7 (4 marks)

1: Aziz, Bella, Chen, Dara, **Eli**, Fran, Gus, Hana, Ivan Gus $>$ Eli, so use the items after Eli

M1

2: Fran, Gus, **Hana**, Ivan Gus $<$ Hana, so use the items before Hana

A1

3: Fran, **Gus** Gus found

A1

Gus is found after 3 comparisons

B1

Answer: Found after 3 comparisons

4 marks

8 (4 marks)

1: 4, 9, 13, 18, 22, **27**, 33, 38, 45, 51 $20 < 27$, so use the items before 27

M1

2: 4, 9, **13**, 18, 22 $20 > 13$, so use the items after 13

3: 18, **22** $20 < 22$, so use the items before 22

A1

4: **18** $20 > 18$, so use the items after 18

A1

With no items left, 20 is not in the list

B1

Answer: Not in the list

4 marks

9 (4 marks)

The middle item (9) needs 1 comparison

B1

The items 5 and 15 (the middle items of the two halves) need 2 comparisons each

M1

The other four items (2, 7, 12, 18) need 3 comparisons each

A1

$$\text{Mean} = \frac{1 + 2 + 2 + 3 + 3 + 3 + 3}{7} = \frac{17}{7} = 2.43$$

A1

Answer: $\frac{17}{7} \approx 2.43$

4 marks

10 (3 marks)

(a) $2^6 = 64 \leq 100 < 128 = 2^7$, so 7

B1

Answer: 7

1 mark

(b) $2^9 = 512 \leq 1000 < 1024 = 2^{10}$, so 10

B1

Answer: 10

1 mark

(c) $2^{19} = 524\,288 \leq 1\,048\,576 = 2^{20}$, so 20

B1

Answer: 20

1 mark

11 (4 marks)

lo = 1, hi = 11; mid = 6, $L[6] = 29 < 41$, so lo = 7

M1

lo = 7, hi = 11; mid = 9, $L[9] = 47 > 41$, so hi = 8

A1

lo = 7, hi = 8; mid = 8, $L[8] = 41 = t$, so found becomes true

A1

The loop ends: t is in position 8

A1

Answer: Found in position 8

4 marks

12 (4 marks)

Guess $\frac{1+100}{2} = 50.5$, so 50: too low, so the number is 51 to 100 M1

Guess 75: too high, so 51 to 74; guess 62: too low, so 63 to 74 A1

Guess 68: too low, so 69 to 74; guess 71: too low, so 72 to 74 A1

Guess 73: correct. The guesses are 50, 75, 62, 68, 71, 73 A1

Answer: 50, 75, 62, 68, 71, 73 4 marks

13 (3 marks)

Items 1 to 20: the middle item is the 11th (right-hand of 10th and 11th) B1

Items 12 to 20: the 16th; items 17 to 20: the 19th; item 20 only: the 20th M1

Positions 11, 16, 19, 20; 4 comparisons, and then no items remain A1

Answer: 11, 16, 19, 20; 4 comparisons 3 marks

14 (6 marks)

(a) 1: 3, 7, 12, 18, 21, 26, **31**, 37, 42, 48, 55, 63 $37 > 31$, so use the items after 31 M1

2: 37, 42, **48**, 55, 63 $37 < 48$, so use the items before 48

3: 37, **42** $37 < 42$, so use the items before 42

4: **37** 37 found A1

4 comparisons B1

Answer: found after 4 comparisons 3 marks

(b) 1: 3, 7, 12, 18, 21, 26, **31**, 37, 42, 48, 55, 63 $50 > 31$, so use the items after 31 M1

2: 37, 42, **48**, 55, 63 $50 > 48$, so use the items after 48

3: 55, **63** $50 < 63$, so use the items before 63

4: **55** $50 < 55$, so use the items before 55

No items remain: 50 is not in the list A1

4 comparisons B1

Answer: not in the list, after 4 comparisons 3 marks

15 (6 marks)

Pass 1 (pivot 33): 27, 14, 9, 18, 22, 6, **33**, 41, 50 M1

Pass 2 (pivots 18, 50): 14, 9, 6, **18**, 27, 22, **33**, 41, **50** A1

Pass 3 (pivots 9, 22): **6**, **9**, **14**, **18**, **22**, **27**, **33**, **41**, **50** A1

1: 6, 9, 14, 18, **22**, 27, 33, 41, 50 $33 > 22$, so use the items after 22 M1

2: 27, 33, **41**, 50 $33 < 41$, so use the items before 41 A1

3: 27, **33** 33 found A1

Answer: Position 7 in the sorted list 6 marks

16 (5 marks)

Linear: 500.5k comparisons for k searches B1

Sort first: at most 499 500 + 10k comparisons B1

Sorting first is better when $499\,500 + 10k < 500.5k$ M1

$490.5k > 499\,500$, so $k > 1018.3$ A1

The smallest whole number is $k = 1019$ A1

Answer: k = 1019 5 marks

17 (4 marks)

31 items: after each comparison at most 15, 7, 3, 1 items remain, so 5 comparisons in all ($2^5 = 32 > 31$) M1 A1

32 items: $2^5 = 32$ is not greater than 32, so 6 comparisons are needed A1

One extra item means that after the first comparison a list of 16 (not 15) may remain, requiring one more halving E1

Answer: 5; 6 4 marks

18 (5 marks)

The list has 7 items, so the first middle item is the 4th, which is 7: the search stops at position 4 **B1**
 Change: when the middle item equals the target, record its position and continue the search with the items before it **M1**
 Items 4, 7, 7: middle item (position 3) is 7, record 3, continue with 4, 7 **M1**
 Items 4, 7: middle item (position 2) is 7, record 2, continue with 4 **A1**
 Item 4 is not 7, nothing more remains: the first 7 is in position 2 **A1**
Answer: Position 2 **5 marks**

19 (5 marks)

lo = 1, hi = 100: mid = 50, f = 2650, so lo = 50 **M1**
 mid = 75, f = 5850 > 5000, so hi = 75; mid = 62, f = 4030, so lo = 62 **A1**
 mid = 68, f = 4828, so lo = 68; mid = 71, f = 5254, so hi = 71 **A1**
 mid = 69, f = 4968, so lo = 69; mid = 70, f = 5110, so hi = 70 **A1**
 hi - lo = 1: the smallest n is 70 **A1**
Answer: n = 70 **5 marks**

20 (4 marks)

(a) $1\,000\,000 \times 3\,\mu\text{s} = 3\,000\,000\,\mu\text{s} = 3\text{ seconds}$ **B1**
Answer: 3 s **1 mark**
 (b) At most 20 comparisons ($2^{19} = 524\,288 \leq 10^6 < 2^{20}$): $20 \times 3 = 60\,\mu\text{s}$ **M1 A1**
Answer: 60 μs **2 marks**
 (c) $3\,000\,000 \div 60 = 50\,000$ times longer **A1**
Answer: 50 000 times **1 mark**

21 (7 marks)

(a) The first comparison deals with 1 item, the middle one, leaving two lists of $2^{k-1} - 1$ items **M1**
 Repeating the argument, the items first used at the i th comparison number 2^{i-1} **E1**
Answer: 2^{i-1} items need i comparisons **2 marks**
 (b) Total comparisons $T = 1 + 2 \times 2 + 3 \times 4 + \dots + k \times 2^{k-1}$ **M1**
 $2T = 1 \times 2 + 2 \times 4 + \dots + (k-1)2^{k-1} + k \times 2^k$, and so $T = k \times 2^k - (1 + 2 + 4 + \dots + 2^{k-1})$ **M1 A1**
 $T = k \times 2^k - (2^k - 1) = (k-1)2^k + 1$, and dividing by $2^k - 1$ gives the mean **A1**
Answer: $\frac{(k-1)2^k + 1}{2^k - 1}$ **4 marks**
 (c) $k = 10: \frac{9 \times 1024 + 1}{1023} = \frac{9217}{1023} = 9.01$ **B1**
Answer: 9.01 **1 mark**

22 (5 marks)

The list has 10 items, so the right-hand middle item is the 6th, 31, not the 5th, 26 **M1 E1**
 1: 5, 9, 14, 20, 26, **31**, 37, 44, 52, 60 $44 > 31$, so use the items after 31 **M1**
 2: 37, 44, **52**, 60 $44 < 52$, so use the items before 52 **A1**
 3: 37, **44** found **A1**
Answer: 3 comparisons (31, 52, 44), not 2 **5 marks**

23 (6 marks)

(a) The middle item is in position $\lfloor \frac{n}{2} \rfloor + 1$, so the items before it number $\lfloor \frac{n}{2} \rfloor$ **M1**
 The items after it number $n - \lfloor \frac{n}{2} \rfloor - 1 \leq \lfloor \frac{n}{2} \rfloor$, so at most $\lfloor \frac{n}{2} \rfloor$ remain **A1**
Answer: At most $\lfloor n/2 \rfloor$ remain **2 marks**
 (b) After one comparison at most $\lfloor \frac{n}{2} \rfloor$ items remain; the worst case is the larger remaining list **M1**
 Repeating, the number of items is $\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{4} \rfloor, \dots$ until 0, which takes one step more than the number **M1**
 of times n can be halved while the result is at least 1
 That number is $\lfloor \log_2 n \rfloor$, so $f(n) = \lfloor \log_2 n \rfloor + 1$ **A1**

Answer: $\lfloor \log_2 n \rfloor + 1$ **3 marks****(c) $f(1000) = 9 + 1 = 10$; $f(1024) = 10 + 1 = 11$** **B1****Answer: 10; 11****1 mark****24 (6 marks)**1: apple, banana, cherry, damson, elder, fig, grape, **guava**, kiwi, lemon, lime, mango, melon, olive, peach lychee > guava, so use the items after guava **M1**2: kiwi, lemon, lime, **mango**, melon, olive, peach lychee < mango, so use the items before mango **A1**3: kiwi, **lemon**, lime lychee > lemon, so use the items after lemon **A1**4: **lime** lychee > lime, so use the items after lime **A1**Nothing remains, so lychee is not in the list, after 4 comparisons **A1**(Alphabetical comparison: lychee comes after lime but before mango and melon) **B1****Answer: Not in the list; 4 comparisons****6 marks****25 (6 marks)**[3, 4]: $x = 3.5$, $x^2 - 10 = 2.2500$ **M1**[3, 3.5]: $x = 3.25$, $x^2 - 10 = 0.5625$ **A1**[3, 3.25]: $x = 3.125$, $x^2 - 10 = -0.2344$ [3.125, 3.25]: $x = 3.1875$, $x^2 - 10 = 0.1602$ **A1**[3.125, 3.1875]: $x = 3.15625$, $x^2 - 10 = -0.0381$ **A1**The final interval is [3.15625, 3.1875], of length $0.03125 < 0.05$, so $\sqrt{10}$ lies in [3.15625, 3.1875] **A1 A1****Answer: $3.15625 < \sqrt{10} < 3.1875$ (interval of length 0.03125)****6 marks****26 (7 marks)**First stage: positions 1, 2, 4, 8, 16, 32 hold smaller items, and position 64 holds a larger item, so the positions compared are 1, 2, 4, 8, 16, 32, 64: 7 comparisons **M1 A1**Binary search of positions 33 to 64 (32 items): middle (right) item is position 49, 37 is before it, leaving positions 33 to 48 (16 items) **M1**Middle position 41: 37 is before it, leaving positions 33 to 40; middle position 37: found **M1 A1**Binary stage uses 3 comparisons, total $7 + 3 = 10$ **A1**Check that 32 items need at most 6 comparisons, and 3 are used **B1****Answer: 10 comparisons in total****7 marks****27 (8 marks)****(a)** The target may be in any of the n positions, or absent, which gives $n + 1$ different results **M1**Each comparison has 2 outcomes, so after m comparisons there are at most 2^m different sequences of outcomes **M1**Hence $2^m \geq n + 1$ **A1**so $m \geq \log_2(n + 1)$, and m is a whole number so $m \geq \lceil \log_2(n + 1) \rceil$ **E1****Answer: $m \geq \lceil \log_2(n + 1) \rceil$** **4 marks****(b)** Let $2^k \leq n < 2^{k+1}$, so that $\lfloor \log_2 n \rfloor = k$ **M1**Then $2^k < n + 1 \leq 2^{k+1}$, so $k < \log_2(n + 1) \leq k + 1$, so $\lceil \log_2(n + 1) \rceil = k + 1$ **M1 A1**So the greatest number of comparisons for binary search equals the lower bound in part (a): no algorithm can do better in the worst case **E1****Answer: Equal, so binary search is optimal****4 marks****28 (8 marks)****(a)** $lo = 1$, $hi = 10$: $mid = 5$, $L[5] = 20 < L[6] = 26$, so $lo = 6$ **M1** $mid = 8$: $15 > 8$, so $hi = 8$; $mid = 7$: $21 > 15$, so $hi = 7$; $mid = 6$: $26 > 21$, so $hi = 6$ **A1** $lo = hi = 6$: output 6 (the largest item is 26) **A1****Answer: 6****3 marks****(b)** If $L[mid] < L[mid + 1]$ the list is still rising at mid , so the largest item is after mid : $lo = mid + 1$ keeps it in range **M1**Otherwise the list is falling after mid (or at its top), so the largest item is at mid or before it: $hi = mid$ keeps it in range **A1**

The range of positions gets shorter each time, so the loop ends when $lo = hi$

Answer: Invariant: the largest item is in positions lo to hi

E1

3 marks

(c) Each pass leaves at most $\lceil \frac{\text{size}}{2} \rceil$ positions, so at most $\lceil \log_2 1000 \rceil = 10$ passes

M1

A linear search needs 999 comparisons: the binary approach works because the list is in a known order (increasing then decreasing)

E1

Answer: 10

2 marks

29 (8 marks)

(a) $27 \rightarrow 9 \rightarrow 3 \rightarrow 1$ (three stages), and each stage uses 2 comparisons: 6 comparisons

M1 A1

Binary search: $\lceil \log_2 27 \rceil + 1 = 5$ comparisons, so binary search is quicker here

A1

Answer: 6 against 5

3 marks

(b) m stages, each with 2 comparisons: $2m$ comparisons, where $m = \log_3 n$

M1

$\log_3 n = \frac{\log_2 n}{\log_2 3}$, so the number is $\frac{2}{\log_2 3} \log_2 n$

A1

$\frac{2}{\log_2 3} = \frac{2}{1.585} = 1.26$

A1

Answer: $1.26 \log_2 n$

3 marks

(c) The binary search needs about $\log_2 n$ comparisons and the ternary search about $1.26 \log_2 n$

M1

So the ternary search needs about 26% more comparisons

A1

Answer: About 26% more

2 marks

30 (8 marks)

(a) First comparison: position 7 (1 item). The two remaining lists have 6 and 5 items

M1

Second: positions 4 and 10 (2 items). Third: positions 2, 6, 9, 12 (4 items)

A1

Fourth: positions 1, 3, 5, 8, 11 (5 items); the numbers at each level are 1, 2, 4, 5

A1

Answer: Levels: 1, 2, 4, 5 items

3 marks

(b) Total = $1 \times 1 + 2 \times 2 + 3 \times 4 + 4 \times 5 = 1 + 4 + 12 + 20 = 37$

M1

Mean = $\frac{37}{12} = 3.08$

A1

Answer: $\frac{37}{12} \approx 3.08$

2 marks

(c) Mean = $\frac{10 \times 4 + 3 \times 3}{13} = \frac{49}{13} = 3.77$

M1 A1

The 3-comparison cases are targets just above the items in positions 6, 9 and 12: those items have no item to their right in the tree, so the list is empty after 3 comparisons

E1

Answer: $\frac{49}{13} \approx 3.77$

3 marks

Bin Packing Algorithms

1 (2 marks)

Bin 1: 4, 2

M1

Bin 2: 7

Bin 3: 5

Bin 4: 6

A1

Answer: 4 bins: 4, 2; 7; 5; 6

2 marks

2 (2 marks)

$47 \div 12 = 3.92$

M1

At least 4 bins are needed

A1

Answer: 4

2 marks

3 (1 mark)

Descending order: 8, 7, 5, 3, 2

B1

Answer: 8, 7, 5, 3, 2

1 mark

4 (2 marks)

Bin 1: 6, 3

M1

Bin 2: 5, 4

Bin 3: 2

A1

Answer: 3 bins: 6, 3; 5, 4; 2

2 marks

5 (2 marks)

The algorithms are heuristics: they follow a simple rule so may give a solution which is not optimal

E1

They are quick to apply, even for large numbers of items

E1

Answer: Heuristic: quick but not always optimal

2 marks

6 (2 marks)Free space = $20 - 16 = 4$

M1

Items 3 and 4 fit (individually); 5 does not

A1

Answer: 3 and 4

2 marks

7 (8 marks)**(a)** Total = 42; $42 \div 15 = 2.80$

M1

At least 3 bins

A1

Answer: 3

2 marks

(b) Bin 1: 7, 5, 3

M1

Bin 2: 9, 4

A1

Bin 3: 6, 8

A1

Answer: 3 bins

3 marks

(c) Descending order: 9, 8, 7, 6, 5, 4, 3

M1

Bin 1: 9, 6

M1

Bin 2: 8, 7

Bin 3: 5, 4, 3

A1

Answer: 3 bins

3 marks

8 (3 marks)

Bin 1: 1.8, 2.4, 0.6

M1

Bin 2: 3.1, 1.6

Bin 3: 2.7, 1.3

A1

Bin 4: 3.4, 1.2

Bin 5: 2.2

A1

Answer: 5 bins

3 marks

9 (4 marks)**(a)** Bin 1: 12, 8

M1

Bin 2: 15, 5

Bin 3: 9, 6

A1

Bin 4: 11

A1

Answer: 4 bins

3 marks

(b) First-fit gives 12, 8; 15, 5; 9, 6; 11: 4 bins

B1

Answer: 4 bins

1 mark

10 (4 marks)

Descending: 9, 8, 7, 6, 5, 4, 3, 2

M1

Bin 1: 9, 5 (total 14)

M1

Bin 2: 8, 6 (total 14)

Bin 3: 7, 4, 3 (total 14)

Bin 4: 2 (total 2)

A1

The total is 44 and $44 \div 14 = 3.14$, so at least 4 bins are needed; as 4 bins are used the solution is optimal

E1

Answer: 4 bins; optimal

4 marks

11 (3 marks)

First-fit gives 5, 5, 5 (15); 6, 6 (12); 6: 3 bins

M1

Total = 33 and $33 \div 16 = 2.06$, so at least 3 bins are needed

M1

So the claim is correct: 3 bins is the minimum (no better packing exists)

A1

Answer: Claim correct: 3 bins is minimal

3 marks

12 (4 marks)Total = 56; $56 \div 20 = 2.80$, so at least 3 boxes

M1 A1

Bin 1: 14, 6

M1

Bin 2: 11, 9

Bin 3: 8, 5, 3

A1

Answer: 3 boxes

4 marks

13 (4 marks)

Bin 1 holds 11 and 6, a total of 17

B1

x is not placed in bin 1, so $17 + x > 20$, giving $x > 3$

M1

Bin 2 holds 9, and x fits there, so $9 + x \leq 20$, giving $x \leq 11$

M1

So $4 \leq x \leq 11$

A1

Answer: $4 \leq x \leq 11$

4 marks

14 (8 marks)**(a)** Total = 82, and $82 \div 25 = 3.28$

M1

At least 4 bins

A1

Answer: 4

2 marks

(b) Bin 1: 14, 9 (total 23)

M1

Bin 2: 6, 11, 5 (total 22)

Bin 3: 8, 12 (total 20)

A1

Bin 4: 7, 10 (total 17)

A1

Answer: 4 bins

3 marks

(c) Descending: 14, 12, 11, 10, 9, 8, 7, 6, 5

M1

Bin 1: 14, 11 (total 25)

M1

Bin 2: 12, 10 (total 22)

Bin 3: 9, 8, 7 (total 24)

Bin 4: 6, 5 (total 11)

A1

Answer: 4 bins

3 marks

15 (6 marks)**(a)** Next-fit: 6, 4; 7, 3; 8, 2; 5 (4 bins)

M1 A1

Answer: 4 bins

2 marks

(b) First-fit: 6, 4, 2; 7, 3; 8; 5 (4 bins)

M1 A1

Answer: 4 bins

2 marks

(c) Worst-fit: 6, 4; 7, 3; 8, 2; 5 (4 bins)

M1 A1

Answer: 4 bins

2 marks

16 (4 marks)Total = 44, and $44 \div 16 = 2.75$, so at least 3 bins are needed

M1 A1

A packing: 9, 7 (16); 8, 6, 2 (16); 5, 4, 3 (12)

M1

This uses 3 bins, so 3 is the minimum

A1

Answer: 3 bins

4 marks

17 (5 marks)14 and 6 share bin 1, so $n \geq 20$

B1

9 did not fit in bin 1, so $14 + 9 > n$, giving $n < 23$

M1

11 did not fit in bin 1 (already 20), which is consistent, and $9 + 11 = 20 \leq n$

M1

5 fits in neither full bin, so $20 + 5 > n$, giving $n < 25$, which is already satisfied

M1

So $n = 20, 21$ or 22

A1

Answer: $n = 20, 21$ or 22

5 marks

18 (4 marks)

Bin 1: 0.6, 0.3 (0.9)

M1

Bin 2: 0.45, 0.25, 0.2 (0.9); Bin 3: 0.7, 0.15 (0.85); Bin 4: 0.5 (0.5)

A1

Total packed = $0.6 + 0.3 + 0.45 + 0.25 + 0.7 + 0.2 + 0.5 + 0.15 = 3.15$

M1

Free space = $4 - 3.15 = 0.85$

A1

Answer: 4 bins; free space 0.85

4 marks

19 (5 marks)

First-fit (list order): 10, 20, 30; 40; 50

M1 A1

First-fit decreasing: 50, 10; 40, 20; 30

M1 A1

Total = 150, $150 \div 60 = 2.5$ so at least 3 bins; 3 bins are used, so the student is right

A1

Answer: Agree: 3 bins is the minimum

5 marks

20 (5 marks)

First-fit: 8, 2; 3, 4; 5; 9; 6 (5 bins)

M1 A1

Total = 37, so at least 4 bins are needed

B1

A better packing: 9; 8, 2; 6, 4; 5, 3 (4 bins)

M1 A1

Answer: 4 bins

5 marks

21 (8 marks)

(a) Bin 1: 5, 4, 6 (total 15)

M1

Bin 2: 11, 7 (total 18)

Bin 3: 8, 10 (total 18)

A1

Bin 4: 8 (total 8)

A1

Answer: 4 bins

3 marks

(b) Descending: 11, 10, 8, 8, 7, 6, 5, 4

M1

Bin 1: 11, 8 (total 19)

M1

Bin 2: 10, 8 (total 18)

Bin 3: 7, 6, 5 (total 18)

Bin 4: 4 (total 4)

A1

Answer: 4 bins

3 marks

(c) Total = 59, so at least 3 bins; a packing in 3 bins: 11, 8; 10, 6, 4; 8, 7, 5

M1

Each bin total is at most 20, so 3 bins suffice and neither algorithm (4 bins) is optimal

A1

Answer: 3 bins

2 marks

22 (5 marks)Total = 34 and $34 \div 10 = 3.4$, so at least 4 bins

M1 A1

Bin 1: 7, 3 (total 10)

M1

Bin 2: 6, 4 (total 10)

Bin 3: 5, 4 (total 9)

A1

Bin 4: 3, 2 (total 5)

A1

Answer: 4 bins

5 marks

23 (6 marks)Total = 310 m; $310 \div 100 = 3.10$ so at least 4 rolls

B1

Descending: 52, 45, 40, 35, 30, 28, 25, 22, 18, 15

M1

Bin 1: 52, 45 (total 97)

M1

Bin 2: 40, 35, 25 (total 100)

Bin 3: 30, 28, 22, 18 (total 98)

Bin 4: 15 (total 15)

A1

Waste = $4 \times 100 - 310 = 90$ m

M1

First-fit decreasing is a heuristic; a different packing may use fewer rolls

E1

Answer: 4 rolls; 90 m wasted

6 marks

24 (5 marks)

$14 + 9 = 23 > 20$, so 9 cannot go in bin 1

M1 E1

Correct first-fit: 14, 6; 9, 11; 5, 8

M1 A1

Bin 2 in the student's work holds $6 + 11 + 5 = 22 > 20$ which is also impossible

E1

Answer: 14, 6; 9, 11; 5, 8

5 marks

25 (6 marks)

Suppose two bins X and Y are both at most half full, with X used before Y

M1

Then the items in Y are each at most $C/2$, and X had at least $C/2$ free space

M1 A1

The first item put in Y would have fitted in X, contradicting first-fit

E1

So at most one bin is at most half full; every other bin is more than half full

A1

If b bins are used, total size $> (b - 1)C/2$, so $b < 2 \times (\text{total} \div C) + 1$

E1

Answer: At most one bin is half full or less

6 marks

26 (6 marks)

Bin 1 holds 0.5 and 0.02 (0.52); the next 0.5 does not fit, so it starts bin 2

M1

Each bin holds one 0.5 and one 0.02, so next-fit uses 10 bins

A1

Optimal: the ten 0.5s fill 5 bins exactly (two per bin)

M1

The ten 0.02s total 0.2, which fit in one further bin: 6 bins

A1

Next-fit uses 10 bins, 4 more than necessary

A1

Total size = 5.2, so at least 6 bins: 6 is optimal

B1

Answer: 10 bins; optimal 6

6 marks

27 (10 marks)

(a) Let the bins be B and B' (B' the next). The first item placed in B' did not fit in B

M1

so (contents of B) + (that item) $> C$

M1

and the total in B and B' includes both, so exceeds C

A1

Answer: The total in any two consecutive bins is greater than C

3 marks

(b) Suppose b bins are used. Pair up bins 1&2, 3&4, ...: there are at least $\lfloor b/2 \rfloor$ pairs, each of total $> C$

M1

So total size $> \lfloor b/2 \rfloor C$

M1

$\lfloor b/2 \rfloor \geq (b - 1)/2$, so total $> (b - 1)C/2$

A1

Hence $b < 2(\text{total} \div C) + 1$

A1

Answer: Number of bins $b < 2 \times (\text{total size} \div C) + 1$

4 marks

(c) $15 \div 4 = 3.75$, so the optimal number is at least 4

M1

With b bins, total $> \lfloor b/2 \rfloor \times 4$. For $b = 8$ this needs total > 16 , which is impossible, so at most 7 bins

M1

7 bins can occur: seven items of size 2.1 and one of size 0.3 (total 15) give 7 bins, so the greatest is 7, compared with the lower bound of 4

A1

Answer: 7 bins; lower bound 4

3 marks

28 (8 marks)

(a) Total = 59, so at least 3 boxes

B1 M1

First-fit decreasing: 9, 8, 3; 7, 7, 6; 5, 5, 4, 3, 2

M1 A1

Answer: 3 boxes; lower bound 3

4 marks

(b) The two smallest items 2 and 3 are replaced by 5: the total is unchanged, so the lower bound is unchanged at 3

M1 A1

Sorted list becomes 9, 8, 7, 7, 6, 5, 5, 5, 4, 3: first-fit decreasing gives 9, 8, 3; 7, 7, 6; 5, 5, 5, 4

M1

No change in the number of boxes (3), so nothing is gained

A1

Answer: The lower bound is unchanged at 3; first-fit decreasing gives 3 boxes, the same as before

4 marks

29 (8 marks)**(a)** Sizes: 0.51, 0.26 and 0.23**B1**

The six 0.23s: four fit in bin 1 (0.92); the other two go in bin 2

M1

The six 0.26s: two fit with the 0.23s in bin 2 (0.98); the other four go three to bin 3 (0.78) and one to bin 4

M1No 0.51 fits with another 0.51; one fits in bin 4 ($0.26 + 0.51 = 0.77$), so the other five need a bin each**M1**

Total number of bins = 9

A1**Answer: 9 bins****5 marks****(b)** Each bin: $0.51 + 0.26 + 0.23 = 1.00$ **M1 A1**

6 bins hold all 18 items, so 6 bins are enough; first-fit uses more

E1**Answer: 6 bins****3 marks****30 (11 marks)****(a)** Pass 1 (pivot 27): 31, 35, **27**, 23, 8, 14, 6, 19, 11, 4**M1**Pass 2 (pivots 35, 6): **35**, 31, **27**, 23, 8, 14, 19, 11, **6**, 4**A1**Pass 3 (pivot 14): **35**, **31**, **27**, 23, 19, **14**, 8, 11, **6**, **4****A1**Pass 4 (pivots 19, 11): **35**, **31**, **27**, **23**, **19**, **14**, **11**, **8**, **6**, **4****A1****Answer: 35, 31, 27, 23, 19, 14, 11, 8, 6, 4****4 marks****(b)** Bin 1: 35, 14 (total 49)**M1**

Bin 2: 31, 19 (total 50)

Bin 3: 27, 23 (total 50)

A1

Bin 4: 11, 8, 6, 4 (total 29)

A1**Answer: 4 sticks****3 marks****(c)** Total = 178 MB and $178 \div 50 = 3.56$, so at least 4 sticks are needed**M1**

4 sticks are used, so the packing is optimal

A1**Answer: optimal****2 marks****(d)** Time = $0.9 \times \frac{12\,000 \log 12\,000}{400 \log 400}$ **M1**

= 42.3 seconds

A1**Answer: 42.3 seconds****2 marks**